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Simulating the nonlinear wave mode evolution in the problem of the falling thin layer of a viscous liquid

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Abstract. The work is devoted to numerical simulation of the dynamics of nonlinear periodic waves on the surface of freely flowing liquid film realized with the use of the previously discovered symmetry of model equations. The calculation results have been presented for the dynamics of amplitudes of significant harmonics and for the wave surface evolution. They prove to be in good agreement with previously calculated steady-state traveling solutions obtained on the full and the reduced bases. In addition, the work has identified intervals of wavenumbers in which the evolution of perturbations with small initial amplitude leads to a wave signal periodically changing in time rather than to the steady-state traveling solution.

1. Introduction

The flow of a thin layer of viscous liquid (film) over an inclined solid plane has been studied for over a hundred years [1]. Until the middle of the last century, the interest in the falling films was almost exclusively utilitarian. In 1941, Friedman and Miller [2] noted: "In the general field of diffusional processes, including absorption, extraction, heat transfer, humidification, and distillation, the flow of thin liquid films is often encountered". In accordance with practical tasks, research works were mainly experimental in nature and were focused on a turbulent flow regime. Precision measurements of the fluid velocity on the surface of very thin laminar liquid layers freely flowing over a vertical tube [2] have revealed its significant exceedence over the velocity calculated with the trivial Nusselt solution for a smooth film. A qualitative explanation of this consisted in the necessity to take into account the wave formation on the liquid surface.

The impetus to a theoretical study of wave flows of liquid films was given in a series of brilliant works of Kapitza [3]. Notable photos of the film surface showed the existence of many regular wave flow regimes that were much more stable than the plane-parallel Nusselt flow. The sustained efforts of subsequent researchers to describe such regimes have been rewarded by the profound results that reveal the general nature of the hydrodynamic instability and have direct application in various fields of physics of continuous media. It is suffice to mention the widely known equation of Kuramoto-Sivashinskii, derived for the first time to simulate the very wave evolution on the liquid film surface [4]. The interaction of effects of long-wave pumping, short-wave dissipation and perturbation nonlinearity (each of them is expressed by only one term of the K-S equation) leads to an unusually rich picture of steady-state traveling wave modes, formed due to the instability of the trivial solution, as well as to the extremely complex dynamics of perturbations, demonstrating main features of the spatial-temporal deterministic chaos.

The full formulation of the problem of the wavy film flow includes the Navier – Stokes and the continuity equations with respective kinematic and dynamic boundary conditions. The major problem is the uncertainty of the moving boundary position, which is determined in the course of solving. If the effects of droplet entrainment and solid surface drying are excluded from consideration, the fluid flow area turns out to be simply connected. The presence of surface tension ensures no sharp edges on the film surface. Under these conditions, the function determining the surface position is often unambiguous. Then, there exists a continuously differentiable coordinate transformation, mapping the area of the fluid flow on the band of constant thickness:

$$x = x, \quad \eta = y/h(x,t), \quad t = t. \tag{1}$$

New variables (1) are not orthogonal, so the conventional formulation of the motion equations in vector form is not applicable. The transformation (1) is realized with the use of new variables in the equations, written in tensor form invariant to the coordinate systems. With the use of such approach for the film, freely falling over a vertical plane, in [5] the following system was obtained in the long-wave approximation:

$$\begin{gathered}\frac{\partial(hu)}{\partial t} + \frac{\partial(hu^2)}{\partial x} + \frac{\partial(huv)}{\partial \eta} = \frac{\sigma}{\rho}\, h\, \frac{\partial^3 h}{\partial x^3} + \frac{\mu}{\rho h}\frac{\partial^2 u}{\partial \eta^2} + gh,\\ \frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial \eta} = 0.\end{gathered} \tag{2}$$

Here, h is the film thickness, u and v are the contravariant components of the longitudinal and transversal velocities, respectively, ρ is the density, μ is the dynamic viscosity, and σ is the surface tension of the liquid.

