Generalizations of polylogarithms for Feynman integrals

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Abstract. In this talk, we discuss recent progress in the application of generalizations of polylogarithms in the symbolic computation of multi-loop integrals. We briefly review the Maple program MPL which supports a certain approach for the computation of Feynman integrals in terms of multiple polylogarithms. Furthermore we discuss elliptic generalizations of polylogarithms which have shown to be useful in the computation of the massive two-loop sunrise integral.

1. Motivation: Multiple polylogarithms and Feynman integrals

Classical polylogarithms Li_n are obtained as a generalization of the logarithm function

$$\text{Li}_1(z) = -\ln(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j}, \ |z| < 1,$$

by allowing for higher integer powers of the summation variable in the denominator:

$$\text{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}, \ |z| < 1.$$

These functions can be expressed in terms of integrals. For the dilogarithm, Leibniz [54] already found the identity

$$\text{Li}_2(z) = -\int_0^z \frac{dx}{x} \ln(1-x).$$

In general, for weights $n \geq 2$, we have

$$\text{Li}_n(z) \ = \ \int_0^z \frac{dx}{x} \text{Li}_{n-1}(x). \tag{1}$$

If we write all integrations on the right-hand side of this equation explicitly, we obtain an iterated integral

$$\text{Li}_n(z) = \int_0^z \frac{dx_n}{x_n}...\int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1}. \tag{2}$$

In this talk, we denote iterated integrals by

$$[\omega_r|...|\omega_2|\omega_1] = \int_0^z \omega_r(x_r)... \int_0^{x_3} \omega_2(x_2) \int_0^{x_2} \omega_1(x_1)$$

where the ω_i are differential 1-forms in some given set. In eq. 2 we see that the set

$$\Omega_{\mathrm{P}} = \left\{ \frac{dx}{x}, \frac{dx}{1-x} \right\}$$

suffices to construct the classical polylogarithms.

Generalizations of polylogarithms can be obtained by either generalizing the terms in the sum representation or by extending the set of differential 1-forms. In both ways, one arrives at multiple polylogarithms. They are defined as the series [44, 45]

$$\mathrm{Li}_{n_1,...n_k}(z_1,...,z_k) = \sum_{0 < j_1 < ... < j_k} \frac{z_1^{j_1}...z_k^{j_k}}{j_1^{n_1}...j_k^{n_k}} \text{ for } |z_i| < 1$$

and they can be expressed in terms of iterated integrals known as hyperlogarithms [61, 52, 53]. These are obtained from differential 1-forms of the set

$$\Omega_{\mathrm{Hyp}} = \left\{ \frac{dx}{x}, \frac{dx}{x-y_i} \middle| i = 1,...,k \right\}. \tag{3}$$

Some powerful methods and computer programs for the analytical computation of Feynman integrals rely on either the sum representation or on an integral representation of multiple polylogarithms. In section 2 we review the computer program MPL which supports an approach based on iterated integrals. For Feynman integrals which can not be expressed in terms of multiple polylogarithms, we are in search of alternatives. In section 3 we briefly recall the concept of elliptic functions and in section 4 we discuss an elliptic generalization of polylogarithms which arises from the computation of the massive sunrise integral.

2. Iterated integrals and the program MPL

As an alternative to hyperlogarithms, we consider a class of iterated integrals over differential 1-forms in the set

$$\Omega_{\mathrm{MPL}} = \left\{ \frac{dx_1}{x_1},..., \frac{dx_k}{x_k}, \frac{d(p_{a,b})}{p_{a,b}} \middle| 1 \le a \le b \le k \right\} \tag{4}$$

where

$$p_{a,b} = \prod_{a \le i \le b} x_i - 1. \tag{5}$$

In order to obtain a framework of well-defined functions of the k variables x_1, ..., x_k, we construct only iterated integrals which are homotopy invariant. In general, an iterated integral admits this property, if and only if it satisfies the condition [40]

$$\mathcal{D}[\omega_1|...|\omega_m] = 0 \tag{6}$$

where the operator $\mathcal{D}$ is defined by

$$\mathcal{D}[\omega_1|...|\omega_m] = \sum_{i=1}^m [\omega_1|...|\omega_{i-1}|d\omega_i|\omega_{i+1}|...\omega_m] + \sum_{i=1}^{m-1} [\omega_1|...|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|...|\omega_m]. \tag{7}$$

