

Nonlinear dynamics of bimodality in vehicular traffic

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Abstract. We provide a global model for the bimodal distribution of a one-dimensional vehicular traffic flow. Our model captures the essential features of bimodality, namely, asymptotically decaying tails, asymmetry of the bimodal peaks, and their oscillatory exchange in a twenty-four cycle of traffic flows. We analyse our model from the perspective of nonlinear dynamics, and show that in a phase portrait, bimodality is implied by fixed points, closed loops, local periodicity and homoclinic paths. We also find the conditions for the asymmetry in a bimodal function, and for a bimodal-to-unimodal transition.

1. Introduction

Bimodal distributions have two distinct modes in a single frequency distribution [1]. In standard statistical treatments, such distributions are seen as a mixture of two normal distributions, but here we develop a mathematical model that gives a single global description of bimodal phenomena. Although our modelling work is founded on the bimodal distribution of data pertaining to the case of one-dimensional vehicular traffic flows, its implications go much beyond any specific context, so long as the basic principle of bimodality, observed in diverse phenomena (see [2, 3] and all references therein), is maintained. Studying the dynamics of a bimodal distribution, by applying standard methods of nonlinear dynamics to our model [4, 5], we see how the symmetry of a bimodal system can be broken and how the controlling parameters can be adjusted for a bimodal-to-unimodal transition (like a bifurcation)

2. Bimodality in traffic flows and its mathematical model

Studies of vehicular traffic flows aim to understand interactions among streams of vehicles, with the practical motive of eliminating the problem of traffic congestion and devising effective methods of traffic control. In our model, we take a macroscopic view of the one-dimensional vehicular traffic of a city at a given hour in the day — in effect, the bulk of traffic as a function of time. The data we need are taken from the repository of the Alabama Department of Transportation (ALDOT) ¹, pertaining to a city named Jackson in Alabama State, USA. This city is comparatively small in size, and so the data about its traffic, free of the complications associated with the traffic of very large cities, have been systematically recorded. The data we have chosen are the average values of all the Wednesday traffic data of Jackson city in the year, 2007. Wednesday is in the middle of the week, and it is in this period of the week that all city traffic rids itself of any weekend effect and reaches a local mid-week equilibrium. Taken together, these factors give us the advantage of simplicity in building our mathematical model.

¹ <http://aldotgis.dot.state.al.us/atd/default.aspx>



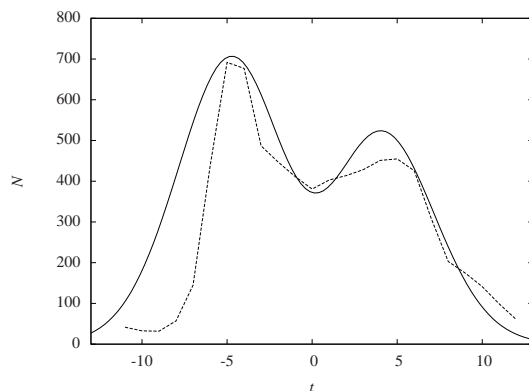


Figure 1. Bimodal distribution of traffic flow due west, plotting traffic volume, N , at time, t (hours). The time is scaled to set $t = 0$ at midday, 12:00 hours. The dotted broken curve joins the real data points, and the continuous curve traces the model given by Eq. (1), with $A = 44.0$, $\mu = 8.53$, $\lambda = 0.19$ and $\beta = -0.09$.

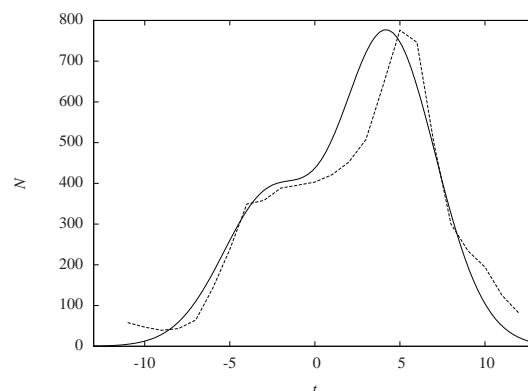


Figure 2. Bimodal distribution of traffic flow due east. The plotting follows the method used for the westward traffic, with $A = 44.1$, $\mu = 10.5$, $\lambda = 0.22$ and $\beta = 0.24$. The sign and the absolute magnitude of β make the most significant difference between the two bimodal distributions.