The boundary conditions of liquid no-slip and impermeability on the surface $\eta = 0$ are:

$$u(x,0,t) = 0, \;\; v(x,0,t) = 0. \tag{3}$$

On the surface $\eta = 1$, the conditions of impermeability and shear stress absence take the form:

$$v(x,1,t) = 0, \;\; \frac{\partial u}{\partial \eta}(x,1,t) = 0. \tag{4}$$

Let's perform a shifting transformation along the transverse coordinate in the considered problem:

$$\eta' = \eta - 1. \tag{5}$$

Now the flow region takes the band $\eta' \in [-1, 0]$. It is easy to notice that the equations (2) are invariant under the transformation:

$$\begin{gathered}\eta' \to -\eta',\\ u(x,\eta',t) \to u(x,-\eta',t),\\ v(x,\eta',t) \to -v(x,-\eta',t).\end{gathered} \tag{6}$$

2. Calculation method

For a numerical study of the problem, the equations (2) and the boundary conditions are written in the following dimensionless form [6]:

$$\begin{gathered}
\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{h}\right)+\frac{\partial}{\partial \eta'}\left(\frac{QV}{h}\right)=\frac{1}{\varepsilon \mathrm{Re} h^2}\frac{\partial^2 Q}{\partial \eta'^2}+\frac{3h}{\varepsilon \mathrm{Re}}+\frac{18}{5}\varepsilon \mathrm{Re} h\frac{\partial^3 h}{\partial x^3},\\
\frac{\partial h}{\partial t}+\frac{\partial Q}{\partial x}+\frac{\partial V}{\partial \eta'}=0,\\
Q=V=0, \quad \text{at} \quad \eta'=-1,\\
\frac{\partial Q}{\partial \eta'}=0, \quad V=0, \quad \text{at} \quad \eta'=0.
\end{gathered} \tag{7}$$

In the system (7), there are new functions $Q = hu$, $V = hv$ relative to which the second equation is linear. Here $\mathrm{Re} = \mathrm{g}h_0^3/3\nu^2$ is the Reynolds number, and $\varepsilon = h_0/l_0$ is the wavelength parameter. As characteristic scales the following are selected: the average thickness h_0 and the superficial velocity of a waveless Nusselt flow $u_0 = gh_0^2/3\nu$. The scales of the length l_0 and time l_0/u_0 are determined from the relationship:

$$\frac{h_0}{l_0}=\varepsilon=\sqrt{\frac{18}{5}\frac{\mathrm{Re}^{5/3}}{3^{1/3}\mathrm{Fi}^{1/3}}},$$

where $\mathrm{Fi} = \sigma^3/\rho^3 \mathrm{g}\nu^4$ is the film number, characterizing liquid properties. As it is shown in [6], this choice provides the value of neutral wave number $\alpha_n \approx 1$.

The problem was solved by the pseudospectral (collocation) method. Functions depending on transverse coordinate η' were expanded in a series on Chebyshev polynomials T_i. The work [6] showed that at moderate values of the Reynolds numbers the steady-state traveling solutions of the system (7) in the area extended over the transverse coordinate $\eta' \in [-1, 1]$ are characterized by the following symmetry, connected with transformation (6):

$$u(x,\eta',t)=u(x,-\eta',t),\ v(x,\eta',t)=-v(x,-\eta',t). \tag{8}$$

This allows choosing only even Chebyshev polynomials for the function Q, and only odd ones for the function V:

$$\begin{gathered}
Q(x,\eta,t)=\sum_{i=1}^{M}Q_{2i}\left(x,t\right)\left(T_{2i}\left(\eta'\right)-1\right),\\
V(x,\eta,t)=\sum_{i=1}^{M}V_{2i+1}\left(x,t\right)\left(T_{2i+1}\left(\eta'\right)-\eta'\right).
\end{gathered} \tag{9}$$

Here, M is a number of Chebyshev nodes.

