Symmetry analysis and invariant solutions of the multipoint infinite systems describing turbulence

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Abstract. The present work concerns multipoint description of turbulence in terms of the probability density functions (pdf's) and the characteristic Hopf functional. Lie symmetries of infinite systems of equations for the pdf's and the Hopf functional equation are discussed. Based on symmetries, invariant solutions for turbulence statistics are calculated.

1. Introduction

With respect to turbulence research three complete statistical descriptions of turbulence, treated as a stochastic field, are known, namely the infnite hierarchy of the multi-point correlation equations (so-called Friedmann-Keller (FK) hierarchy), the infnite hierarchy of the multipoint probability density functions (pdf's) equations (Lundgren-Monin-Novikov (LMN) equations, [1]) and finally the Hopf functional approach [2]. The two latter approaches will be discussed below.

The n-point velocity pdf $f_n(\boldsymbol{x}_{(1)}, \boldsymbol{v}_{(1)}, \boldsymbol{x}_{(2)}, \boldsymbol{v}_{(2)}, \ldots, \boldsymbol{x}_{(n)}, \boldsymbol{v}_{(n)}, t)$ contains information about all statistics up to n-point statistics of infinite order which can be calculated from the pdf by integration over the sample space variables, for example

$$\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)}, t) U_{i_{(2)}}(\boldsymbol{x}_{(2)}, t) \cdot \cdots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)}, t)\rangle = \int v_{i_{(1)}} v_{i_{(2)}} \cdot \cdots \cdot v_{i_{(n)}} f_n \mathrm{d}\boldsymbol{v}_{(1)} \ldots \mathrm{d}\boldsymbol{v}_{(n)}.$$

The infinite hierarchy of equations for multipoint pdf's was derived in [1]. E. Hopf introduced another very general approach to the description of turbulence. He considered the case where the number of points in pdf goes to infinity, so that the probability density function becomes a probability density functional and one deals with a continuous set of sample space variables $\boldsymbol{v}(\boldsymbol{x})$. It is more convenient to consider a functional Fourier transform of the probability density functional called the characteristic functional $\Phi([\boldsymbol{v}(\boldsymbol{x})], t)$. The approach of Hopf was later generalised in Ref. [3] where the full space-time functional formalism was introduced and the functional $\Phi([\boldsymbol{v}(\boldsymbol{x}, t)])$ was defined as

$$\Phi([\boldsymbol{v}(\boldsymbol{x}, t)]) = \left\langle \exp\left(i \iint \boldsymbol{U}(\boldsymbol{x}, t) \cdot \boldsymbol{y}(\boldsymbol{x}, t) \mathrm{d}t \mathrm{d}\boldsymbol{x} \right) \right\rangle \tag{1}$$

where $\boldsymbol{y}(\boldsymbol{x},t)$ is an arbitrary vector field which vanishes at spatial infinity. With this definition moments of velocity can be calculated as the functional derivatives of Φ at the origin [3]

$$\left.\frac{\delta^n\Phi}{\delta y_{i_(1)}(\boldsymbol{x}_{(1)},t_1)\cdots\delta y_{i(n)}(\boldsymbol{x}_{(n)},t_n)}\right|_{\boldsymbol{y}=0}=i^n\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t_1)U_{i_{(2)}}(\boldsymbol{x}_{(2)},t_2)\cdot\cdots\cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)},t_n)\rangle \quad (2)$$

where the functional derivative $\delta/\delta y_i(\boldsymbol{x})$ can be understood as a generalisation of a gradient, for the case of infinitely many variables, e.g. we have

$$\frac{\partial}{\partial s_i}\sum_{j=-\infty}^{\infty}f_js_j=\sum_{j=-\infty}^{\infty}f_j\delta_{ij}=f_i$$

and, for the functional derivative

$$\frac{\delta}{\delta s(\boldsymbol{x})}\int f(\boldsymbol{x}')s(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'=\int f(\boldsymbol{x}')\delta(\boldsymbol{x}'-\boldsymbol{x})\mathrm{d}\boldsymbol{x}'=f(\boldsymbol{x}).$$

Evolution equation for the characteristic functional was derived in Ref. [3]. It is only one equation (not a hierarchy) which embodies the statistical properties of the fluid flow in a very concise form.

