

# Automating QCD amplitudes with on-shell methods

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**Abstract.** We review some of the modern approaches to scattering amplitude computations in QCD and their application to precision LHC phenomenology.

## 1. Introduction

New understanding of the structure of scattering amplitudes has led to efficient algorithms for high multiplicity final states which are now common in next-to-leading order simulations of LHC collisions. To keep up with the experimental accuracy expected during Run II, new theoretical methods are required to look at high multiplicity final states at higher accuracy. In this contribution we take a brief look at some new on-shell methods attempting to tackle this problem. As a specific example we consider applications of momentum twistor technology for the parameterisation of multi-leg kinematics.

Over the last ten years or so an increasingly large number of theoretical physicists have turned their attention to understanding the underlying structure of amplitudes in gauge, gravity and string theory. Much of this can be traced to Witten's 2003 paper on  $\mathcal{N} = 4$  super-symmetric Yang-Mills theory as a string theory in twistor space [1] though on-shell unitarity methods [2, 3] have been hard at work in collider physics applications for much longer. Perhaps the most remarkable example of on-shell simplicity was the realisation of Parke and Taylor [4] that tree-level amplitudes for multi-gluon scattering amplitudes with specific external helicity states can be written in a compact form,

$$A_n^{(0)}(1^+, 2^+, 3^+, \dots, n^+) = 0, \quad (1)$$

$$A_n^{(0)}(1^-, 2^+, 3^+, \dots, n^+) = 0, \quad (2)$$

$$A_n^{(0)}(1^-, 2^-, 3^+, \dots, n^+) = \frac{i\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle}. \quad (3)$$

This famous representation for the maximal helicity violating configuration at the amplitude level, due to later work by Mangano, Parke and Xu [5] and Berends and Giele [6], resulted from understanding the power of both colour ordering to exploit the  $SU(N_c)$  group structure and the spinor-helicity formalism to simplify the on-shell kinematics. The motivation for studying these high multiplicity tree-level expressions came from collider physics applications where multi-jet backgrounds were still impossible even at leading order in perturbation theory.

Though the number of Feynman diagrams involved in a given computation are sometimes used as measure of the complexity, with a high degree of automation they can easily be applicable to



relevant processes. Obtaining compact and efficient amplitude representations with this method is fairly hopeless however. In Table 1 we compare the set of Feynman graphs for the full colour amplitude  $\mathcal{A}_n^{(0)}$  with the reduced colour ordered set of graphs which contribute to the partial amplitudes  $A_n^{(0)}$  defined by,

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 \sigma(a_2) \sigma(a_3)} \dots \tilde{f}^{\sigma(a_{n-2}) \sigma(a_{n-1}) a_n} A_n^{(0)}(1, \sigma(2), \dots, \sigma(3), n) \quad (4)$$

Here we have used the compact colour decomposition into adjoint traces [8] which incorporates additional colour symmetry coming from Kleiss-Kuijf identities [9] and runs over  $(n-2)!$  permutations.  $\tilde{f}^{abc} = \sqrt{2} f^{abc}$  are the usual structure constants in  $SU(N_c)$ . By exploiting the symmetry in colour space the amplitude can be constructed from a minimal set of gauge invariant kinematic building blocks. The number of diagrams  $N_A(n)$  in the full colour amplitudes can be derived from a simple algorithm using differential operators [10]. If we let  $t$  be the number of three-gluon vertices and  $g$  be the number of internal and external gluons then,

$$N_A(n) = \left( t g^3 \frac{\partial}{\partial g} + g \frac{\partial}{\partial t} \right)^{n-3} t g^3 \Big|_{t=g=1}, \quad (5)$$

where the first derivative adds a gluon to each gluon line via a three vertex and the second derivative adds a gluon to each three vertex via a four-point vertex. The colour ordered diagrams can be quickly computed using the recursive approach of Berends and Giele [6].

$$N_A(n) = \sum_{k=1}^{n-2} N_A(k+1) N_A(n-k) + \sum_{k=1}^{n-3} \sum_{l=k+1}^{n-3} N_A(k+1) N_A(l-k+1) N(n-l) \quad (6)$$

