Confusing aspects regarding the symmetry of the vacuum state in theories with "spontaneous symmetry breaking"

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Abstract. The topic known as spontaneous symmetry breaking is afflicted with a high level of confusion. Here the topic is discussed in some detail focusing on the various sources of the confusion. A picture is presented, in which, with a small exception, and in contrast with standard presentations, the vacuum state is symmetric, and yet, the successful phenomenology usually associated with the subject is fully recovered.

1. Introduction

Spontaneous symmetry breaking (SSB) is regarded as a fundamental aspect in the understanding of multiple situations in various areas of theoretical physics. The general idea is often described as characterizing situations where the vacuum, or lowest energy state in the theory, does not share all the symmetries of the theory. When the symmetry is a continuous one that condition is then associated with the emergence of Nambu-Goldstone Bosons, or the Higgs mechanism, when Gauge Symmetries are broken.

Despite its importance, the topic is plagued by a serious level of confusion and by multiple misunderstandings, that can be found not only in research articles, but also in many leading textbooks. Moreover, the discussions found in the literature are often rather unclear about the general context in which they are carried on. In fact the SSB appellation is one that invokes things occurring in time: "everything was symmetric when suddenly and spontaneously a fundamental asymmetry simply appeared". If that was what one was referring to, one might then want to inquire about the timing of the symmetry breaking? and what exactly was then the cause of that occurrence? Here, again confusion can arise if we mix up the simple kind of classical statistical fluctuations, such as those associated with Brownian motion, and involve processes taking place in time, with the more fundamental quantum fluctuations, that are nothing more that the uncertainties that quantum mechanics normally associates with any given observable in an arbitrary quantum state, and as such refer to no explicit process in time. This line of inquire would rapidly take us into the realm of foundations of quantum mechanics, a topic that would require an extensive discussion on its own. Fortunately however, and despite the wording, the subject matter that concerns us here is something much simpler, namely, the question of under what conditions does the lowest energy state of a system exhibit the symmetries of the system's dynamics. It is also clear that the question itself is only well posed when one is

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dealing with situations in which the notion of energy is well defined, something that, as we know, requires stationarity of the space-time, and thus, from the onset, more general situations including the cosmological setting itself, would lie beyond the regime under consideration. Nonetheless confusion is abundant. Take the very important case of SSB in Gauge Theories which is, in fact, one of those associated with the most serious misunderstandings. Let us consider the following point: Gauge theories are theories with constrains (first order), and as it is well known, the Dirac rules for quantization of such theories require the physical states to be annihilated by the constraints. However the constrains are also the generators of infinitesimal gauge transformations, and thus, all physical states are necessary invariant. If the vacuum were not invariant, then it would follow that the vacuum could not be a physical state! Something seems clearly wrong in the standard accounts.

There are some people that understand these issues very well, but given the vast state of confusion that prevails in the literature on the subject it is also clear that there are many colleagues for whom this article could be helpful. In fact, before our original work on the subject [6] my colleague and I where quite confused ourselves. In order to help clarify the issues, in the present review I discuss the relevant ideas and sources of misunderstandings in the following contexts: Classical Mechanics, Quantum Mechanics, Global Symmetries in QFT for finite spatial extension, Global Symmetries in QFT in the limit of infinite spatial extension, Gauge Theories, and Statistical Mechanics. The goal would then be to identify and understand the fundamental differences regarding SSB in the various contexts, when they do exist, addressing the sources of confusion in each situation.

The manuscript is organized as follows: Section 2 deals with the simple case of discrete symmetries both at classical and quantum mechanical level, while section 3 extends the discussion for the continuous ones. Then we start addressing the case of infinite number of degrees of freedom in section 4, which is devoted to quantum field theory and global (or rigid) exact symmetries. The case of approximate symmetries in field theory is treated in section 5, while the situation in gauge theories is briefly discussed in section 6. Section 7 is devoted to the important differences between the previous treatments and those that one usually finds discussed in the statistical mechanical context. We end with a very brief discussion.

2. Classical and Quantum Mechanics, discrete symmetries.

Consider a system with a simple discrete symmetry P. In the classical theory there are two possibilities: either the lowest energy state is invariant under P, or there is a degeneracy in the state with lowest energy.

In the quantum theory the characterization of the situation is given by the existence of a unitary (or anti-unitary, a possibility we ignore here for simplicity) operator $\hat{P}$ acting on the Hilbert space $\mathcal{H}$ of the theory, $\hat{P} : \mathcal{H} \to \mathcal{H}$, which commutes with the Hamiltonian $PHP^{\dagger} = H$.

Consider for instance a particle in one dimension with a symmetric double well potential. Let the states $|R\rangle$ & $|L\rangle$, represent the localized wave-packets at the bottom of each well. The symmetry is reflected by

$$\hat{P}|R\rangle = |L\rangle, \qquad \hat{P}|L\rangle = |R\rangle. \tag{1}$$

The vacuum state must be of the form $|0\rangle = \frac{1}{\sqrt{2}}(|R\rangle + e^{i\alpha}|L\rangle)$. Now let us consider how do we determine α?