The objectives of the present work are to discuss the classical and new statistical Lie symmetries that were first found for the FK hierarchy [4] and are also present in the LMN hierarchy [5] and the Hopf equation. Based on symmetries, invariant solutions of the considered systems can be derived [6]. Such solutions and their realizability will be addressed within the present work. In particular, a possible route to derive invariant solutions for the characteristic functional of turbulence will be outlined.

The Lie symmetry transformation is such transformation of the independent and dependent variables, which does not change the functional form of a considered equation [6], $F(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}),\underset{1}{\boldsymbol{y}}(\boldsymbol{x}),\underset{2}{\boldsymbol{y}}(\boldsymbol{x}),\dots)=0$, where $\underset{1}{\boldsymbol{y}}(\boldsymbol{x})$ denotes any first-order derivative of $\boldsymbol{y}$, $\underset{2}{\boldsymbol{y}}(\boldsymbol{x})$ any second-order derivative etc. The transformed variables

$$\boldsymbol{x}^*=\boldsymbol{g}(\boldsymbol{y}(\boldsymbol{x}),\epsilon),\;\;\boldsymbol{y}^*=\mathbf{h}(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x}),\epsilon)$$

are functions of $\boldsymbol{x}$, $\boldsymbol{y}$ and a group parameter ϵ. The functional form of a considered equation does not change when written in the new variables, i.e. we have

$$F(\boldsymbol{x}^*,\boldsymbol{y}^*,\underset{1}{\boldsymbol{y}^*},\underset{2}{\boldsymbol{y}^*},\dots)=0.$$

The transformations can also be written in infinitesimal forms after the Taylor series expansion about ϵ:

$$\boldsymbol{x}^*=\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{y})\epsilon+\mathcal{O}(\epsilon^2),\;\;\boldsymbol{y}^*=\boldsymbol{y}+\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{y})\epsilon+\mathcal{O}(\epsilon^2). \quad (3)$$

It follows from the Lie first theorem that knowing the infinitesimal forms ξ and η uniquely determines the global form of the group transformation $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. With the use of infinitesimals invariant solutions of the considered equation may be derived [6]. In fluid mechanics these solutions often represent attractors of the instantaneous fluctuating solutions of the Navier-Stokes equations, i.e. the scaling laws for turbulence statistics. From the Lie symmetry analysis of the LMN hierarchy it followed that the new symmetries are connected with intermittent laminar/turbulent flows [5]. The outcome of the symmetry analysis are the invariant solutions for turbulence statistics and new possibilities to improve turbulence closures, such that invariance under the whole set of symmetries is accounted for.

2. Symmetries of the LMN hierarchy

Symmetries of the LMN hierarchy were investigated in Ref. [5]. Therein, it was shown that the hierarchy is invariant under the classical symmetries of the Navier-Stokes, equations, in particular, time and space translations $t^* = t + a_t$ and $\boldsymbol{x}_i^* = \boldsymbol{x}_i + \boldsymbol{a}_x$, Galilean invariance $t^* = t$, $\boldsymbol{x}_i^* = \boldsymbol{x}_i + \boldsymbol{v}_0\, t$, $\boldsymbol{v}_i^* = \boldsymbol{v}_i + \boldsymbol{v}_0$ and, for non-zero viscosity, one scaling group

$$t^* = \mathrm{e}^{2a_2}t,\;\; \boldsymbol{x}_{(l)} = \mathrm{e}^{a_2}\boldsymbol{x}_{(l)},\;\; \boldsymbol{v}^*_{(l)} = \mathrm{e}^{-a_2}\boldsymbol{v}_{(l)}\; f_n^* = \mathrm{e}^{-3na_2}f_n,\; f_{n+1}^* = \mathrm{e}^{-3(n+1)a_2}f_{n+1}. \tag{4}$$