A more modern technique for the evaluation of tree-level amplitudes purely from on-shell building blocks is the recursive method of Britto, Cachazo, Feng and Witten [11, 12]. By counting the number of on-shell BCFW diagrams that contribute to a given helicity configuration, one can see that there is considerable simplification from the ordered diagrams to a fully on-shell amplitude. Automated packages which solve the on-shell recursion relation to obtain compact expression for all tree amplitudes in massless QCD have been developed [13, 7]. The complexity of helicity amplitudes then becomes clear with the addition of more and more negative helicity gluons.

For applications in collider phenomenology an individual helicity amplitude is unfortunately of little use and both sums of the helicity and colour configurations of the ordered partial amplitudes must be performed efficiently. Given the precision for the LHC experiments, leading order predictions in QCD are insufficient for four or even five particle final states, so it is necessary to perform this task at least to one-loop order.

| $n$                     | 3 | 4 | 5  | 6   | 7    | 8     | 9      | 10       |
|-------------------------|---|---|----|-----|------|-------|--------|----------|
| Feynman diagrams        | 1 | 4 | 25 | 220 | 2485 | 34300 | 559405 | 10525900 |
| Ordered diagrams        | 1 | 3 | 10 | 38  | 154  | 654   | 2871   | 12925    |
| MHV                     | 1 | 1 | 1  | 1   | 1    | 1     | 1      | 1        |
| BCFW NMHV               | - | - | -  | 3   | 6    | 10    | 15     | 21       |
| BCFW N <sup>2</sup> MHV | - | - | -  | -   | -    | 20    | 50     | 105      |
| BCFW N <sup>3</sup> MHV | - | - | -  | -   | -    | -     | -      | 175      |

**Table 1.** The number of graphs appearing in tree-level  $n$ -gluon scattering using different approaches. The number of terms in BCFW are chosen for the most complicated alternating helicity case and were obtained using Bourjaily's `bcfw` package [7].

Nevertheless a host of new algorithms have been developed in recent years which exploit this tree-level simplicity at the loop level and can provide the necessary information to the Monte Carlo event generators like SHERPA [14], MADGRAPH5\_aMC@NLO [15] and HERWIG7 [16] used in experimental analyses. A few important insights were necessary to make this possible some of which I will expand on the next sections.

## 2. Unitarity, generalised unitarity and integrand reduction

One of the key insights which led to the automation of one-loop amplitudes is that a purely algebraic algorithm would kinematic algebra to be performed numerically and avoid a traditional bottleneck in multi-leg amplitude computations.

The starting point for this approach is the well known decomposition of one-loop amplitudes into a basis of known box, triangle and bubble integrals,

$$A_n^{(1)} = \sum_i c_i I_i^{4-2\epsilon} + \text{rational} + \mathcal{O}(\epsilon). \quad (7)$$

In this formula for one-loop primitive (i.e. colour ordered) amplitudes  $c_i$  are rational functions while  $I_i^{4-2\epsilon}$  are loop integrals in the dimensional regularisation parameter  $\epsilon$ . The generalised unitarity algorithm uses complex momenta to systematically solve the multiple cut conditions and isolate the coefficients  $c_i$  from products of tree-level amplitudes [17, 18, 19]. The remaining rational term can be extracted from modified BCFW recursion relations [20, 21, 22, 23, 24] or  $D$ -dimensional unitarity cuts [25].

Another approach developed at the same time was the integrand reduction method of Ossola, Papadopoulos and Pittau [26]. Here the loop momentum dependence of the integrand is completely parameterised in terms of the scalar integral basis and additional 'spurious' terms which integrate to zero. The sum over topologies remains the same while the numerator is decomposed into monomials of irreducible scalar products  $\Delta_i(k)$ :

$$A_n^{(1)} = \int d^{4-2\epsilon} \sum_i \frac{\Delta_i(k)}{\prod_{\alpha \in i} D_\alpha(k)}, \quad (8)$$

where  $D_\alpha(k)$  are the propagators for the  $i$ th topology. The integrand reduction procedure is compatible with  $D$ -dimensional generalised unitarity cuts [27, 28]. A complete reconstruction of the integrand, including spurious terms, is important to allow the algorithm to be implemented efficiently using fixed precision numerics. Further details on these algorithms can be found in reference [29].