If $\langle R|H|L\rangle = c = |c|e^{i\beta} \neq 0$ then the minima of energy corresponds to $\alpha = \pi - \beta$. The point is that quantum interference can reduce the energy! By judiciously selecting the overall phase one can see that, of course, the vacuum is symmetric under P. However what happens if $c = 0$? Then α is undetermined and the vacuum is degenerate. Even some asymmetric states such as $|R\rangle$ or $|L\rangle$, are "vacuum states". In this case a system prepared in the state $|R\rangle$, has zero probability of tunneling to $|L\rangle$.

All this is very simple and clear, however we must note that, if c is fixed by the physical situation and the case $c = 0$ represents just an idealization, say, in the double well potential, the limit in which the barrier becomes infinite, the degeneration related with $c = 0$ is just a singular aspect of the limit: Regardless of how small $c \neq 0$ is, the vacuum is unique and symmetric, and it also determines, in general, a unique and symmetric vacuum in the limit. We should not confuse the **limit** $c \to 0$, with the **case** $c = 0$. In general there is no "continuity" in this limiting procedure. It is only when $c = 0$ represents the actual physical situation rather than an unphysical limit (for instance if the barrier is actually infinite) that we do have a true degeneration in the lowest energy states and the possibility of a non-symmetric vacuum. However, even then, among the multiplicity of vacua, there will be some that are symmetric states.

Of course the actual state of the system, will depend on how it was prepared. In this case energetic arguments do not allow one to select a unique quantum state. One cannot talk about the symmetry of " the vacuum" because there are symmetric and non-symmetric ones.

3. Mechanics Classical and Quantum: continuous symmetries

Consider now a system with two degrees of freedom and with a continuous $O(2)$ or $U(1)$ symmetry. Let the lagrangian of the theory be:

$$L = \frac{1}{2}m(\dot{X}_1^2 + \dot{X}_2^2) - \lambda(X_1^2 + X_2^2 - v^2)^2 \,. \tag{2}$$

The classical vacua form a degeneracy circle corresponding to $X_1 + iX_2 = ve^{i\theta}$. Each one of these classical states breaks the symmetry. What happens at the quantum level? The Hamiltonian, in polar coordinates is:

$$H = \frac{1}{2m}\, P_r^2 + \frac{1}{2mr^2} P_\theta^2 + \lambda(r^2 - v^2)^2. \tag{3}$$

Note however that θ is not a good coordinate on all the configuration space manifold. The vacuum wave function is of the form $\Psi_0(r, \theta) = \frac{1}{\sqrt{2\pi}}\Phi_0(r)$, where $\Phi_0(r)$ is the vacuum of the "radial hamiltonian",

$$H^{(radial)} = \frac{1}{2m}\, P_r^2 + \lambda(r^2 - v^2)^2, \tag{4}$$

as all dependence on θ will increase the energy simply because the operator $\frac{1}{2mr^2} P_\theta^2$ is a positive definite operator. We thus have $H^{(radial)}|\Phi_0\rangle = E_0|\Phi_0\rangle$. Now, introducing the effective Hamiltonian for variable θ as :

$$H_{eff}(\theta, P_\theta) = \langle\Phi_0|H|\Phi_0\rangle = \langle\Phi_0|H^{(radial)}|\Phi_0\rangle + \langle\Phi_0|\frac{1}{2mr^2}P_\theta^2|\Phi_0\rangle \approx E_0 + \frac{1}{2mv^2}P_\theta^2, \tag{5}$$

we see that it corresponds to a free particle (or a harmonic oscillator with zero spring constant). This is the analog of a Nambu-Goldstone boson, a particle with zero mass. Note however that the vacuum state is symmetric, and in particular $\langle\Psi_0|X_1|\Psi_0\rangle = \langle\Psi_0|X_2|\Psi_0\rangle = 0$. There are Goldstone bosons even though there is no breakdown of the symmetry! However it is clear that $\langle\Psi_0|X_1^2|\Psi_0\rangle \neq \langle\Psi_0|X_1|\Psi_0\rangle^2$. That is, the state is characterized by strong quantum mechanical "correlations".

Consider next a system with three degrees of freedom to illustrate the analog of the (bosonic) mass generation mechanism usually associated with SSB. We take the system's Lagrangian to be:

$$L' = \frac{1}{2}m(\dot{X}_1^2 + \dot{X}_2^2) + (\mu/2)\dot{X}_3^2 - \lambda(X_1^2 + X_2^2 - v^2)^2 - \alpha(X_1^2 + X_2^2)X_3^2 \,. \tag{6}$$

We note that there is no quadratic term in X_3, so it is not a true harmonic oscillator.

The Hamiltonian (with the same change of variables as before) is:

$$H' = \frac{1}{2m}\, P_r^2 + \frac{1}{2mr^2} P_\theta^2 + \frac{1}{2\mu} P_3^2 + \lambda(r^2 - v^2)^2 + \alpha r^2 X_3^2. \tag{7}$$

The vacuum wave function is now: $\Psi_0'(r, \theta, X_3) = \frac{1}{2\pi}\Phi_0'(r, X_3)$, where $\Phi_0'(r, X_3)$ is the vacuum wave function for the two degrees of freedom system with Hamiltonian:

$$H^{(2)} = \frac{1}{2m} {P_r}^2 + \lambda(r^2 - v^2)^2 + \frac{1}{2\mu} P_3^2 + \alpha r^2 X_3^2. \tag{8}$$

Again, the vacuum is thus symmetric, as any dependence of the wavefunction on θ can only increase the energy. Again $\langle\Psi_0|X_1|\Psi_0\rangle = \langle\Psi_0|X_2|\Psi_0\rangle = 0$ and therefore there is no SSB!.