Moreover, it was shown in [5] that two additional symmetries of the LMN hierarchy exist which transform a pdf of a turbulent signal into the pdf of an intermittent laminar-turbulent flow. These symmetries are equivalent to the additional symmetries derived first for the multipoint velocity correlations in [4]. Written for the the one-point pdf these symmetries are

$$f_1^*(\boldsymbol{v};\boldsymbol{x},t) \;\;=\;\; \delta(\boldsymbol{v}) + \mathrm{e}^{a_s}(f_n - \delta(\boldsymbol{v})) \tag{5}$$

and

$$f_1^*(\boldsymbol{v};\boldsymbol{x},t) = f_1(\boldsymbol{v};\boldsymbol{x},t) + \psi(\boldsymbol{v}),\;\; \text{where} \;\; \int \psi(\boldsymbol{v})\mathrm{d}\boldsymbol{v} = 0. \tag{6}$$

The transformation (5) forms a semigroup, as the parameter a_s is not arbitrary but is restricted to $a_s \geq 0$.

With the use of new symmetries series of invariant solutions for turbulence statistics can be derived from the characteristic equation [4]. A turbulent/laminar solution for the mean velocity in the plane Poiseuille channel flow was derived in [5]. For this purpose it was necessary to use the classical scaling symmetry (4), the new statistical scaling (5) and translation (6) written for a channel flow (in a modified form due to the presence of boundaries). For the mean velocity these transformations have the following form

$$x_2^* = \mathrm{e}^{k_2}x_2,\;\; \langle U\rangle^* = \mathrm{e}^{a_s-k_2}\langle U\rangle + C_1(1 - {x_2}^2/H^2)$$

where x_2 is the wall-normal coordinate, or, in the infinitesimal forms (3)

$$\xi_{x_2} = k_2 x_2, \quad \eta_{\langle U\rangle} = (a_s - k_2)\langle U\rangle + C_1(1 - {x_2}^2/H^2). \tag{7}$$

Invariant solution for velocity can be found from the solution of the characteristic equation [5]

$$\frac{\mathrm{d}\langle U\rangle}{\eta_{\langle U\rangle}} = \frac{\mathrm{d}x_2}{\xi_{x_2}},\;\Rightarrow\;\; \frac{\mathrm{d}\langle U\rangle}{(a_s - k_2)\langle U\rangle + C_1(1 - {x_2}^2/H^2)} = \frac{\mathrm{d}x_2}{k_2 x_2}. \tag{8}$$

The system for $a_s = k_2$ has the following solution

$$\langle U\rangle = \frac{C_1}{k_2}\ln(x_2) + \frac{C_1}{2k_2}\left(1 - \frac{x_2^2}{H^2}\right) + \mathcal{C} \tag{9}$$

where, C_1, k_2 and $\mathcal{C}$ are constants and $a_s = k_2 \geq 0$. This solution is a sum of the turbulent (logarithmic) and laminar velocity profiles and follows from a mathematical analysis of the governing equations. In particular, the logarithmic solution for the turbulent part is obtained without the use of the mixing-length assumption and without specification of the mixing-length.

Within the present work the realizability of the solution (9) will be confirmed. Let us consider a flow in a channel with Re close to its critical value where the laminar-turbulent transition takes place. In the channel both laminar and turbulent flow is possible with a certain probability. A realisable solution of an intermittent laminar-turbulent flow is given by

$$\langle U\rangle = \gamma u_\tau \frac{1}{\kappa}\ln(x_2) + (1-\gamma)U_L\left(1-\frac{x_2^2}{H^2}\right) + \mathcal{C}' \tag{10}$$

where where u_τ is the friction velocity and H- the channel half-width, κ is the von Karman constant, U_L is the centerline velocity of a laminar profile in a channel and γ is the intermittency factor. The value $\gamma = 1$ corresponds to a fully turbulent and $\gamma = 0$ to a fully laminar profile, hence for (10) to be realizable, γ must be contained within the bounds $0 \le \gamma \le 1$.