Since the information required for any loop amplitude has now been parameterised in terms of rational functions, the generation of the input to these algorithms can then be performed using tree-level recursive techniques. An important development in this area was the extension of off shell recursive techniques to tensor integrands using the OPENLOOPS method [30]. Variations of these techniques have subsequently been implemented into computer codes such as GOSAM [31], OPENLOOPS [30], MADLOOP<sup>1</sup> [32] and RECOLA [33] that can handle a wide variety of complicated processes including QCD and Electro-weak corrections GOSAM [31] combines efficient Feynman diagram generation together with the integrand reduction algorithm NINJA [34].

The BLACKHAT [19] and NJET [35] one-loop amplitude providers are more specialised but use a fully on-shell approach that is currently able to attack higher multiplicity processes than other algorithms. These codes have been interfaced with the SHERPAMonte-Carlo to produce NLO

<sup>1</sup> MADLOOP is part of the MADGRAPH5\_aMC@NLO Monte Carlo event generator which offers fully automated NLO simulations interfaced with parton showers.

predictions for high multiplicity processes like  $pp \rightarrow 5j$  [36],  $pp \rightarrow WW+3j$  [37],  $pp \rightarrow \gamma\gamma+3j$  [38] and  $pp \rightarrow W+5j$  [39]. The bottleneck in these extreme configurations is no longer in the virtual corrections since most of the event generation time is spent in the real radiation phase-space integration. Nevertheless, without the advanced recursive tree amplitude algorithms in COMIX [40] and SHERPA's automated dipole subtraction method [41] such predictions wouldn't be possible at all.

### 3. $D$ -dimensional generalised unitarity for multi-loop integrands

Extending the current multi-loop methods to higher multiplicity represents a serious challenge. The  $D$ -dimensional generalised unitarity cut algorithm has recently been extended to multi-loop integrands using integrand reduction [26] and elements of computational algebraic geometry [42, 43, 44, 45, 46, 47, 48, 49]. In contrast to the one-loop case, the basis of integrals obtained through this method is not currently known analytically and is much larger than the set of basis functions defined by standard integration-by-parts (IBP) identities. Nevertheless, new results in non-supersymmetric theories have been obtained for five gluon scattering amplitudes with all positive helicities [49, 50]. The maximal unitarity method [51], which incorporates IBP identities, has been applied to a variety of two-loop examples in four dimensions [52, 53, 54, 55, 56]. This approach can be seen as an extension of the generalised unitarity methods of Britto, Cachazo and Feng [17] and Forde [18]. Efficient algorithms to generate unitarity compatible IBPs are a key ingredient in both approaches and has been the focus of on-going investigations [57, 58, 59, 60].

Constructing loop amplitudes from trees for non-supersymmetric theories like QCD require the identification of the additional components of the dimensionally regulated loop momenta. At two-loops this information can be extracted from six-dimensional trees which are computed efficiently using the six-dimensional spinor-helicity formalism [61]. These cuts can then be dimensionally reduced to  $4 - 2\epsilon$  in an analogous way to the one-loop approaches considered previously [62, 63].

Breaking loop amplitudes into trees also has benefits for colour decompositions and representations of the non-planar sector since tree-level amplitude relations such as Kleiss-Kuijf [9] and Bern-Carrasco-Johansson [64] can be applied at the level of the cut amplitude. This technique has been shown to be a powerful tool in  $\mathcal{N} = 4$  Yang-Mills and  $\mathcal{N} = 8$  super-gravity computations [65], but recently been exploited in the computation of the complete five gluon all-plus amplitude at two loops in pure Yang-Mills [50]. Combining  $D$ -dimensional generalised unitarity cuts, BCJ relations and integrand reduction a remarkably compact form was obtained,