This choice provides an automatic satisfaction of boundary conditions on the surface and on a solid wall. So, in a number of Chebyshev nodes:

$$\eta'_j=-\cos\left(\frac{j\pi}{2(M+1)}\right),\ j=1,...,M,$$

we obtain the set of equations:

$$\begin{aligned}
&\sum_{i=1}^{M} \frac{\partial Q_{2i}}{\partial t} \left(T_{2i}(\eta'_j) - 1\right) = \frac{1}{\varepsilon \mathrm{Re} h^2} \sum_{i=1}^{M} Q_{2i} \frac{d^2 T_{2i}}{d\eta'^2}(\eta'_j) + \frac{18}{5} \varepsilon \mathrm{Re} h h_{xxx} \\
&+ \frac{3}{\varepsilon \mathrm{Re}} h - \frac{\partial}{\partial x} \left(\frac{\sum_{i=1}^{M} Q_{2i} \left(T_{2i}(\eta'_j) - 1\right) \sum_{i=1}^{M} Q_{2i} \left(T_{2i}(\eta'_j) - 1\right)}{h} \right) \\
&- \frac{1}{h} \frac{\partial}{\partial \eta'} \left(\sum_{i=1}^{M} Q_{2i} \left(T_{2i}(\eta'_j) - 1\right) \sum_{i=1}^{M} V_{2i+1} \left(T_{2i+1}(\eta'_j) - \eta'_j\right) \right),
\end{aligned} \tag{10}$$

$$\frac{\partial h}{\partial t} = - \frac{\partial q}{\partial x}.$$

Here $q(x,t) = \int_{-1}^{0} Q(x,\eta,t) d\eta'$ is a flow rate.

Functions that depend on the longitudinal coordinate were expanded into spatial Fourier series:

$$\begin{aligned}
Q_{2i}(x,t) &= \sum_{k=-\infty}^{\infty} Q_{2i}^{k}(t)\ e^{I\alpha k x}, \\
V_{2i+1}(x,t) &= \sum_{k=-\infty}^{\infty} V_{2i+1}^{k}(t)\ e^{I\alpha k x}, \\
h(x,t) &= \sum_{k=-\infty}^{\infty} H^{k}(t)\ e^{I\alpha k x}.
\end{aligned} \tag{11}$$

Here α is the wave number of the periodic solution, I is the imaginary unit. The functions are real-valued, so $Q_{2i}^{-k} = \overline{Q}_{2i}^{k}, \quad V_{2i+1}^{-k} = \overline{V}_{2i+1}^{k}, \quad H^{-k} = \overline{H}^{k}$. The overline denotes the operation of complex conjugation.

Substituting representations (11) in (10) and limiting to finite numbers of harmonics we obtain the system of differential equations which was solved by the Runge-Kutta method of the 4-th order.

3. The calculation results

The evolution of periodic waves on freely falling thin layer of water ($\mathrm{Fi} = 4 \cdot 10^{10}$) for the Reynolds number $\mathrm{Re} = 20$ has been simulated for various values of the dimensionless wave number α. Plane-parallel Nusselt flow with harmonic perturbation of free surface $|H^1| = 0.45$ was selected as initial condition. Typical scenarios of the evolution of perturbations with small initial amplitude are presented in figures from 1 to 4. Figure 1 presents the evolution of modulus of amplitudes of the first three harmonics at the value $\alpha = 0.51$. It is seen that after the transient stage the amplitude of harmonics tends to some stationary value. At that forms the steady-state travelling wave, which profile is shown in figure 2. The wave profile and velocity fully agree with the calculations of steady-state travelling solutions in [6].

A different scenario is presented in figure 3 at the value $\alpha = 0.3$. It is seen that the modulus of harmonics' amplitudes changes periodically. Figure 4 shows the wave profile at time $t = 200$.

4. Conclusion

The numerical simulation of the freely-falling liquid film has been realized using the previously discovered symmetry of model equations. The calculation results have been presented for the

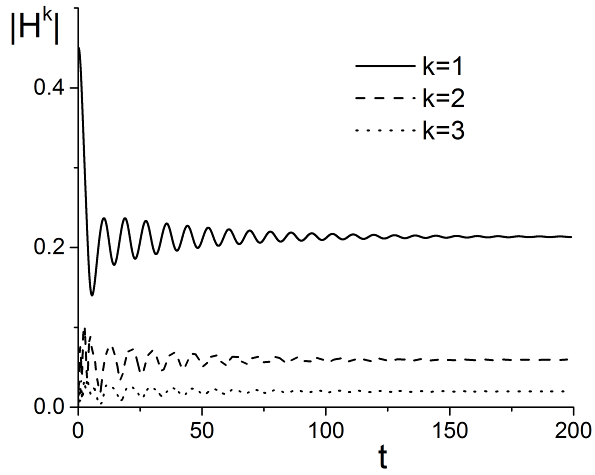

Figure 1. Evolution of the modulus of harmonics' amplitudes at $\alpha = 0.51$ (k is the number of harmonic).