Comparing (7) and (3) we see that $H' = H + \frac{1}{2\mu}P_3^2 + \alpha r^2 X_3^2$ and from (8) and (4) we see that $H^{(2)} = H^{(radial)} + \frac{1}{2\mu}P_3^2 + \alpha r^2 X_3^2$.

Consider now the effective Hamiltonian for the X_3 degree of freedom: $H'_{eff}(X_3, P_3) = \langle\Psi_0|H'|\Psi_0\rangle$ where Ψ_0 is the vacuum state of the previous example, (i.e. the state described bellow eq. (3)). In this case we thus have,

$$H'_{eff}(X_3, P_3) = \langle\Psi_0|H|\Psi_0\rangle + \langle\Psi_0|(\frac{1}{2\mu}P_3^2 + \alpha r^2 X_3^2)|\Psi_0\rangle \approx E_0 + \frac{1}{2\mu}P_3^2 + \alpha v^2 X_3^2, \tag{9}$$

that is the X_3 degree of freedom (D.O.F) has now become a true harmonic oscillator with $K = 2\alpha v^2$. We note the analogy with the mass generation mechanism, again occurring without the breakdown of the symmetry by the vacuum state.

4. Quantum Field Theory and global symmetries

In this section we consider what part of the previous results extends to this situation? what does not ?, and then, why? Consider the paradigmatic case corresponding to a theory with n scalar fields $\Phi_i, i = 1, ...n$, and a global $O(n)$, symmetry described by the Lagrangian density:

$$\mathcal{L} = (1/2)\Sigma_{i=1}^n \partial_\mu \Phi_i \partial^\mu \Phi_i + V(\Sigma_{j=1}^n \Phi_j \Phi_j). \tag{10}$$

Let us focus on the case $n = 2$ where the potential has the standard Mexican Hat shape $V = \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 - v^2)^2$. It is convenient to make a change of field variables so that $\Phi_1 = (\rho + v)\cos(\theta)$, and $\Phi_2 = (\rho + v)\sin(\theta)$. The conjugate momenta to these fields are: $\pi_\rho = \dot{\rho}$ and $\pi_\theta = (\rho + v)^2\dot{\theta}$, and the Hamiltonian density takes the form:

$$\mathcal{H} = \frac{1}{2}\pi_\rho^2 + \frac{1}{2(\rho + v)^2}\pi_\theta^2 + \frac{1}{2}(\partial_i \rho)^2 + \frac{1}{2}(\rho + v)^2(\partial_i \theta)^2 + \frac{\lambda}{4}(\rho^2 + 2v\rho)^2. \tag{11}$$

All this is standard [1], [2]. More nuanced treatments also involve discussions of stability under external perturbations and other considerations [3]. However, we want to focus on the simple and very specific issue of the symmetry of the vacuum of the theory in the absence of external factors, as a precise mathematical physics question. To this end let us now consider the situation in more detail.

In order to quantize the theory we separate into free and interaction Hamiltonian densities. The free part is:

$$\mathcal{H}_f = \frac{1}{2}\pi_\rho^2 + \frac{1}{2v^2}\pi_\theta^2 + \frac{1}{2}(\partial_i \rho)^2 + \frac{1}{2}v^2(\partial_i \theta)^2 + \frac{1}{2}m^2\rho^2 \tag{12}$$

with $m^2 = 2\lambda v^2$. We will now review the quantization procedure focusing on the treatment of the "zero mode"- the part which is position independent- of the field θ, which is the DOF which is supposed to acquire a non-vanishing expectation value in association with the SSB.

In order to proceed we consider the theory "in a box" with sides of length L with periodic boundary conditions. At the end we will explore the limit $L \to \infty$. We note that even with L finite, this is a theory with an infinite number of DOF, an issue that is sometimes mentioned as leading to a fundamental difference with Quantum Mechanics regarding the symmetry of the vacuum. It is well known that the question of number of DOF implies some differences between the QFT and Quantum Mechanics. The question we must consider here is whether there is a difference regarding the symmetry of the vacuum. Let us examine this in detail.