From the comparison between (9) and (10) it follows that

$$\frac{C_1}{k_2} = \gamma u_\tau \frac{1}{\kappa},\ \ \frac{C_1}{2k_2} = (1-\gamma)U_L. \tag{11}$$

The following relations between the driving pressure gradient $\nabla\langle P\rangle$ and u_τ and U_L result from the averaged Navier-Stokes equations for the laminar and turbulent flow, respectively

$$U_L = -\frac{1}{2\nu\rho}\nabla\langle P\rangle H^2,\ \ u_\tau = \sqrt{-\frac{H}{\rho}\nabla\langle P\rangle}. \tag{12}$$

We can also define the Reynolds number, based on the pressure gradient as $Re = \sqrt{-H\nabla\langle P\rangle/\rho}H/\nu$. When the flow is turbulent this corresponds to $Re = Re_\tau = u_\tau H/\nu$. It then follows from (12) that $u_\tau/U_L = 2/Re$. Comparing (11) with (12) we obtain

$$\gamma = \frac{\kappa Re}{1+\kappa Re} = \frac{1}{1+(\kappa Re)^{-1}}. \tag{13}$$

Such value of γ is, correctly, contained within the bounds $0 \le \gamma \le 1$, independently of the value of Re, moreover it is a monotonic function which increases with Re. Hence, the realizability of the solution (9) which follows from the Lie group analysis can be confirmed. It is difficult to comment on a possible predictive property of (13) (in terms of the dependence of γ on Re) as the laminar-turbulent transition depends on various factors (e.g. the level of external disturbances) which are not included in the analysis. Still, it is interesting that Eq. (13) could be obtained as a result which follows from the Lie-group invariant solution (9). For $Re = 150$ where low-Re turbulent flows are observed Eq. (13) predicts $\gamma \approx 0.98$, which is a reasonable result. However, for very low Re, γ obtained from Eq. (13) is overpredicted. Possibly, Eq. (13) could be valid within a certain range of Re.

3. Symmetries of the Hopf equation

In this section, symmetries of the Hopf functional formulation for turbulence will be discussed. The evolution of the space-time functional, as defined in Eq. (1) is governed by the following equation [3]

$$\iint \eta_k(\boldsymbol{x},t)\left[\frac{\partial}{\partial t}\frac{\delta\Phi}{\delta y_k(\boldsymbol{x},t)} - i\frac{\partial}{\partial x_l}\frac{\delta^2\Phi}{\delta y_k(\boldsymbol{x},t)\delta y_l(\boldsymbol{x},t)} - \nu\frac{\partial^2}{\partial x_l\partial x_l}\frac{\delta\Phi}{\delta y_k(\boldsymbol{x},t)} + \frac{\partial\Pi}{\partial x_k}\right]\mathrm{d}t\mathrm{d}\boldsymbol{x} = 0, \tag{14}$$

where Π is the pressure functional and $\eta_k(\boldsymbol{x},t)$ is a testing field. If η_k satisfies the condition $\partial\eta_k/\partial x_k = 0$, then, the pressure functional term can be eliminated by integrating (14) by parts with $\int \partial\eta_k/\partial x_k \Pi \mathrm{d}\boldsymbol{x} = 0$. The continuity condition for the incompressible flow in the functional formulation reads

$$\frac{\partial}{\partial x_k}\frac{\delta\Phi}{\delta y_k(\boldsymbol{x},t)} = 0. \tag{15}$$

Moreover, Φ must satisfy the relations

$$\Phi(0) = 1, \;\; \overline{\Phi([y])} = \Phi([-y]), \;\; |\Phi| \leq 1, \tag{16}$$

where $\bar{\cdot}$ denotes the complex conjugate.

The Hopf functional equation (14) is invariant under the following transformation of variables

$$\Phi^* = \Phi, \quad \boldsymbol{x}^* = \mathrm{e}^{k_2}\boldsymbol{x}, \quad t^* = \mathrm{e}^{2k_2}t, \quad y_i^* \mathrm{d}\boldsymbol{x}^*\mathrm{d}t^* = \mathrm{e}^{k_2} y_i \mathrm{d}\boldsymbol{x}\mathrm{d}t, \quad \boldsymbol{y}^* = \mathrm{e}^{-4k_2}\boldsymbol{y}, \tag{17}$$

where we note that the functional derivative $\delta/\delta y_k(\boldsymbol{x}, t)$ is sometimes denoted by $\partial/(\partial y_k(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}\mathrm{d}t)$ which indicates that its dimension is $1/([y][L][T])$. Hence,

$$\frac{\delta}{\delta y_k^*(\boldsymbol{x}, t)} = \mathrm{e}^{-k_2}\frac{\delta}{\delta y_k(\boldsymbol{x}, t)}.$$