For example, among the two integrals

$$I_1 = \left[\frac{dx_3}{x_3} + \frac{dx_2}{x_2} \,\middle|\, \frac{d(x_2 x_3)}{x_2 x_3 - 1} \right], \quad I_2 = \left[\frac{dx_3}{x_3} \,\middle|\, \frac{d(x_2 x_3)}{x_2 x_3 - 1} \right],$$

only I_1 is homotopy invariant while I_2 fails eq. 6. For any number of variables k, we can apply algorithms described in [22, 23] to construct a basis of all homotopy invariant iterated integrals over 1-forms in Ω_{MPL}. Together with certain boundary conditions at a tangential basepoint (see [23]), this construction provides a $\mathbb{Q}$-vectorspace $V(\Omega_{\mathrm{MPL}})$ of functions, including the class of multiple polylogarithms.

MPL [21] is a Maple program for computations with this class of functions. Its main algorithms [23] rely on the mathematical theory developed in [31]. One of the main purposes of the program is the computation of definite integrals of the type

$$I = \int_0^1 dx_n \frac{q}{\prod_j p_j^{a_j}} f \tag{8}$$

where $f \in V(\Omega_{\mathrm{MPL}})$, q is some arbitrary polynomial in x_n, all $a_j \in \mathbb{N}$ and all p_j are polynomials of the type of eq. 5. For example, the program computes analytically

$$\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{x_1^4 (1-x_1)^4 \, x_2^9 (1-x_2)^4 \, x_3^4 (1-x_3)^4}{(1-x_1 x_2)^5 (1-x_2 x_3)^5} = -\frac{11424695}{144} + 66002\zeta(3).$$

Such integrals appear in various contexts. Examples are given in [34, 21, 23].

The other main purpose of the program MPL is the analytical computation of a certain class of scalar Feynman integrals. For some Feynman graph G, consider the D-dimensional, scalar L-loop integral

$$I(\Lambda) = \Gamma(\nu - LD/2) \left(\prod_{i=1}^N \int_0^\infty \frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_i)} \right) \delta(H) \frac{\mathcal{U}^{\nu - (L+1)D/2}}{(\mathcal{F}(\Lambda))^{\nu - LD/2}}, \tag{9}$$

where N is the number of edges of G, ν_i are integer powers of the Feynman propagators, ν is the sum of all ν_i, Λ is a set of kinematical invariants and masses and $H = 1 - \sum_{i \in S} x_i$ for some choice of $S \subseteq \{1, ..., N\}$. The terms $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials in the Feynman parameters $x_1, ..., x_N$ (see e.g. [25]). Applying the methods of [58, 56, 35] we can expand such a possibly divergent integral as a Laurent series in a parameter ϵ of dimensional regularisation,

$$I = \sum_{j=-2L}^\infty I_j \epsilon^j,$$

such that the integrals I_j are finite. The integrands of these I_j will involve Symanzik polynomials of G and of related graphs.

In general, Symanzik polynomials are more complicated than the polynomials of eq. 5. Therefore, the computation involves some additional steps. Before each integration, MPL attempts to express the integrand in the form of eq. 8 by an appropriate change of variables. Then the integral is computed and the result is mapped back to Feynman parameters, as a preparation of the next integration. In order for all Feynman parameters to be integrated out in this way, the (Symanzik) polynomials in the original integrand have to satisfy the condition of linear reducibility as discussed in [32, 33, 60]. This and two further conditions can be checked by the program. If they are satisfied, the integral can be computed automatically with MPL. Examples are given in [21] and in a manual obtained with the program.
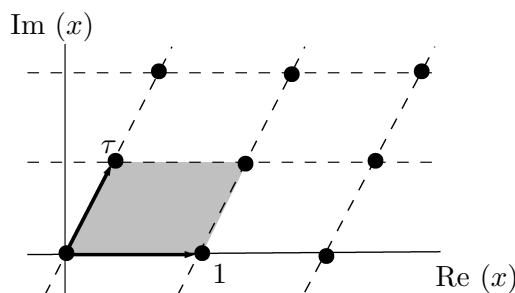
Figure 1. A lattice in the complex plane

A similar approach is followed by Panzer's program HyperInt [59], based on hyperlogarithms, which is publicly available as well, and related methods are applied by programs discussed in [1, 2, 5, 3, 4, 30].