It is convenient at this point to make a canonical transformation to the new variables: $\phi \equiv v\theta$ and $\pi_\phi \equiv (1/v)\pi_\theta$, so that the free Hamiltonian is now:

$$H_f = \int d^3x[\frac{1}{2}\pi_\rho^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\partial_i\rho)^2 + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2\rho^2]. \tag{13}$$

Making a Fourier expansion we have:

$$\rho(\vec{x},t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k L^3}}(a_k(t)e^{-i\vec{k}\vec{x}} + a_k^\dagger(t)e^{i\vec{k}\vec{x}}), \pi_\rho(\vec{x},t) = -i\sum_{\vec{k}} \sqrt{\frac{\omega_k}{2L^3}}(a_k(t)e^{-i\vec{k}\vec{x}} - a_k^\dagger(t)e^{i\vec{k}\vec{x}}) \tag{14}$$

where $\omega_k = \sqrt{\vec{k}\cdot\vec{k} + m^2}$, the components of $\vec{k}$ are those compatible with boundary conditions and $[a_k, a_{k'}^\dagger] = \delta_{k,k'}$ (we use k instead of $\vec{k}$ in the indices to simplify the notation). In the same way we expand the field ϕ, while treating the zero mode separately. Thus we write,

$$\phi = \frac{1}{L^{3/2}}\phi_0(t) + \sum_{\vec{k}\neq 0} \frac{1}{\sqrt{2\omega_k' L^3}}(b_k(t)e^{-i\vec{k}\vec{x}} + b_k^\dagger(t)e^{i\vec{k}\vec{x}}) \tag{15}$$

$$\pi_\phi = \frac{\pi_0(t)}{L^{3/2}} - i\sum_{\vec{k}\neq 0} \sqrt{\frac{\omega_k'}{2L^3}}(b_k(t)e^{-i\vec{k}\vec{x}} - b_k^\dagger(t)e^{i\vec{k}\vec{x}}) \tag{16}$$

where $\omega_k' = |\vec{k}|$. The Hamiltonian is then,

$$H_f = \int \mathcal{H}_f \; d^3x = \frac{1}{2}\pi_0^2 + \sum_{\vec{k}}[\frac{1}{2}((\pi_\phi)_k)^2 + \frac{k^2}{2}(\phi_k)^2] + \sum_{\vec{k}}[\frac{1}{2}((\pi_\rho)_k)^2 + \frac{k^2+m^2}{2}(\rho_k)^2]. \tag{17}$$

This is the Hamiltonian for a free particle plus that of an infinite collection of harmonic oscillators. Note however that this is not exactly correct, because of the fact that that the field coordinate ϕ takes values in the interval $[0, 2\pi v]$ and it is periodical. We ignore the range restrictions on $\phi(x)$ (as those are irrelevant regarding the determination of the vacuum) and proceed with the analysis based on the Hamiltonian of the system which becomes:

$$H_f = \int \mathcal{H}_f d^3x = \frac{1}{2}\pi_0^2 + \sum_{\vec{k}}[\omega_k(a_k^\dagger a_k + \frac{1}{2}) + \omega_k'(b_k^\dagger b_k + \frac{1}{2})]. \tag{18}$$

The lowest energy state or vacuum $|0\rangle$ is then clearly characterized by $a_k|0\rangle = 0, b_k|0\rangle = 0$, & $\pi_0|0\rangle = 0$. The wave functional for the vacuum state is then the product of the wave functions for all the modes:

$$\Psi_0 = \otimes_k \psi_k^\phi[\phi_k] \otimes \psi_k^\rho[\rho_k], \tag{19}$$

where

$$\psi_k^\rho[\rho_k] = N_k \exp[-\frac{\sqrt{\vec{k}^2+m^2}}{2}\rho_k^2], \forall\vec{k}; \qquad \psi_k^\phi[\phi_k] = N_k'\exp[-\frac{\sqrt{\vec{k}^2}}{2}\phi_k^2], \forall\vec{k}\neq 0 \tag{20}$$

with N_k and N'_k normalization factors, while for the zero mode of the fieldϕ we have simply $\psi_0^\phi[\phi_0] = \frac{1}{(2\pi v L^{3/2})^{1/2}}$. This shows that the zero mode of the field has a constant wave function. Therefore there is no breakdown of the symmetry and in particular $\langle 0|\Phi_1|0\rangle = \langle 0|\Phi_2|0\rangle = 0$. Once more we have a situation where $\langle 0|\Phi_1^2|0\rangle \neq \langle 0|\Phi_1|0\rangle^2$, so the vacuum state is characterized by strong quantum correlations.

In fact, and independently of the composite nature of the operators: $\phi_1 = (\rho + v)\cos(\phi/v)$, y $\phi_2 = (\rho + v)\sin(\phi/v)$, one can see that the contribution of the zero mode to the vacuum expectation value of the field operators, vanish for any value of the "cut-off " (L). That is $\langle 0|\phi_i|0\rangle = 0; i = 1, 2$.

Thus, it is clear that this will remain valid also in the limit $L \to \infty$. Therefore this shows that the symmetry is not broken by the "vacuum". In fact one can show that: $\exp(i\epsilon \int \pi_\phi(x)dx)|0\rangle = |0\rangle$, where the operator in the l.h.s is rotation operator $U(1)$ with angle ϵ around the axis of the Mexican Hat potential.

4.1. Zero Mode, causality and "clustering"

In this subsection we briefly discuss a couple of objections to the overall picture we are presenting. The first one concerns the issue of "clustering". That is the often invoked requirement that, in the vacuum state, there should not be correlations of arbitrarily long range. It is clear, that in the examples we have considered, our vacuum fails to satisfy this condition. In our view this requirement is based on misconceptions about causality and is often presented with erroneous justifications. The usual argument is that unless the clustering requirement is enforced there would be violations of causality. One can think of simple examples of physical situations where there are long range correlations and where there are no violations of causality, such as those arising in standard EPR-B experiments. Of course there is then the nontrivial issue of how the state with such long correlations is prepared, or how did the system come into being set in such a state. We certainly acknowledge that this is an interesting question but it is also clear that this is a completely separate question from the identification of the vacuum or lowest energy state, and whether that state shares the symmetries of the theory.

