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Deconvolving the detector in Fourier space

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Abstract. We discuss algorithms for the analysis of hadronic final states, with application to single top t-channel production, heavy Higgs decaying as $H \to W^+W^-$ in the lepton plus jets mode, and others. In our examples, nature has arranged for the triple differential decay rates in angles we call θ_1, θ_2, ϕ to be a short finite series in orthogonal functions, $\sqrt{2\pi}a^m_{k,l}Y^m_k(\theta_1,\phi)Y^m_l(\theta_2,0)$ (summation implied), where the $a^m_{k,l}$ are complex constants. This observation can be exploited in two ways; first, a technique called orthogonal series density estimation may be employed to extract coefficients $a^m_{k,l}$ of the decay and physics parameters related to these coefficients; second, an angular analog of the convolution theorem may be employed to analytically deconvolve detector resolution effects from an observed signal.

The angular distributions of the decay products of massive elementary particles are a window on fundamental physics. Nowadays, the main goal of precision measurements in Higgs physics and top quark physics is to determine the strength and structure of the couplings amongst fermionic and bosonic fields. One example is the coupling of the Higgs boson to the charged W bosons, for which the effective interaction Lagrangian[1] is:

$$\mathcal{L}_{HWW} = m_W^2\left(\sqrt{2}G_F\right)^{1/2}(1-\frac{v^2}{2\Lambda^2}f_{\Phi,2})HW^+_\mu W^{-\mu} + \frac{g^2 v}{2\Lambda^2}\frac{f_W}{2}(W^+_{\mu\nu}W^{-\mu}\partial^\nu H) - \frac{g^2 v}{2\Lambda^2}f_{WW}W^+_{\mu\nu}W^{-\mu\nu}$$

(h.c. implied) where the complex constants $f_{\Phi,2}$, f_W, and f_{WW} are *anomalous* couplings. Likewise at the Wtb vertex one has the effective interaction Lagrangian[2]

$$\mathcal{L}_{Wtb} = -\frac{g}{\sqrt{2}}\bar{b}\gamma^\mu(V_L P_L + V_R P_R)tW^-_\mu - \frac{g}{\sqrt{2}}\bar{b}\frac{i\sigma^{\mu\nu}q_\nu}{M_W}(g_L P_L + g_R P_R)tW^-_\mu + h.c;$$

here, the coupling constant $V_L = 1$ in the standard model while the anomalous couplings $V_R = g_L = g_R = 0$. These couplings are affected by new physics and they affect the angular distributions of top quark and Higgs decay products.

$H \to W^+W^-$ and $t \to Wb \to l^+\nu b$ are the two benchmark examples we choose to illustrate the proposed technique for analyzing the angular distribution of Higgs and top quark decay products. In the former example, we assume that the decay occurs in a lepton plus jets mode. In the latter, we assume single top quark production in the t-channel, which produces polarized top quarks at hadron colliders, followed by their semileptonic decay. Each of these modes is characterized by three decay angles: In the case of the Higgs decay, we define θ_1 (θ_2) to be the angle between the W^- (W^+) boson direction in the Higgs frame and the outgoing lepton or quark in the W frame. ϕ is the angle between the decay planes of the two W bosons. In the case of the top quark decay, we define θ_1 to be the angle between a light "spectator" quark from t-channel production, and the W boson in the top quark reference frame; θ_2 is the angle between the W boson direction in the top quark rest frame and the lepton direction in the W boson rest frame. ϕ is the angle between the plane defined by the spectator quark and the W boson, and that defined by the W boson and the lepton.