$$\begin{aligned}
 \mathcal{A}^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+) = & \\
 & ig^7 \sum_{\sigma \in S_5} \sigma \circ I \left[ C \left( \text{Diagram 1} \right) \left( \frac{1}{2} \Delta \left( \text{Diagram 2} \right) + \Delta \left( \text{Diagram 3} \right) + \frac{1}{2} \Delta \left( \text{Diagram 4} \right) \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} \Delta \left( \text{Diagram 5} \right) + \Delta \left( \text{Diagram 6} \right) + \frac{1}{2} \Delta \left( \text{Diagram 7} \right) \right) \right. \\
 & \quad \left. + C \left( \text{Diagram 8} \right) \left( \frac{1}{4} \Delta \left( \text{Diagram 9} \right) + \frac{1}{2} \Delta \left( \text{Diagram 10} \right) + \frac{1}{2} \Delta \left( \text{Diagram 11} \right) \right. \right. \\
 & \quad \left. \left. - \Delta \left( \text{Diagram 12} \right) + \frac{1}{4} \Delta \left( \text{Diagram 13} \right) \right) \right. \\
 & \quad \left. + C \left( \text{Diagram 14} \right) \left( \frac{1}{4} \Delta \left( \text{Diagram 15} \right) + \frac{1}{2} \Delta \left( \text{Diagram 16} \right) \right) \right]. \quad (9)
 \end{aligned}$$

where the  $C$  functions are the colour factors in terms of adjoint  $\tilde{f}$  structure constants and  $\Delta$  are the irreducible numerators. Further details can be found in reference [50].

#### 4. Momentum Twistors

In this section we collect some useful formulae for parameterising external kinematics using Hodges' momentum twistor formalism [66]. Momentum twistors are extremely useful for studying the geometric properties of amplitudes but they also have a practical property that can be used in general amplitude computations: these variable linearise the momentum conservation conditions and give a rational parameterisation of the phase-space.

The topic is somewhat more specialised than the techniques reviewed in the previous sections. The connection is that rational parameterisation of the kinematics can allow analytic computations to be performed in exactly the same way as the numerical algorithms described above and hence is a useful tool when combined with unitarity cuts and integrand reduction.

The notation used in this section relies on the spinor-helicity formalism of which good introductions can be found in references [67, 68, 69, 70]. The first observation stems from the fact that massless momenta can be decomposed in two component Weyl spinors,

$$p^2 = 0 \Rightarrow (\sigma \cdot p)^{\alpha\dot{\alpha}} = \lambda^\alpha(p) \tilde{\lambda}^{\dot{\alpha}}(p). \quad (10)$$

Hence the on-shell conditions are manifest when using the spinor-helicity formalism. Momentum conservation,

$$\sum_i p_i^\mu = 0, \quad (11)$$

still impose some non-trivial conditions on the amplitudes. Hodges [66] introduced momentum twistors as a natural extension of Penrose's twistor formalism. We begin by defining dual momentum co-ordinates,  $x_i^\mu$ ,

$$p_i^\mu = x_i^\mu - x_{i-1}^\mu \quad (12)$$

which can be inverted up to a fixed point,  $x_0^\mu$ ,

$$x_i^\mu = x_0^\mu + \sum_{k=1}^i p_k^\mu. \quad (13)$$

The momentum twistor is then constructed from these dual co-ordinates and the holomorphic Weyl spinors:

$$Z_{iA} = (\lambda_\alpha(i), \mu^{\dot{\alpha}}(i) = \lambda_\alpha(i) (\sigma \cdot x_i)^{\alpha\dot{\alpha}}), \quad (14)$$