The Galilean invariance reads $t^* = t$, $\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{U}_0 t$ and $\boldsymbol{U}^* = \boldsymbol{U} + \boldsymbol{U}_0$. With this, the functional Φ transforms as follows

$$\Phi^* = \left\langle \mathrm{e}^{i\iint \boldsymbol{U}^*(\boldsymbol{x},t)\cdot \boldsymbol{y}^*(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}^*\mathrm{d}t^*} \right\rangle = \left\langle \mathrm{e}^{i\iint \boldsymbol{U}(\boldsymbol{x},t)\cdot \boldsymbol{y}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t} \right\rangle \mathrm{e}^{i\int \boldsymbol{y}(\boldsymbol{x},t)\cdot \boldsymbol{U}_0 \mathrm{d}\boldsymbol{x}\mathrm{d}t} = \Phi C([\boldsymbol{y}(\boldsymbol{x}, t)], \tag{18}$$

where the functional $C([\boldsymbol{y}(\boldsymbol{x}, t)]) = \exp(i \iint \boldsymbol{y}(\boldsymbol{x}, t) \cdot \boldsymbol{U}_0 \mathrm{d}\boldsymbol{x}\mathrm{d}t)$ satisfies the condition $C(0) = 1$, hence, also $\Phi^*(0) = 1$ as required in Eq. (16).

The n-th derivative of the transformed functional Φ^* at $\boldsymbol{y} = 0$ gives

$$\left.\frac{\delta^n \Phi^*}{\delta y_{i_{(0)}}(\boldsymbol{x}, t) \cdots \delta y_{i_{(n-1)}}(\boldsymbol{x}, t)}\right|_{\boldsymbol{y}=0} = i^n (U_{i_{(0)}}(\boldsymbol{x}, t) + U_{0i_{(0)}}) \cdots (U_{i_{(n-1)}}(\boldsymbol{x}, t) + U_{0i_{(n-1)}}) \tag{19}$$

as expected for the Galilean invariance. In the Galilean transformation, the space derivatives in equation (14) transform as $\partial/\partial x_i^* = \partial/\partial x_i$ and the integral over time and space $\int \mathrm{d}\boldsymbol{x}^*\mathrm{d}t^* = \int \mathrm{d}\boldsymbol{x}\mathrm{d}t$. As the variables $\boldsymbol{y}^* = \boldsymbol{y}$, also the functional derivative remains unchanged $\delta/\delta(y_i(\boldsymbol{x}))^* = \delta/\delta y_i(\boldsymbol{x})$. The derivative $\partial/\partial t$ can be presented as

$$\frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} + \frac{\partial x_i^*}{\partial t}\frac{\partial}{\partial x_i^*} = \frac{\partial}{\partial t^*} + U_{0k}\frac{\partial}{\partial x_k}. \tag{20}$$

The transformed functional equation (14) reads

$$\iint \eta_k(\boldsymbol{x}, t) \left[\frac{\partial}{\partial t^*} \frac{\delta C([\boldsymbol{y}(\boldsymbol{x}, t)])\Phi}{\delta y_k(\boldsymbol{x}, t)} - i\frac{\partial}{\partial x_l}\frac{\delta^2 C([\boldsymbol{y}(\boldsymbol{x}, t)])\Phi}{\delta y_k(\boldsymbol{x}, t)\delta y_l(\boldsymbol{x}, t)} - \nu \frac{\partial^2}{\partial x_l \partial x_l}\frac{\delta C([\boldsymbol{y}(\boldsymbol{x}, t)])\Phi}{\delta y_k(\boldsymbol{x}, t)} \right] \mathrm{d}t\mathrm{d}\boldsymbol{x} = 0. \tag{21}$$

With (20) the first term in bracket in (21) is

$$\frac{\partial}{\partial t^*}\frac{\delta C\Phi}{\delta y_k(\boldsymbol{x}, t)} = \left(\frac{\partial}{\partial t} - U_{0l}\frac{\partial}{\partial x_l}\right)\left(C\frac{\delta \Phi}{\delta y_k(\boldsymbol{x}, t)} + \Phi\frac{\delta C}{\delta y_k(\boldsymbol{x}, t)}\right) = C\left(\frac{\partial}{\partial t} - U_{0l}\frac{\partial}{\partial x_l}\right)\frac{\delta \Phi}{\delta y_k(\boldsymbol{x}, t)} \tag{22}$$

where the second equality follows from the fact that $\delta C/\delta y_k = iU_{0k}C$ does not depend explicitly on $\boldsymbol{x}$ or t.