We use the following shorthand notation. The product of spherical harmonics, $M^m_{k,l}(\theta_1,\theta_2,\phi) \equiv \sqrt{2\pi}Y^m_k(\theta_1,\phi)Y^m_l(\theta_2,0)$ is referred to as an M-function. Properties of the M-functions derive from those of those of spherical harmonics. The M-functions are orthonormal,

$$\int M^m_{k,l}(\theta_1,\theta_2,\phi)M^{m'*}_{k',l'}(\theta_1,\theta_2,\phi)d\Omega^M = \delta_{k,k'}\delta_{l,l'}\delta_{m,m'} \tag{1}$$

where $d\Omega^M = \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi$; their complex conjugates obey

$$M^{m*}_{k,l}(\theta_1,\theta_2,\phi) = M^{-m}_{k,l}(\theta_1,\theta_2,\phi),$$

and they obey a version of Gaunt's theorem,

$$M^m_{k,l}(\theta_1,\theta_2,\phi)M^{m'}_{k',l'}(\theta_1,\theta_2,\phi) = W^{m,m',M}_{k,l,k',l',L,K}M^M_{K,L}(\theta_1,\theta_2,\phi) \tag{2}$$

where

$$W^{m,m',M}_{k,l,k',l',L,K} = \sqrt{2\pi}G^{m,m',M}_{k,k',K}G^{m,m',M}_{l,l',L},$$

and

$$G^{m,m',M}_{l,l',L} = \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}C^{m,m',M}_{l,l',L}C^{0,0,0}_{l,l',L}$$

and the constants $C^{m,m',M}_{l,l',L}$ are ordinary Clebsch-Gordan coefficients. In this paper summation over repeated indices is always implied unless otherwise indicated.

The two benchmark decays, as well as plenty more in B-physics such as $B^0 \to J/\psi K^*$, are characterized by three decay angles and nature arranges for the triple differential decay rate at tree level to have a simple form when expressed in M-functions:

$$\rho(\theta_1,\theta_2,\phi) \equiv \frac{1}{\Gamma}\frac{d\Gamma}{d\Omega^M} = a^m_{k,l}M^m_{k,l}(\theta_1,\theta_2,\phi) \tag{3}$$

In $H \to W^+W^-$, the complex coefficients $a^m_{k,l}$ can be expressed in terms of the amplitudes A_{00}, A_{LL}, and A_{RR} for the Higgs to decay into a pair of longitudinal, left handed, or right handed W bosons. All other transitions are forbidden by the conservation of angular momentum. From the helicity formalism one obtains:

$$\begin{aligned}
a^0_{0,0} &= \frac{1}{\sqrt{8\pi}}\left(|A_R|^2+|A_L|^2+|A_0|^2\right) & a^0_{2,2} &= \frac{1}{40\sqrt{2\pi}}\left(|A_R|^2+|A_L|^2+4|A_0|^2\right)\\
a^0_{0,1} &= a^0_{1,0} = \sqrt{\frac{3}{32\pi}}\left(|A_L|^2-|A_R|^2\right) & a^1_{1,1} &= \frac{3}{\sqrt{128\pi}}\left(A_0A_R^*+A_LA_0^*\right)\\
a^0_{0,2} &= a^0_{2,0} = \frac{1}{\sqrt{160\pi}}\left(|A_R|^2+|A_L|^2-2|A_0|^2\right) & a^1_{1,2} &= a^1_{2,1} = \frac{3}{\sqrt{640\pi}}\left(A_LA_0^*-A_0A_R^*\right)\\
a^0_{1,1} &= \frac{3}{\sqrt{128\pi}}\left(|A_R|^2+|A_L|^2\right) & a^1_{2,2} &= \frac{3}{40\sqrt{2\pi}}\left(A_0A_R^*+A_LA_0^*\right)\\
a^0_{2,1} &= a^0_{1,2} = \sqrt{\frac{3}{640\pi}}\left(|A_L|^2-|A_R|^2\right) & a^2_{2,2} &= \frac{3}{20\sqrt{2\pi}}A_LA_R^*.
\end{aligned}$$