What if a given Feynman integral does not satisfy the criterion of linear reducibility? In some cases, this problem is just an artefact of the parametrization and after some clever change of variables, the above approach can still be applied[1]. However, there are as well Feynman integrals, which can not be expressed in terms of multiple polylogarithms, no matter which parameters or classes of iterated integrals we try to apply. For such Feynman integrals, we have to turn to other frameworks of functions. The given success with multiple polylogarithms suggests to give further generalizations of polylogarithms a try.

3. Elliptic generalizations

Let us recall the basic concept of an elliptic function. In the complex plane of a variable $x \in \mathbb{C}$ we consider the lattice $L = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ (the points in fig. 1). A function $f(x)$ is called elliptic with respect to L if

$$f(x) = f(x + \lambda) \text{ for } \lambda \in L. \tag{10}$$

It makes sense to consider such a function f only in one cell of the lattice (the grey area in fig. 1), as its behaviour in all other cells are just copies. If τ is the quotient of two periods ψ_1, ψ_2 of an elliptic curve E, this cell of the lattice is isomorphic to E and we can consider f as a function on the elliptic curve.

Now we introduce a change of variables, considering the function $f'(z)$ of $z \in \mathbb{C}^\star$ given by

$$f'\left(e^{2\pi i x}\right) = f(x).$$

Clearly, if f is elliptic with respect to L, then with respect to the new variable z, eq. 10 implies

$$f'(z) = f'(z \cdot q) \text{ where } q = e^{2\pi i \lambda} \text{ for } \lambda \in L. \tag{11}$$

Now there is a simple idea for the construction of such elliptic functions. If $f'(z)$ can be defined with the help of some other function $g(z)$ as

$$f'(z) = \sum_{n \in \mathbb{Z}} g\left(z \cdot q^n\right),$$

it satisfies eq. 11 by construction.

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[1] An example for such a case is the graph found to be irreducible in [24] and later computed in [48, 58].

This concept can be applied to define elliptic generalizations of polylogarithms. A first version of an elliptic dilogarithm was defined in [17] for the single-valued Bloch-Wigner dilogarithm. Later the concept was generalized in various directions (see [66, 15, 55, 43]). Let us refer particularly to [36] where elliptic polylogarithms of the form

$$E_m(z) = \sum_{n \in \mathbb{Z}} u^n \mathrm{Li}_m\left(z \cdot q^n\right) \tag{12}$$

with a damping factor u are considered and where the concept is furthermore generalized to establish multiple elliptic polylogarithms.

In the following section, a related class of functions appears in the context of a Feynman integral.

4. The massive sunrise integral
The massive sunrise integral

$$S(D,\,t) = \int \frac{d^D k_1 d^D k_2}{\left(i\pi^{D/2}\right)^2} \frac{1}{\left(-k_1^2 + m_1^2\right)\left(-k_2^2 + m_2^2\right)\left(-(p - k_1 - k_2)^2 + m_3^2\right)}$$

is a Feynman integral which can not be expressed in terms of multiple polylogarithms. This integral was extensively considered in the literature [11, 12, 13, 14, 16, 27, 39, 37, 38, 46, 47, 51, 62, 63, 65, 64, 26, 50, 41, 42]. In a recent computation of the case $D = 2$ and equal masses, $m_1 = m_2 = m_3$, for the first time an elliptic polylogarithm was applied explicitly to express a Feynman integral [20]. Here we discuss further cases of the sunrise integral where elliptic generalizations of polylogarithms arise.