The functional derivative of Φ^* in (21) reads $C\delta\Phi/\delta y_k(\boldsymbol{x}) + \Phi\delta C/\delta y_k$, and we note that Laplacian ∇_x^2 of the second term is zero as this term is not a function of $\boldsymbol{x}$. Hence, the last RHS term of equation (21) inside the integral is $C\nu\nabla_x^2\delta\Phi/\delta y_k$. Further, the second functional derivative of Φ^* reads

$$C\frac{\delta^2 \Phi}{\delta y_k(\boldsymbol{x}, t)\delta y_l(\boldsymbol{x}, t)} + \frac{\delta \Phi}{\delta y_k(\boldsymbol{x}, t)}\frac{\delta C}{\delta y_l(\boldsymbol{x}, t)} + \frac{\delta \Phi}{\delta y_l(\boldsymbol{x}, t)}\frac{\delta C}{\delta y_k(\boldsymbol{x}, t)} + \Phi\frac{\delta^2 C}{\delta y_k(\boldsymbol{x}, t)\delta y_l(\boldsymbol{x}, t)}. \tag{23}$$

Again, the derivative $\partial/\partial x_l$ of the last term is zero, as it does not depend on $\boldsymbol{x}$. In addition we also have

$$\frac{\partial}{\partial x_l}\left[\frac{\delta\Phi}{\delta y_l(\boldsymbol{x},t)}\frac{\delta C}{\delta y_k(\boldsymbol{x},t)}\right] = \frac{\delta C}{\delta y_k(\boldsymbol{x},t)}\frac{\partial}{\partial x_l}\frac{\delta\Phi}{\delta y_l(\boldsymbol{x},t)} = 0, \tag{24}$$

where the first equality follows from the fact that the derivative of C does not depend explicitly on $\boldsymbol{x}$ and the second from the continuity condition (15).

The second term in Eq.. (23), introduced into Eq. (21), leads to

$$-\frac{\delta C}{\delta y_l(\boldsymbol{x})}\left[i\frac{\partial}{\partial x_l}\frac{\delta\Phi}{\delta y_k(\boldsymbol{x},t)}\right] = U_{0l}C\left[\frac{\partial}{\partial x_l}\frac{\delta\Phi}{\delta y_k(\boldsymbol{x},t)}\right], \tag{25}$$

which cancels with the last term in (22). This finally proves the Galilean invariance of Eq. (14).

An analogue of the new statistical scaling and translation symmetries (found for the multipoint correlations in [4] and discussed in [5] for pdf's, see also Eqs. (5,6)) can also be derived for the Hopf functional. Mathematically, they follow from the linearity of the considered functional equation (14) and the normalisation condition (16), and have the folowing form [7]

$$\Phi^* = 1 + \mathrm{e}^{k_s}(\Phi - 1) \quad \text{and} \quad \Phi^* = \Phi + \Psi([\boldsymbol{y}(\boldsymbol{x})]), \tag{26}$$

where Ψ is a functional which satisfies Eq. (14) such that $\Psi(0) = 0$. Substituting (26) into (14) the invariance of (14) under these transformations is easily confirmed.