Vehicular traffic in Jackson city is bi-directional, i.e. its flows are only in the eastward and westward directions. The data for the westward traffic is plotted in Fig. 1, with the broken line. Similarly, the plot of the eastward traffic is shown in Fig. 2. In both these plots, the time, t , at which the traffic volume, $N(t)$, is given, is scaled to coincide $t = 0$ with midday, 12:00 hours. This allows us look for any possible symmetry in the distribution of the data, especially if there is a symmetry about $t = 0$. In both Figs. 1 & 2, however, only asymmetric bimodal distributions are seen, with two peaks of unequal height in either plot. Fig. 1 suggests that the morning traffic volume is much higher in the western direction than in the reverse direction, while the behaviour of the bimodal curve in Fig. 2 is the opposite, with the eastward traffic being disproportionately high in the evening. Evidently, there is a clear commuter preference in the traffic flow as a function of time, and the traffic volume oscillates back and forth between the eastern and western limits of the city, once in a cycle of twenty four hours.

We plot the continuous curves in Figs. 1 & 2 with the model function,

$$N(t) = A (\mu + t^2) \exp \left[-(\lambda t - \beta)^2 \right]. \quad (1)$$

The foregoing function captures the essential features of bimodality, the first of which is that there should be three turning points, with a local minimum flanked by two peaks. This is a fundamental property of any bimodal function, and in Eq. (1), this is controlled by the parameter, μ . Secondly, for $t \rightarrow \pm\infty$, there is an asymptotic decay of N , which is given by the Boltzmann tail of Eq. (1). Finally, there is a broken symmetry in the function about $t = 0$, in consequence of which, the two peaks of the bimodal distribution have uneven heights. In addition, we see that the uneven peaks also exchange their respective positions in Figs. 1 & 2. In Eq. (1) both the breaking of symmetry and the exchange of the peaks in the bimodal curve are controlled by the parameter, β . The former condition is given by the absolute magnitude of β , while the latter is given by its sign.

3. The dynamics of the bimodal model

The form of Eq. (1) is $N \equiv N(t)$, in which μ and β collectively play a crucial part to cause an asymmetric bimodality. This part can be known more clearly from the perspective of a dynamical system, and accordingly we cast Eq. (1) in a first-order non-autonomous form as $\dot{N} = f_1(t)N$, where $f_1(t) = 2[t(\mu + t^2)^{-1} - \lambda(\lambda t - \beta)]$ and the overdot on N implies a time derivative. Further, the second

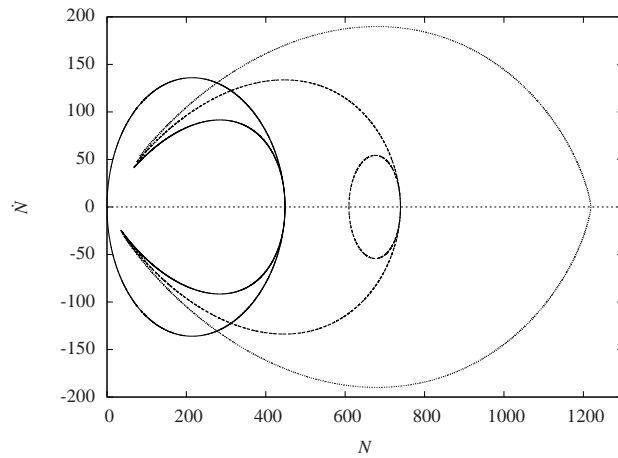


Figure 3. A symmetric phase portrait with $\beta = 0$. When $\mu = 0$, the phase trajectories are shown by the continuous curve at the extreme left. The closed loop is the largest in this case. When $\mu = 0.5\lambda^{-2}$, the loop size shrinks, as the dotted curve shows in the centre of the plot. When $\mu = \lambda^{-2}$, the loop vanishes as the curve on the extreme right shows. The closed loops are a signature of bimodality. When $\mu = \lambda^{-2}$, a bimodal-to-unimodal transition occurs, and the closed loop disappears, i.e. bimodality is lost. There is a fixed point at $N = \dot{N} = 0$, joined to itself by a homoclinic trajectory.