We note that when the vacuum wave function has a "width" $\neq 0$ in the variable ϕ_0, the "clustering" property is violated. In fact the correlation Δ is

$$\Delta(\vec{x}, \vec{y}) \equiv \langle \phi(\vec{x}, t)\phi(\vec{y}, t)\rangle - \langle \phi(\vec{x}, t)\rangle\langle \phi(\vec{x}, t)\rangle = \langle \phi_0^2\rangle/L^3 + \delta(\vec{x}, \vec{y}) - \langle \phi_0\rangle^2/L^3 = \sigma_{\phi_0}^2/L^3 + \delta(\vec{x}, \vec{y}) \tag{21}$$

where σ_{ϕ_0} is the uncertainty on the value of the zero mode.

Thus, in the vacuum we do have long distance correlations: characterized by $\Delta(\vec{x}, \vec{y}) = \frac{\pi^2 v^2}{3} + \delta(\vec{x}, \vec{y})$ which are independent of our "universe' s size". Thus the clustering property is violated, independently of the size of the universe. If this was a real problem, it would not be resolved by taking the "vacuum" as a state sharply "picked" on ϕ_0 (as normally considered in the context of SSB), because such a state (which is not the true vacuum) would have violation of "clustering" property for the n-point functions for $\pi(x)$ instead of those of for $\phi(x)$. It is clear that, due Heisenberg's uncertainty relation between ψ_0 and π_0, one can not get rid of both. The violation of "clustering" is a manifestation of the quantum nature of the zero mode, and it survives in the $L \to \infty$ limit.

Another issue that is sometimes raised in the present context is weather one can simply ignore the zero mode, or just treat it classically. In fact in the compact case the zero mode must be treated quantum mechanically, otherwise one violates causality. Let us evaluate the commutator:

$$[\phi(x), \phi(y)] = \sum_{\vec{k}\neq 0} \frac{1}{2\omega'_k L^3} e^{(-ik\cdot(x-y))} - \sum_{\vec{k}\neq 0} \frac{1}{2\omega'_k L^3} e^{(ik\cdot(x-y))} + 2i\frac{(y^0 - x^0)}{L^3}, \tag{22}$$

where the last term comes from the zero mode. The point is that an explicit calculation shows that for $x-y$ space-like, the vanishing of the commutator arises due to cancellations between the contribution coming from the sum and that of the zero mode: The conclusion is that the zero mode **must** be treated quantum mechanically.

4.2. The $L=\infty$ case vs. the $L\to\infty$ limit.

For the case where L is finite, the energetic cost involved in localizing the zero mode of the field θ with an uncertainty $\delta\theta$ is $\delta E = \frac{1}{2v^2L^3(\delta\theta)^2}$ which tends to 0 as $L\to\infty$. Here we should recall the lessons from the Quantum Mechanical examples discussed at the beginning involving a potential barrier: As we saw there, it was essential to consider whether the barrier is very big and finite, or if it is truly infinite.

In the case $L=\infty$ the QFT construction faces a technical problem: We do not have a way to normalize all functions on R (or R^n). It is then argued in many treatments of the issue that we should only be interested on the functions of compact support as they characterize the excitations that "can be experimentally produced in a (finite) laboratory". Following this logic, one would, in effect, leave without a quantum treatment the zero mode of the field!. We should be careful here and avoid a potential confusion: It is **not** that the zero mode becomes classical, but that we do not have the mathematical tools to treat it quantum mechanically. However we should note that it would be incongruous to construct a theory for the excitations of compact support and use it to say something (like argue that there is SSB) about excitations which are not in this class.

If we justify this way of proceeding on our supposed limitation of interest to discuss only those excitations that we can produce in the laboratory, we could not then, come around and pretend to discuss the global behavior of the fields (including effects that might have cosmological causes). And particularly we would not be able to argue that those global modes that are left untreated determine something we do see in the laboratory, like the masses of elementary particles.

In fact, we must consider what is the true relevance of the $L=\infty$ case? Our universe is clearly not the infinite Minkowski space-time. That characterization is used only as an approximation for a region which is very large compared to the size of our measuring apparatus. What is relevant for such situations is then the limit, rather than the case of infinite Minkowski space-time.

When the limit is singular, as we saw, the appropriate thing to do is not "to consider the Hamiltonian in the limit", (among other things because we can't treat the zero mode) and then construct the theory, find the vacuum and do calculations, but rather, the appropriate procedure is: first construct the theory for finite L, then find the vacuum, do all calculations, and then take the limit $L\to\infty$. As we saw, in that case the treatment is transparent, the vacuum is always invariant, and there is nothing like SSB.