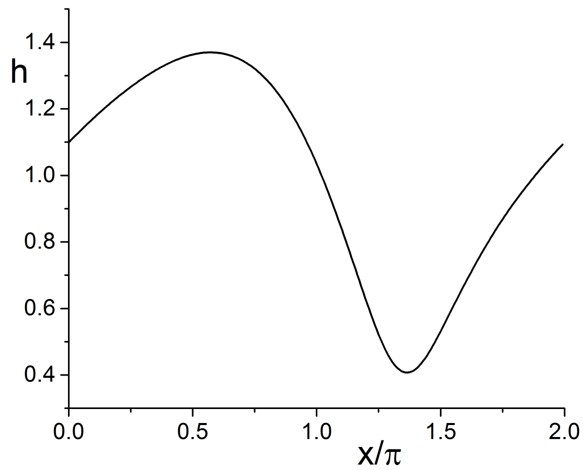

Figure 2. The steady state wave profile ($t = 200$, $\alpha = 0.51$).

dynamics of amplitudes of significant harmonics and for the wave surface evolution. They prove to be in good agreement with previously calculated steady-state traveling solutions obtained on the full and the reduced bases. In addition, the work has identified intervals of wavenumbers in which the evolution of perturbations with a small initial amplitude leads to a wave signal periodically changing in time rather than to the steady-state traveling solution.

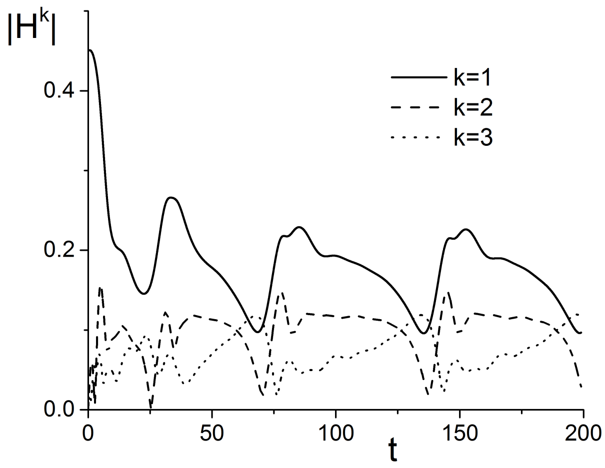

Figure 3. Evolution of the modulus of harmonics' amplitudes at $\alpha = 0.3$ (k is the number of harmonic).

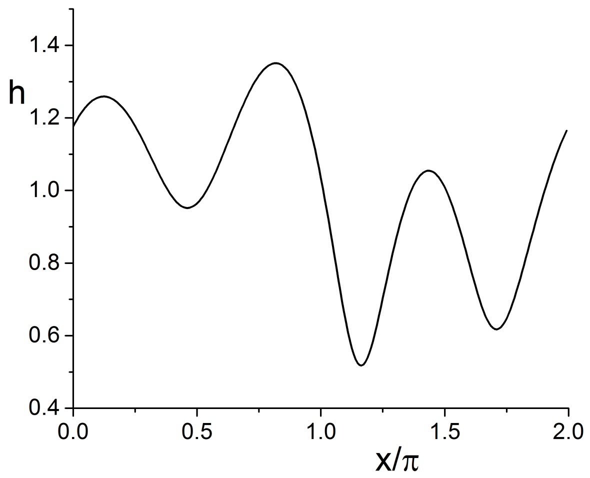

Figure 4. The wave profile ($t = 200$, $\alpha = 0.3$).

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