derivative of N gives a second-order non-autonomous equation as $\ddot{N} = f_2(t)N$, where $f_2(t)$ is another polynomial function of t . The turning points of N are obtained when $\dot{N} = 0$, and the fixed points in the phase portrait of $N-\dot{N}$ are obtained when both $\ddot{N} = \dot{N} = 0$. The simplest way to understand these features is to consider the symmetric limit of $\beta = 0$ and to solve for $\dot{N} = 0$. This delivers three roots of t at $t = 0$ and $t = \pm(\lambda^{-2} - \mu)^{1/2}$. It is easy to show that none of these roots satisfies the condition for $\ddot{N} = 0$, and so we conclude that there is only one fixed point in the bimodal system at $N = N^* = 0$. Any other solution of $\dot{N} = 0$ gives only turning points. At $t = 0$ a turning point of N is given by $N_0 = A\mu$, as Eq. (1) shows, while the two non-zero roots of t give two coinciding turning points, $N_T = A\lambda^{-2} \exp(\mu\lambda^2 - 1)$. All three roots of t and their corresponding turning points are real under the condition, $\mu < \lambda^{-2}$, by means of which, we can also argue that $N_T > N_0$. As $\mu \rightarrow \lambda^{-2}$, the positions of the turning points approach each other, and when $\mu = \lambda^{-2}$, there is a bifurcation-like behaviour, due to which, the turning point, N_0 , and one of the turning points, N_T , annihilate each other. What survives this mutual annihilation is the second “concealed” turning point, given also by N_T . Subsequently, for $\mu > \lambda^{-2}$, only this turning point continues its existence, and the distribution becomes unimodal.

This behaviour of the turning points over the range, $0 < \mu < \lambda^{-2}$, is demonstrated in Fig. 3. We see clearly from this plot that bimodality is closely related to the formation of a closed loop by a phase trajectory in the $N-\dot{N}$ portrait. When $\mu \rightarrow 0$, bimodality is very pronounced, and when $\mu \rightarrow \lambda^{-2}$, bimodality becomes progressively enfeebled, disappearing altogether when $\mu = \lambda^{-2}$. In the former case, the sole fixed point at $N = N^* = 0$ coincides with the turning point, $N_0 = 0$, while in the latter case, bimodality is eradicated because of a mutual annihilation between the turning points, $N_0 (\neq 0)$ and N_T . Once bimodality disappears and the distribution becomes unimodal, the phase-portrait also shows the existence of a homoclinic trajectory, that joins the fixed point, $N^* = 0$, to itself. Further, the closed loop in the phase portrait indicates a locally periodic behaviour (implying that the two peaks in the bimodal distribution give a local scale), and when $\mu < \lambda^{-2}$, the time scale of this local periodicity is λ^{-1} .

We now look into the breaking of symmetry, when $\beta \neq 0$. What happens in this case is that the two coinciding turning points at $N = N_T$ are separated from each other. If $|\beta| \ll 1$, then by perturbing about the turning point N_T (determined for $t^2 = \lambda^{-2} - \mu$), we linearise all terms involving β in Eq. (1), and obtain the two separated roots as $N_{T\pm} \simeq N_T[1 \pm 2\beta(1 - \mu\lambda^2)^{1/2}]$. As β increases in magnitude, the position of N_{T+} shifts to higher values of N on $\dot{N} = 0$, while the position of N_{T-} slides in the opposite direction. This pair of roots corresponds to the two asymmetric peaks in the bimodal curve, with N_{T+} tracing the locus of the higher peak, and N_{T-} the lower. Between them, these peaks flank a local minimum in the bimodal distribution, which itself turns out to be the third turning point on $\dot{N} = 0$. When $\beta = 0$, this point is at $N = N_0$, with $N_0 < N_T$, but when $\beta \neq 0$, the position of this third unpaired turning point shifts to larger values of N , and for sufficiently high values of β , it suffers a separate pair