5. QFT , approximate symmetries. SU(2) & SU(3) flavor and the mesons.

The paradigmatic case is that of the effective strong interactions of light hadrons (see for instance [3]). The fundamental theory is QCD with a gauge group $SU(3)$ and global approximate symmetry $SU(2)_L\times SU(2)_R$: or Chiral symmetry (often one considers instead $SU(3)_L\times SU(3)_R$). The lagrangian density for this theory is :

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] + \sum_{i=u,d}[\bar{\psi}_R^{(i)}D_\mu\gamma^\mu\psi_R^{(i)} + \bar{\psi}_L^{(i)}D_\mu\gamma^\mu\psi_L^{(i)}] + \sum_{i=u,d} m_i[\bar{\psi}_R^{(i)}\psi_L^{(i)} + \bar{\psi}_L^{(i)}\psi_R^{(i)}] \quad (23)$$

where $F_{\mu\nu}$ are the field strength tensors for the gauge fields (and are Lie-Algebra valued) and $\psi_{R,L}^{(i)}$ are the left and right components of the quark fields of various flavors i. Here D_μ are color-covariant derivatives γ^μ are the Dirac matrices, and m_i are the light masses for the quarks.

The Lagrangian has various internal symmetries: $SU(3)$ color (indices not shown) an 2(3) $U(1)$ flavor symmetries.

If we set the 2 quark masses equal, we have an extra $SU(2)$ (flavor symmetry), and if we set them $m_i = 0$, we have an even larger symmetry: $SU(2)_L \times SU(2)_R$. In this case there should be two sets of $SU(2)$ conserved currents, denoted $J_V^{a\mu}(x)$ and $J_A^{a\mu}(x)$. The $J_V^{a\mu}(x)$ are associated with $SU(2)$ identical & simultaneous transformations of the left and right fermion field components, while, $J_A^{a\mu}(x)$, are related to opposite & simultaneous transformations of the left and right fermion field components.

The $J_A^{a\mu}(x)$ has the same quantum numbers of the pion, allowing the successful phenomenological identification known as (PCAC) [4]:

$$< 0|J_A^{a\mu}(x)|\pi(k), c >= ik^\mu f_\pi e^{-ikx}\delta^{ac} \tag{24}$$

where $|\pi(k), c >$ is the state with one pion of momentum k and flavor quantum number $SU(2) : c$. The proportionality constant f_π, turns out to be connected to the weak decay of the pion, and is thus measurable.

As far as we understand, the issue of the symmetry of the vacuum arises when considering objects which are obtained through formal manipulations such as:

$$f_\pi^2 \delta^{ab} = -\frac{i}{3}\int d^4x \langle 0|T(J_L^a(x) \cdot J_R^b(0))|0\rangle. \tag{25}$$

The point is that if the symmetry was 100% respected by the vacuum then the r.h.s should be 0, as that expression is not invariant under the $SU(2)$, axial transformations (note however that it is invariant under the vector ones). The question is then: what happens in the un-physical limit, in which the quark masses tend to zero ? The phenomenological argument is that the l.h.s. (i.e. f_π) can not be expected to tend to zero in this limit, because its value $130MeV$ is much larger than the value of $m_i \approx 5MeV$. This argument supports a point of view according to which, the symmetry would need to be broken even if $m_i = 0$, a situation where the Lagrangian is fully invariant.

The conclusion would be that the symmetry is spontaneously broken. This argument seems to ignore that in general, such limits are singular (as we saw in the QM example). Let us see a very clear example: Consider a free particle with mass M in 1-d: the state of lowest energy corresponds to a plane wave with zero momentum, i.e. a constant function. In such situation the expectation value $\langle 0|X|0\rangle$ is ill defined, but the state reflects the translational symmetry of the lagrangian, (a different issue is that of its normalization and the fact that $|\phi(X)|^2 X$ is even more divergent than $|\phi(X)|^2$).

Next let us consider a quadratic potential: $V(X) = \frac{K}{2}(X-5)^2$. The vacuum wave function is $\Phi(X) = Ne^{-\frac{(x-5)^2}{\sigma^2}}$, with $\sigma = \frac{1}{\sqrt{KM}}$. The expectation value is now well defined: $\langle 0|X|0\rangle = 5$ and the translational symmetry is clearly broken. However we note that the expectation value is independent of K, and thus we have: $\lim_{K\to 0}\langle 0|X|0\rangle = 5$ which **cannot be used** to conclude that, even in the **case** $K = 0$, the ground state should break the translational symmetry.

6. QFT Gauge Symmetries

This case has in fact been treated in detail by various authors [**?**, 8] and thus we will only devote a little space to them here. The central point is that these theories have, even at the classical level, a very serious problem: At the pure mathematical level they do not have a well-posed initial value formulation. This is easy to see by considering a given solution of the evolution equation with certain initial conditions at say the hypersurface $t = 0$ in Minkowski space-time. One can consider a gauge transformation which is the identity in a neighborhood of the initial hypersurface and becomes nontrivial elsewhere. The result will be again a solution of the equations of motion corresponding to the same initial conditions as the first one, and differing from it elsewhere in the space-time. The problem disappears once one identifies the gauge invariance as a redundancy in the description: Two solutions of the field equations that differ by a gauge transformation represent the same physical solution!

At the quantum level, we can think of the states of the system as wave functionals $\Psi[\phi, A_\mu]$. It is clear that for the theory to make sense, when evaluating the functional on two configurations that differ by a gauge transformation, the functional Ψ should give the same value. That is, all physical states must be gauge invariant.