For single top quark t-channel production the transition amplitudes are $A_{-1,-\frac{1}{2}}$, $A_{1,\frac{1}{2}}$, $A_{0,-\frac{1}{2}}$, and $A_{0,\frac{1}{2}}$ to a left handed W boson and a left handed b quark, a right handed W boson and a right handed b quark, a longitudinal W boson and a left handed b quark, and a longitudinal W boson and a right handed b quark. The angular coefficients are:

$$a^0_{0,0} = \frac{1}{\sqrt{8\pi}}\left(|A_{1,\frac{1}{2}}|^2 + |A_{0,\frac{1}{2}}|^2 + |A_{0,-\frac{1}{2}}|^2 + |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^0_{0,1} = +\frac{\sqrt{3}}{2}\left(|A_{1,\frac{1}{2}}|^2 - |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^0_{0,2} = +\frac{1}{2\sqrt{5}}\left(|A_{1,\frac{1}{2}}|^2 - 2|A_{0,\frac{1}{2}}|^2 - 2|A_{0,-\frac{1}{2}}|^2 + |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^0_{1,0} = +P\frac{1}{\sqrt{3}}\left(|A_{1,\frac{1}{2}}|^2 - |A_{0,\frac{1}{2}}|^2 + |A_{0,-\frac{1}{2}}|^2 - |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^0_{1,1} = +P\frac{1}{2}\left(|A_{1,\frac{1}{2}}|^2 + |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^0_{1,2} = +P\frac{1}{2\sqrt{15}}\left(|A_{1,\frac{1}{2}}|^2 + 2|A_{0,\frac{1}{2}}|^2 - 2|A_{0,-\frac{1}{2}}|^2 - |A_{-1,-\frac{1}{2}}|^2\right)$$

$$a^1_{1,1} = -P\frac{1}{\sqrt{2}}\left(A_{1,\frac{1}{2}}A^*_{0,\frac{1}{2}} + A^*_{-1,-\frac{1}{2}}A_{0,-\frac{1}{2}}\right)$$

$$a^{-1}_{1,1} = -P\frac{1}{\sqrt{2}}\left(A^*_{1,\frac{1}{2}}A_{0,\frac{1}{2}} + A_{-1,-\frac{1}{2}}A^*_{0,-\frac{1}{2}}\right)$$

$$a^1_{1,2} = -P\frac{1}{\sqrt{10}}\left(A_{1,\frac{1}{2}}A^*_{0,\frac{1}{2}} - A^*_{-1,-\frac{1}{2}}A_{0,-\frac{1}{2}}\right)$$

$$a^{-1}_{1,2} = -P\frac{1}{\sqrt{10}}\left(A^*_{1,\frac{1}{2}}A_{0,\frac{1}{2}} - A_{-1,-\frac{1}{2}}A^*_{0,-\frac{1}{2}}\right),$$

P being the top quark polarization. In both cases, a real PDF requires that $a^m_{k,l} = a^{-m*}_{k,l}$.

To estimate unknown angular coefficients $a^m_{k,l}$ from a dataset $\mathcal{D} = \{(\theta_{1i}, \theta_{2i}, \phi_i)\}$, $i = 1, N$, the usual recourse is some variant of a maximum likelihood fit. Another approach relies upon the orthonormality of the M-functions. By projecting the density function $\rho(\theta_1, \theta_2, \phi)$ onto a basis of M-functions we can express the coefficients as

$$a^m_{k,l} = \int \rho(\theta_1, \theta_2, \phi) M^{m*}_{k,l}(\theta_1, \theta_2, \phi) d\Omega^M \tag{4}$$

The Monte Carlo estimate of the integral is simply the average of the function $M^{m*}_{k,l}(\theta_1, \theta_2, \phi)$, over a realization of the PDF $\rho(\theta_1, \theta_2, \phi)$. The dataset $\mathcal{D}$ is, precisely, the desired realization. So, the estimate of the coefficients $a^m_{k,l}$ comes down simply to, first, computing the value of $M^{m*}_{k,l}(\theta_1, \theta_2, \phi)$ for each event in the dataset and, second, taking the average. A complete covariance matrix between all of the $a^m_{k,l}$ can be computed at the same time. The technique is referred to as Orthogonal Series Density Estimation[3] (OSDE).

To illustrate this we show, in Fig. 1 on the left, an image of the American film actor Humphrey Bogart (1899-1957). The image defines a two-dimensional PDF. In the middle of Fig. 1 we show a realization of the distribution, generated using Von Neumann rejection. 100K points were generated. On the right we show an OSDE estimate of the distribution, obtained using harmonic functions as a basis.