The two component object  $\mu^{\dot{\alpha}}(i)$  is used instead of the  $\tilde{\lambda}^{\dot{\alpha}}(i)$  spinor to define the kinematics for the  $n$ -particle system  $i = 1, n$ . The  $\tilde{\lambda}(i)_{\dot{\alpha}}$  spinor is defined through the dual twistor,

$$W_i^A = (\tilde{\mu}_\alpha(i), \tilde{\lambda}^{\dot{\alpha}}(i)) = \frac{\varepsilon^{ABCD} Z_{(i-1)B} Z_{iC} Z_{(i+1)D}}{\langle i-1i \rangle \langle ii+1 \rangle} \quad (15)$$

from which we can find the definition of the anti-holomorphic spinor,

$$\tilde{\lambda}(i)^{\dot{\alpha}} = \frac{\langle i-1i \rangle \mu^{\dot{\alpha}}(i+1) + \langle i+1i-1 \rangle \mu^{\dot{\alpha}}(i) + \langle ii+1 \rangle \mu^{\dot{\alpha}}(i-1)}{\langle i-1i \rangle \langle ii+1 \rangle}. \quad (16)$$

By use of the Schouten identity it is easy to show that any  $\tilde{\lambda}(i)_{\dot{\alpha}}$  defined through the dual twistor will automatically satisfy momentum conservation. Amplitudes can then be written in terms of holomorphic spinor products and momentum twistor 4-brackets:

$$\langle ijkl \rangle = \varepsilon^{ABCD} Z_{iA} Z_{jB} Z_{kC} Z_{lD}. \quad (17)$$

The  $4 \times n$  matrix  $Z_{iA}$  has  $3n - 10$  independent entries and we can generate rational phase-space points (for complex momenta) by filling the matrix with random integers. We can even pick rational functions to fill the momentum twistor matrix, which amounts to picking a special frame to evaluate general kinematics. A convenient choice is,

$$Z_i = \begin{pmatrix} \frac{\Sigma_i}{s_{12}} \\ 1 - \delta_{1i} \\ \frac{\langle 123i \rangle \langle 34 \rangle [23]}{\langle 1234 \rangle \langle 1i \rangle [12]} \\ \frac{-\langle 13 \rangle \langle 124i \rangle + \langle 14 \rangle \langle 123i \rangle}{\langle 1234 \rangle \langle 1i \rangle} \end{pmatrix} \quad (18)$$

where

$$\frac{\Sigma_i}{s_{12}} = \begin{cases} 0 & i = 1 \\ \frac{\langle 13 \rangle \langle 2i \rangle}{\langle 23 \rangle \langle 1i \rangle} & \text{otherwise} \end{cases} \quad (19)$$

One additional subtlety in using this technique is that phase information ensures parity invariance is discarded in favour of rational functions. This can be easily restored as a pre-factor. In the explicit parameterisation above this is:

$$\Phi_n(1^{h_1}, \dots, n^{h_n}) = \left( \frac{\langle 13 \rangle}{[12] \langle 23 \rangle} \right)^{-h_1} \prod_{i=2}^n \left( \frac{\langle 1i \rangle^2 [12] \langle 23 \rangle}{\langle 13 \rangle} \right)^{-h_i} \quad (20)$$

where  $h_i$  are the helicities of the external gluons. As an example the MHV amplitude in this language would read,

$$\frac{A_n^{(0)}(1^-, 2^-, 3^+, \dots, n^+)}{\Phi_n(1^-, 2^-, 3^+, \dots, n^+)} = \frac{i s_{12}^{n-2}}{(\Sigma_4 - 1) \prod_{i=4}^{n-1} (\Sigma_i - \Sigma_{i+1})} \quad (21)$$

The power of this representation is that the above expression can be obtained by simply plugging in the parameterisation (18) into the thousands of ordered Feynman diagrams - assuming your computer has enough memory to perform the multi-variate factorisation. The MHV is a special case of course since being a single term the factorisation is guaranteed to land on the compact expression.

## 5. Conclusions

In these proceedings we have given a very brief overview of the on-shell methods used in automated loop amplitude computations. The success of this approach to the one-loop problem has motivated activity in applying the methods at higher loops and some useful progress has been made. Some subtleties of the integrand reduction method beyond one-loop have been identified in particular the non-minimal basis of integrals compared to those generated by traditional IBP reduction to master integrals.

We highlighted one of the new techniques used in modern high multiplicity loop computations. Though momentum twistors were originally applied in the context of maximally super-symmetric Yang-Mills theory these computations show they can be applied just as easily to general gauge theories.

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