At first, let us consider the integral with three different particle masses as Laurent series at two and around four dimensions:

$$S(2 - 2\epsilon,\, t) = S^{(0)}(2,\, t) + S^{(1)}(2,\, t)\epsilon + \mathcal{O}\left(\epsilon^2\right), \tag{13}$$

$$S(4 - 2\epsilon,\, t) = S^{(-2)}(4,\, t)\epsilon^{-2} + S^{(-1)}(4,\, t)\epsilon^{-1} + S^{(0)}(4,\, t) + \mathcal{O}(\epsilon). \tag{14}$$

Here we have used $t = p^2$. We begin with the result for exactly $D = 2$ dimensions, $S^{(0)}(2,\, t)$. In this case, the Feynman parametric representation (eq. 9) of the sunrise integral only involves the second Symanzik polynomial

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_1 x_3\right)$$

whose zero-set intersects the integration domain at three points P_1, P_2, P_3. Together with each possible choice of one of these points as the origin, this zero-set defines an elliptic curve.

In [7] the following functions are introduced:

$$\mathrm{ELi}_{n;m}(x; y; q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \frac{y^k}{k^m} \mathrm{Li}_n(q^k x), \tag{15}$$

$$\mathrm{E}_{n;m}(x;\, y;\, q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x; y; q) - \mathrm{ELi}_{n;m}(x^{-1}; y^{-1}; q)\right) & \text{for } n + m \text{ even,} \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x; y; q) + \mathrm{ELi}_{n;m}(x^{-1}; y^{-1}; q) & \text{for } n + m \text{ odd.} \end{cases} \tag{16}$$

Notice that these definitions are related to the basic ideas recalled in section 3 but slightly differ[2] from the functions of eq. 12. By use of the differential equation of second order [57] for $S^{(0)}(2, t)$, we obtain

$$S^{(0)}(2,\, t) = \frac{\psi_1(q)}{\pi} \sum_{i=1}^{3} \mathrm{E}_{2;\,0}(w_i(q);\, -1;\, -q) \text{ where } q = e^{i\pi\frac{\psi_2}{\psi_1}}. \qquad (17)$$

Here ψ_1 and ψ_2 are periods of the elliptic curve defined by $\mathcal{F}$, which are given by complete elliptic integrals of the first kind [6]. The three arguments $w_i(q)$, $i = 1, 2, 3$, are directly related to the three intersection points P_1, P_2, P_3 (see [7]).

While all terms in eq. 17 can be nicely related to the underlying elliptic curve, the situation becomes considerably more complicated for the higher coefficients of eqs. 13 and 14. Here the integrands under consideration depend on both Symanzik polynomials. However, the functions defined in eqs. 15 and 16 remain to be useful. Generalizing this concept, we furthermore introduce the multi-variable functions [10]

$$\mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}}(x_1, ..., x_l; y_1, ..., y_l; q)$$

$$= \sum_{j_1=1}^{\infty} ... \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} ... \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} ... \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} .. \frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1 k_1 + ... + j_l k_l}}{\prod_{i=1}^{l-1}(j_i k_i + ... + j_l k_l)^{o_i}}. \qquad (18)$$

By use of this set-up of functions, the coefficients $S^{(1)}(2, t)$ and $S^{(0)}(4, t)$ are computed in [8]. Furthermore it is shown in [10], that in the case of equal masses, every higher coefficient of the two-dimensional case $S(2 - 2\epsilon, t)$ can be expressed in terms of these functions as well. This result includes an explicit algorithm for the computation of these coefficients.

We want to point out that other elliptic generalizations of polylogarithms recently found further applications to the two- and the three-loop sunrise graph [18, 19] and to integrals arising in string theory [49, 28, 29].

5. Conclusions

With their double nature as nested sums and iterated integrals, multiple polylogarithms provide a very useful framework for the computation of Feynman integrals. The Maple program MPL serves for the computation of a certain class of Feynman integrals in terms of these functions. The program is publicly available and supports computations with a class of iterated integrals, which arise in other contexts as well.

The massive two-loop sunrise integral is a Feynman integral which can not be expressed in terms of multiple polylogarithms. For the computation of various cases of this integral, a class of elliptic generalizations of polylogarithms has shown to be useful. These and other appearances of elliptic generalizations give rise to the hope, that when we have to leave the realm of multiple polylogarithms, we might not have to dispense with all of its advantages.

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