Returning now to physics, we have used the Pythia Monte Carlo[4] to generate a sample of heavy Higgs bosons with a mass of 200 GeV decaying to W boson pairs. An OSDE analysis of the data sample was performed to determine the coefficients. Results are shown in Fig. 2. In Fig. 3 we implement simple acceptance cuts and smearing (mostly due to missing energy effects). These spoil the simple picture, but we can recover using a signal processing trick akin to the convolution theorem. The convolution theorem says that the Fourier transform of a convolution of two functions, $\widetilde{f \circ g}$, is the product of the Fourier transforms of each function:

$$\widetilde{f \circ g} = \tilde{f} \cdot \tilde{g}.$$

An analogous theorem, the Funk-Hecke theorem[5, 6] treats isotropic smearing on a sphere. If

$$\rho(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c^m_l Y^m_l(\theta, \phi) \qquad \text{and} \qquad \mathcal{R}(\Theta) = \sum_{l=0}^{M} r_l P_l(\cos\Theta)$$

Figure 1. Demonstration of OSDE. Left: original image. Middle: data generated according to the original image. Right: estimate of the light density obtained with OSDE.

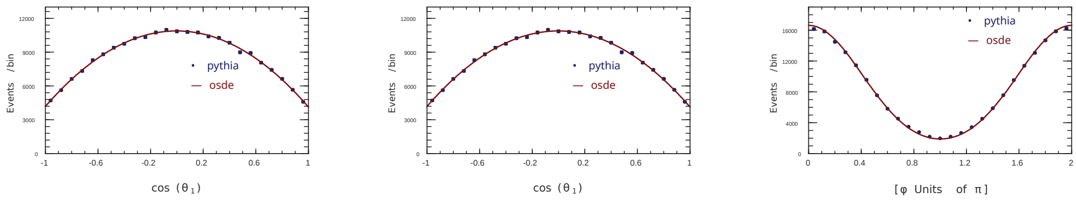

Figure 2. Projections of the joint PDF $\rho(\theta_1, \theta_2, \phi)$ determined using OSDE onto $\cos\theta_1$ and $\cos\theta_2$ (left, middle) and ϕ (right), for a 200 GeV Higgs decaying to W^+W^-.

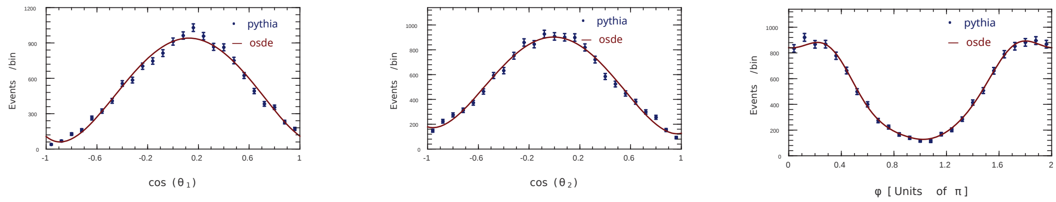

Figure 3. Projection of the joint PDF $\rho(\theta_1, \theta_2, \phi)$ with the inclusion of detector effects determined using OSDE. Obvious distortions occur.

then the convolution is

$$(\rho \circ \mathcal{R})(\theta, \phi) = \sum_{l=0}^{M} \sum_{m=-l}^{l} d_l^m Y_l^m(\theta, \phi) \qquad \text{where} \qquad d_l^m \equiv \frac{2}{2l+1} r_l c_l^m \qquad \text{(no summation)}$$

To handle the case of anisotropic smearing of the function $\rho(\theta_1, \theta_2, \phi)$ we develop our own convolution theorem as follows. We describe the joint PDF for production at true angles θ_{1T}, θ_{2T}, and ϕ_T reconstructed at θ_{1R}, θ_{2R}, and ϕ_R, as a series in *product* M-functions:

$$\mathcal{R}(\theta_{1T}, \theta_{2T}, \phi_T, \theta_{1R}, \theta_{2R}, \phi_R) = r_{k,l,m,k',l',m'} M_{k,l}^{m}(\theta_{1T}\theta_{2T}, \phi_T) M_{k',l'}^{m'}(\theta_{1R}, \theta_{2R}, \phi_R). \tag{5}$$