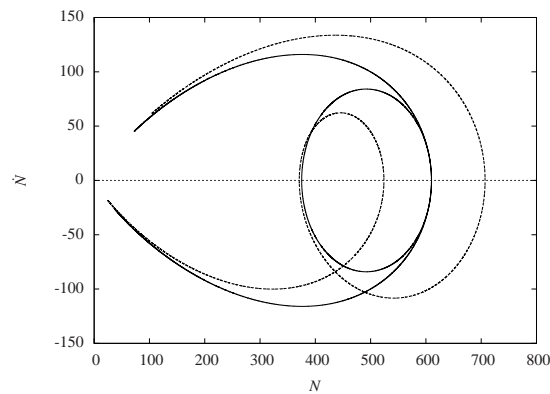


Figure 4. Phase plot of the bimodal distribution of the traffic flow due west. When $\beta = 0$, the resulting plot is shown by the continuous symmetric curve. When $\beta = -0.09$, asymmetry sets in, as shown by the dotted curve, in which a closed loop implies that bimodality prevails nonetheless.

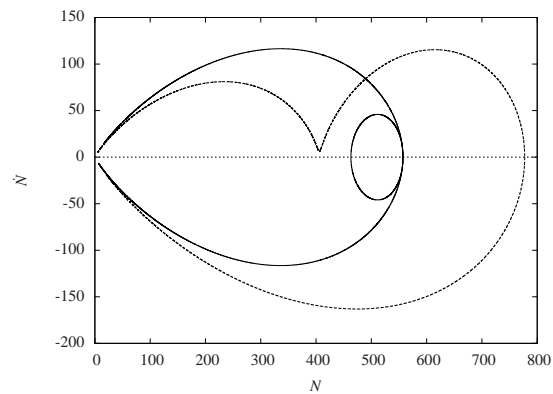


Figure 5. Phase plot of the bimodal distribution of the traffic flow due east. The continuous curve is for $\beta = 0$, while the asymmetric dotted curve is for $\beta = 0.24 (> 0.21)$. A bimodal-to-unimodal transition has occurred, and instead of a closed loop on the dotted curve, there is a cusp just above $\dot{N} = 0$.

annihilation with the shifting turning point at N_{T-} . We support this reasoning by solving for $f_1(t) = 0$, and getting a third-order polynomial in t , which is $t^3 - (\beta/\lambda)t^2 - (\lambda^{-2} - \mu)t - (\mu\beta/\lambda) = 0$, whose analytical roots are extracted by the application of the Cardano-Tartaglia-del Ferro method. Transforming $t = \tau + h$ with $h = \beta(3\lambda)^{-1}$, gives the canonical form of a cubic equation, $\tau^3 + P\tau + Q = 0$, in which P and Q depend on β . The discriminant of this cubic equation also depends on β , and is given as $D = (Q^2/4) + (P^3/27)$. If $D > 0$, τ has a single root, giving a unimodal condition. If, however, $D < 0$, then τ has three real roots, providing the bimodal condition. Knowing τ , leads to t , which, by Eq. (1), gives the value of N , as a turning point on $\dot{N} = 0$.

The critical condition for a bimodal-to-unimodal transition is $D = 0$, for $\beta = \beta_b$. With the values of μ and λ used to calibrate the data of both the east and west-moving traffic, we find that $|\beta_b| \simeq 0.21$ for the eastward traffic, and $|\beta_b| \simeq 0.46$ for the westward traffic. In Fig. 1, the value of $|\beta| = 0.09$ is much less than $|\beta_b| \simeq 0.46$, while in Fig. 2, the reverse is true ($|\beta_b| \simeq 0.21 < |\beta| = 0.24$). Therefore, as opposed to Fig. 1, there is no conspicuous secondary peak in the bimodal distribution of Fig. 2. The implications of both these conditions have been depicted in Figs. 4 & 5. These plots show the dynamics of bimodality for the west-moving and the east-moving traffic, respectively. In both cases, when $\beta = 0$, there is a point on $\dot{N} = 0$, where the curve loops over itself. In either case, when $\beta \neq 0$, this point shifts away from the line of $\dot{N} = 0$, suggesting asymmetric bimodality. Stated concisely, μ causes bimodality, and β causes a breaking of symmetry. Taken together, the result is an asymmetric bimodality. Changing the values of both these parameters can cause a bimodal-to-unimodal transition, like a bifurcation.

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