With the classical scaling (17) and the new symmetries (26) we can attempt to derive invariant solutions for the space-time Hopf functional. For this purpose we can use a procedure analogous to the one used in [8, 9]. We first present the functional in a form of Taylor-series expansion [2]

$$\Phi = 1 + C_1 + C_2 + \cdots \tag{27}$$

where

$$C_n = \int K_{i_{(1)}\dots i_{(n)}}(\boldsymbol{x}_{(1)},\dots,\boldsymbol{x}_{(n)},t_1,\dots,t_n)y_{i_{(1)}}(\boldsymbol{x}_{(1)},t_1)\cdots y_{i_{(n)}}(\boldsymbol{x}_{(n)},t_n)\mathrm{d}\boldsymbol{x}_{(1)}\cdots\mathrm{d}\boldsymbol{x}_{(n)}\mathrm{d}t_1\cdots\mathrm{d}t_n \tag{28}$$

with functions $K_{i_{(1)}\dots i_{(n)}}(\boldsymbol{x}_{(1)},\dots,\boldsymbol{x}_{(n)},t_1,\dots,t_n) = i^n/n!\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t_1)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t_n)\rangle$ which are related to the multipoint, multitime moments of velocity. The three symmetries, (17) and the new symmetries (26), written in the infinitesimal forms lead to the following characteristic system of equations

$$\frac{\mathrm{d}t}{2k_2t} = \frac{\mathrm{d}x_i}{k_2x_i} = \frac{\mathrm{d}(y_i\mathrm{d}\boldsymbol{x}\mathrm{d}t)}{k_2(y_i\mathrm{d}\boldsymbol{x}\mathrm{d}t)} = \frac{\mathrm{d}\Phi}{a_s(\Phi-1)+\Psi},\quad i = 1,2,3, \tag{29}$$

which should hold for each point $\boldsymbol{x}$ and each time t.

The purpose of the present study is to outline a procedure of deriving invariant solutions for the characteristic functional. Hence, one particular invariant solution will be derived, and discussed although it seems that many more possibilities exist. We will consider two different points $\boldsymbol{x}_{(1)}$ and $\boldsymbol{x}_{(2)}$ and two different times t_1 and t_2, and note that the characteristic system (29) can also be rewritten as

$$\frac{\mathrm{d}t_1}{\mathrm{d}s} = 2k_2t_1,\quad \frac{\mathrm{d}t_2}{\mathrm{d}s} = 2k_2t_2,\quad \frac{\mathrm{d}x_{(1)i}}{\mathrm{d}s} = k_2x_{(1)i},\quad i = 1,2,3,\quad \frac{\mathrm{d}\Phi}{\mathrm{d}s} = a_s(\Phi-1)+\Psi \tag{30}$$

$$\frac{\mathrm{d}(y_i(\boldsymbol{x}_{(1)},t_1)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1)}{\mathrm{d}s} = k_2y_i(\boldsymbol{x}_{(1)},t_1)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1,$$

$$\frac{\mathrm{d}(y_i(\boldsymbol{x}_{(2)},t_1)\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2)}{\mathrm{d}s} = k_2y_i(\boldsymbol{x}_{(2)},t_2)\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2.$$

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With further, mathematical transformations Eqs. (30) can be reformulated e.g. to the following form

$$\frac{\mathrm{d}t_1}{2k_2t_1} = \frac{\mathrm{d}t_2}{2k_2t_2} = \frac{\mathrm{d}(t_1-t_2)}{2k_2(t_1-t_2)} = \frac{\mathrm{d}(x_{(1)i}-x_{(2)i})}{k_2(x_{(1)i}-x_{(2)i})} = \frac{\mathrm{d}(y_j(\boldsymbol{x}_{(1)},t_1)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1)}{k_2(y_j(\boldsymbol{x}_{(1)},t_1)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1)} = \tag{31}$$
$$= \frac{\mathrm{d}(y_k(\boldsymbol{x}_{(1)},t_1)y_l(\boldsymbol{x}_{(2)},t_2)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2)}{2k_2(y_k(\boldsymbol{x}_{(1)},t_1)y_l(\boldsymbol{x}_{(2)},t_2)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2)} = \dots = \frac{\mathrm{d}\Phi}{a_s(\Phi-1)+\Psi}, \quad i,j,k,l=1,2,3$$

The first five integration constants, obtained from (31) are

$$a_1 = \frac{t_1}{t_2},\ a_2 = \frac{t_1-t_2}{t_1},\ a_3 = \frac{(x_{(1)1}-x_{(2)1})^2}{t_1-t_2},\ a_4 = \frac{(x_{(1)2}-x_{(2)2})^2}{t_1-t_2},\ a_5 = \frac{(x_{(1)3}-x_{(2)3})^2}{t_1-t_2}.$$