The vacuum must be a physical state and therefore it needs to be gauge invariant. There is, in this sense, simply no room for SSB. However one finds statements affirming that in connection SSB non invariant fields acquire vacuum expectation values and thus the vacuum brakes the symmetry [1]. In fact in the standard model of particle physics the symmetry breaking by the vacuum characterized by a non-vanishing expectation value of the scalar fields in the theory is supposed to be responsible for generating masses for some of the gauge bosons W^+W^- and Z^0 and for the fermions in the theory.

The key to clarify the situation lies in the recognition that much of the treatment that is presented in many places is performed in schemes where one "fixes the gauge". That is, one is no longer working with the full theory but has instead chosen to work with a limited number of representatives of each equivalence class of field configurations (or to one such representative if in fact the gauge was fixed completely, something that is known to be generically impossible [5]).

As we have seen in various examples, the analogous of the mass generating mechanism works perfectly well without the need for the vacuum to break the symmetry, and that is exactly what happens in the present case. However in order to see this explicitly it is clear that one has to work in settings where the gauge has not been fixed. The reader might want to consult [6] to see this at work. The point is that it is incongruent to "fix the gauge" and then argue that somehow the gauge symmetry has been spontaneously broken.

7. Statistical Mechanics, Ferromagnetism.

One should not confuse the situations treated in Statistical Mechanics with those of Quantum Mechanics. They have many similarities but they are not the same. The starting point of the latter is the consideration of a large ensemble of identical systems. One is interested on ensemble averages of some physical quantities. Under certain conditions (ergodicity) one expects those to coincide with time averages for a particular system.

The standard applications involve some additional assumptions, particularly in the treatment of quantum systems. These include the interaction of the system with external D.O.F. often characterized as a thermal bath. Even at $T = 0$, the environment has a fundamental influence: It acts as the physical source of the ergodicity and of the **de-coherence**, which justifies the "random a priori random phases" hypothesis (see for instance [9]) for the elements of the ensemble, which is at the basis of most Quantum Statistical Mechanical treatments. This is required to justify various aspects of the treatments of the quantum system with the use of ensembles (micro-canonical, canonical, and gran canonical).

In contrast with the ordinary Quantum Mechanics, the treatment is fundamentally designed to deal with open systems which justifies in Stat. Phys. the considerations of, for instance, a system in equilibrium at a temperature T, which is represented by the Gibbs density matrix:

$$\rho(T) = e^{-\frac{\hat{H}}{kT}} = \sum_i e^{-\frac{E_i}{kT}} \; |\psi_i\rangle\langle\psi_i|, \tag{26}$$

instead of treatments designed to consider essentially pure states as is often the case in quantum theory. (In the above expression $\hat{H}$ is the system's hamiltonian, $\{|\psi_i\rangle\}$ a basis of its eigenstates with energies E_i and k is Boltzman's constant.) We should be careful not to forget about these when comparing or relating results of one with those of the other.

7.1. Ferromagnetism.

As a very illustrative example we consider a system of N atoms with spin $1/2$ in a fixed arrangement. We further assume that each atom interacts only with an external magnetic field $\vec{B}$ and with the nearest neighbors. The Hamiltonian is thus,

$$H = -J \sum_{a,\ b=1}^{N} \vec{s}_a \cdot \vec{s}_b - \mu \vec{B} \cdot \sum_{a=1}^{N} \vec{s_a}. \tag{27}$$

where J and μ are constants. The analysis is usually based on the canonical ensemble at temperature T: One computes the Helmholtz's free energy $A(T, B)$, and from it one obtains the magnetization trough the standard relation,

$$M(B,T) = -(\frac{\partial A}{\partial B})_{|T} \tag{28}$$

The so called spontaneous magnetization, often invoked as an analogy of SSB in QFT, corresponds to the case in which $M(0,T) \neq 0$ (i.e, magnetization without external B field) which happens when $T < T_c$ where T_c is called the critical temperature. The point is, however, that in this case the derivative does not exist !! This is hidden by the use of spherical coordinates which are bad coordinates at the origin. In fact in order to be precise, the equation above should be written as:

$$\vec{M}(\vec{B},T) = -(\vec{\nabla}_B A)_{|T} \tag{29}$$

where the B appearing in eq. (28) above, would correspond to the radial coordinate representing the magnitude of the vector $\vec{B}$. The feature that we are dealing with here is in fact an extreme

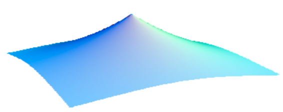

Figure 1: shows the actual shape of the function $A(T, \vec{B})$ in the thermodynamic limit as a function of $\vec{B}$ in a two dimensional case. It is clear that at the origin the function is not differentiable.

sensitivity to external fields that is reached in the thermodynamic limit $N \to \infty$. At any point different from the origin ($\vec{B} = 0$), the function A is differentiable and $\vec{M}$ is nonvanishing, nor does it vanish as $\vec{B} \to 0$. However the limiting behavior is dependent on the direction of approach to the origin, so strictly speaking the limit does not exist. It is nonetheless worth noting that in the realistic cases with finite N, the function A is smooth at the origin ($\vec{B} = 0$) and the symmetry of the situation then implies $\vec{M} = 0$. That is, we do not seem to have here something we might call spontaneous symmetry breaking.