and determine the coefficients from Monte Carlo, using OSDE. Using orthonormality (Eq. 1) and Gaunt's theorem (Eq. 2), one can relate the coefficients of a joint PDF to those, denoted by $g_{\kappa',\lambda',\mu',K',L',M'}$, of a conditional PDF, as follows:

$$a_{\kappa,\lambda,\mu} W_{\kappa',\lambda',\kappa,\lambda,K,L}^{\mu',\mu,M} \cdot g_{\kappa',\lambda',\mu',K',L',M'} = r_{1,K,L,M,K',L',M'} \tag{6}$$

Here, the $a_{\kappa,\lambda,\mu}$ are the known coefficients of $\rho(\theta_{1T}, \theta_{2T}, \phi_T)$ taken from Monte Carlo.

For each value of K, L, M, K', L', M', Eq. 6 is a matrix equation which can be inverted to determine the coefficients $g_{\kappa',\lambda',\mu',K',L',M'}$. To obtain the distribution of reconstructed angles, one must convolve the distribution of true angles with the conditional probability to migrate from the true angles the reconstructed angles. The coefficients of this distribution are:

$$\mathcal{A}_{k,l,m} = g_{K,L,-M,k,l,m} a_{K,L,M}. \tag{7}$$

This is a matrix equation that can be expressed as

$$\vec{\mathcal{A}} = \mathbf{G} \cdot \vec{a} \tag{8}$$

This equation has to be inverted, but the matrix $\mathbf{G}$ is rectangular, and has an infinite number of rows! So, we restrict to a finite subset of reconstructed coefficients and “invert” by minimizing

$$\chi^2(\vec{a}) = (\vec{\mathcal{A}} - \mathbf{G} \cdot \vec{a})^T \cdot \mathbf{W} \cdot (\vec{\mathcal{A}} - \mathbf{G} \cdot \vec{a}) \tag{9}$$

which has the analytic solution

$$\vec{a} = \mathbf{V}\mathbf{G}^T\mathbf{W}\vec{\mathcal{A}} \quad \text{where} \quad \mathbf{V} = \mathrm{Cov}(\vec{a}) = (\mathbf{G}^T\mathbf{W}\mathbf{G})^{-1} \quad \text{and} \quad \mathbf{C} = \mathbf{W}^{-1} = \mathrm{Cov}(\vec{\mathcal{A}}) \tag{10}$$

Increasing the number of coefficients increases the information used in the fit, but also increases the noise. Some experimentation is required to optimize the cutoff.

1. Conclusion

We have outlined here a procedure for using Fourier tricks, particularly a kind of convolution theorem, to deconvolve the effects of a particle detector from a reconstructed signal. We show this procedure in action in Fig. 4. On the left we show reconstructed coefficients obtained by applying the OSDE procedure on simulated data at reconstruction level. On the right we compare the true coefficients with those obtained by the proposed deconvolution procedure.

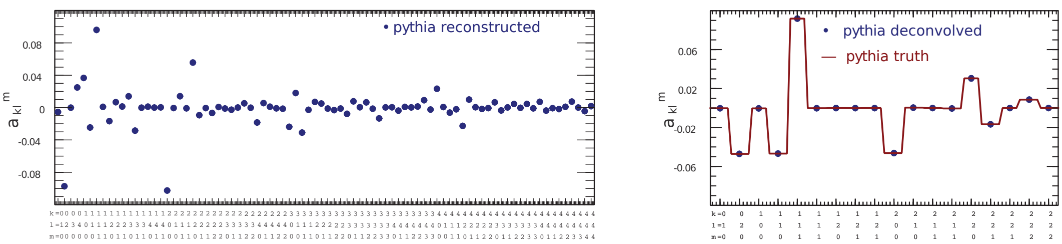

Figure 4. Left: angular coefficients $a_{k,l}^m$ of a simulated $H \to W^+W^-$ signal in a lepton+jets final state, at reconstruction level. Right: deconvolved angular coefficients (blue points) compared to a truth-level estimate (red line). This demonstrates the viability of the method outlined herein.

References

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