A solution, which follows from the comparison of the first and the sixth term in (31) is

$$A_6(\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)},t_1,t_2) = \frac{1}{t_1}F_{ij}(a_1,a_2,a_3,a_4,a_5)y_i(\boldsymbol{x}_{(1)},t_1)y_j(\boldsymbol{x}_{(2)},t_2)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2, \tag{32}$$

for each $\boldsymbol{x}_{(1)}$,$\boldsymbol{x}_{(2)}$,t_1,t_2. Integrating (32) we obtain another constant

$$C_2 = \iint \frac{1}{t_1}F_{ij}(a_1,a_2,a_3,a_4,a_5)y_i(\boldsymbol{x}_{(1)},t_1)y_j(\boldsymbol{x}_{(2)},t_2)\mathrm{d}\boldsymbol{x}_{(1)}\mathrm{d}t_1\mathrm{d}\boldsymbol{x}_{(2)}\mathrm{d}t_2. \tag{33}$$

We next consider the last term in (31) and assume $a_s = 0$ and $\Psi = 0$. We will present the solution Φ as a sum of constants $\Phi = 1 + C_1 + C_2 + \dots$, as in the Taylor-series expansion (27). For the homogeneous isotropic turbulence with zero mean velocity we have $C_1 = 0$, the constant C_2 is given in Eq. (33) and it could also be possible to derive other constants, i.e. C_3, C_4 by an analogous procedure, considering larger number of points $\boldsymbol{x}_{(1)}$, $\boldsymbol{x}_{(2)}$, $\boldsymbol{x}_{(3)}$, $\boldsymbol{x}_{(4)}$ etc. and larger number of times.

The second functional derivative of Φ at the origin can provide a relation for the decay law of the second-order turbulence statistics

$$\left.\frac{\delta^2\Phi}{\delta y_i(\boldsymbol{x}_{(1)},t_1)\delta y_j(\boldsymbol{x}_{(2)},t_2)}\right|_{\boldsymbol{y}=0} = i^2\langle U_i(\boldsymbol{x}_{(1)},t_1)U_j(\boldsymbol{x}_{(2)},t_2)\rangle = \frac{1}{t_1}F_{ij}(a_1,a_2,a_3,a_4,a_5). \tag{34}$$

For example, if $F_{ij} = const$ for a possible choice $\boldsymbol{x}_{(1)} = \boldsymbol{x}_{(2)}$ and $t_1 = t_2$, we obtain from (34) the power-law decay of the kinetic energy in the homogeneous, isotropic turbulence

$$\langle U_i(\boldsymbol{x}_{(1)},t_1)U_i(\boldsymbol{x}_{(1)},t_1)\rangle \sim \frac{1}{t_1}.$$

In the literature, the decay of kinetic energy $k \sim t^{-m}$ is reported [10]. Although there is a discussion on the value of m, several studies suggest that m approaches the value 1 as the initial Re number increases [11].

4. Conclusions

The present work concerns the Lie symmetry analysis of infinite systems describing the statistics of turbulence, the LMN hierarchy for pdf's and the Hopf functional equation. First, the symmetries of the LMN hierarchy, as derived in [5] were addressed, together with an invariant solution for the mean velocity in a laminar-turbulent channel flow. The new contribution of the present work is a discussion of the realizability of this solution. It was shown that the intermittency factor, following from the solution is correctly contained within the bounds $0 \leq \gamma \leq 1$.

Next, the Hopf functional in the spatio-temporal formulation introduced in Ref. [3] was discussed. It was shown that the evolution equation for the functional is invariant under the transformations which follow from the Navier-Stokes equations, in particular, the Galilean invariance was shown. Next, a possible procedure to derive invariant solutions for the Hopf functional was proposed, which is another new contribution of the present work. Studying other invariant solutions and their physical consequences is a perspective for a further study.

Another interesting issue is connected with the numerical study of the multipoint equations. A numerical approach for the multipoint pdf's has recently been devised in [12]. A new method of numerical approximation of nonlinear functionals and functional differential equations has been put forward in [13]. Both novel numerical approaches provide a new perspective for further, numerical study of the multipoint approaches to turbulence.

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