However we must recall that the treatment above refers only to a statistical ensemble, while any consideration of an individual and finite system characterized by (27) leads, even when $\vec{B} = 0$ to a non-vanishing magnetization. Let us consider this last aspect of symmetry breaking, in more detail, in order to identify the differences with what we had faced before.

The first observation is that the Hilbert space of each individual D.O.F. does not have symmetric states: If $|\vec{n},+\rangle$ is a spin pointing in the $+\vec{n}$ direction, we might try to built and rotationally invariant state as the superposition:

$$|\mathbf{invariant}\rangle \equiv \int_{SU(2)} dg\ \mathbf{D}^{\frac{1}{2}}(g)|\vec{n},+\rangle, \tag{30}$$

where the matrix $\mathbf{D}^{1/2}(g)$ represents a Wigner rotation with g in $SU(2)$. The point however is that this gives as a result simply 0, that is the null vector in the Hilbert space. There is also no invariant state for the case of N spins when N is odd. That makes it hard to consider the limit $N \to \infty$. There are of course symmetric states for N even. In the case $N = 2$ these are the well known singlet states. However their misalignment implies an energetic cost according to 27. This seems to be a fundamental difference with the previously considered cases.

A second aspect worth noting refers to the statistical nature of the problem we faced and the fact that we deal in practice with an open quantum system. The ferromagnet is described in Statistical Mechanics by a density matrix (not a pure state), and this CAN be symmetric. For instance, for a single spin :

$$\rho_{inv} \equiv \int_{SU(2)} dg\ \mathbf{D}^{\frac{1}{2}}(g)|\vec{n},+\rangle(\mathbf{D}^{\frac{1}{2}}(g)|\vec{n},+\rangle)^{\dagger} = \frac{1}{2}I \tag{31}$$

where I is the unit matrix, which is in fact rotationally invariant. This illustrates that the conclusions regarding the issue of symmetry of the ground state of an individual quantum system can be very misleading if one relies on a standard statistical mechanical treatment. In particular it is worth noting that the treatment of individual systems composed of large number of spins can in fact be rather complex and cumbersome, as illustrated by works based on Majorana stars[10] or quantum constellations[11].

Analogously, for a system with N spins, the density matrix describing the ferromagnet (with $B = 0$) at low temperature, is spherically symmetric. However each of the possible states corresponding to an element in the ensemble, the one identified by observation, breaks the symmetry, and this corresponds to the selection of one of the degenerate vacua. This associated with the possibility, exemplified with the case $N = 1$, that the appropriate fully symmetric superposition of states, may not exist.

Moreover, when considering, in practice, the occurrence of "SSB" in this case, one in fact is looking for much more than in previous cases: Not only the state should not be symmetric, but it should be macroscopically non-symmetric. That is, somehow small domains of non-vanishing magnetization that could in principle lead to zero spatially averaged magnetization, should instead become correlated leading to a macroscopic magnetization of the individual sample. The fact that this is what one finds in the actual dealings with ferromagnets seems to connected with the EXTREME susceptibility to external fields which we saw in the evaluation of the magnetization.

Finally we should be mindful of the potential for confusions that arise due to the fact that a density matrix can be used to represent various things: i) The state of an ensemble, in which each of the elements is in a pure state, ii) A sub system which has no individual state (because it is part of a larger system) in which there are correlations between our subsystem and the rest. iii) We could have an ensemble of subsystems, each of which is as in ii). The reader is invited to see the discussion concerning *proper* and *improper* mixtures in [12], for further aspects of the importance of these distinction. The point is that our interest is establishing when an individual system's state is symmetric and the treatments involving ensembles tend to be obscure as this issue is concerned simply because, from the onset, one is dealing with a very large collection (generally infinite) of systems.

8. Discussion

As we have seen the treatments of the general subject of SSB and related topics generate a great deal of confusion, even at the text-book level. We touched on various representative aspects of those. In particular we showed that at the level of quantum filed theory, the most popular views are mistaken: In the case of rigid (non gauge) symmetries we saw that for the realistic situations involving finite spatial regions there is no SSB, while the considerations of infinite spatial extent required distinguishing the limit $L \to \infty$ (where again there is no SSB) from the physically irrelevant and mathematically problematic case $L = \infty$.

We showed that a treatment devoted to the case where L is allowed to be extended to the $L \to \infty$ limit, successfully accounts for the standard phenomenological aspects such as the emergence of Nambu-Goldstone bosons and the Higgs mechanism, while presenting a picture where the vacuum state is fully symmetric.

We briefly dealt with the case of gauge symmetries where again we hope to have clarified the situation and concluded there is simply no room for a vacuum state that breaks the symmetry. We ended considering the quantum statistical mechanical treatment case of ferromagnets, as one of the few examples where we truly face something we might call SSB.

Hopefully this manuscript will contribute to some extent in clarifying the various issues that tend to obscure many treatments of this very important topic in theoretical physics. For a more detailed discussion of all these issues we suggest consulting the work [6] and the references therein.

Acknowledgments

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