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Simulating nonlinear steady-state traveling waves on the falling liquid film entrained by a gas flow

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Abstract. The article is devoted to the simulation of nonlinear waves on a liquid film flowing under gravity in the known stress field at the interface. In the case of small Reynolds numbers the problem is reduced to the consideration of solutions of the nonlinear integral-differential equation for film thickness deviation from the undisturbed level. Weakly nonlinear steady-state traveling solutions of the equation with wave numbers in a vicinity of neutral wave numbers are constructed analytically. The nature of the wave branching from the undisturbed solution is investigated. Steady-state traveling solutions, whose wave numbers within the instability area are far from neutral wave numbers, are found numerically.

1. Introduction and problem statement

Simultaneous liquid and gas flow is a classical problem of hydrodynamics. The solution to this problem in a full statement is associated with considerable computational difficulties. So, two stages of simulation are often distinguished: determining gas stresses on the film surface and further calculating the evolution of waves in the liquid. The liquid velocity is much smaller than the characteristic gas velocity, so the boundary surface is supposed to be rigid and stationary. In addition, due to the smallness of film thickness, the influence of interfacial perturbations on gas velocity may be considered as linear. As a result, the problem of computing the normal and shear stresses of gas on the surface is reduced to considering the impact of individual spatial harmonics. At the second stage of simultaneous flow study, the dynamics of nonlinear waves on the liquid film surface is examined.

This paper deals with the second stage of studies, namely simulating the dynamics of nonlinear waves on a liquid film, flowing under the action of gravity in the known stress field at the interface. The gas flow is turbulent and occurs in a vertical channel. Full statement of the problem for liquid includes Navier–Stokes and continuity equations with respective kinematic and dynamic boundary conditions.

If function $y = h(x,t)$ determining the position of the area boundary is single-valued there is a continuously differentiable coordinate transformation that transforms the area of the fluid flow into the band of constant thickness.

$$x = x, \qquad \eta = y / h(x,t), \qquad t = t. \tag{1.1}$$

In the paper [1] the system of hydrodynamic equations was presented in tensor form invariant under coordinate systems. Its use enabled obtaining a model system of equations for the considered flow to describe the evolution of long-wave perturbations of the interface at moderate Reynolds numbers.

In the case of low flow rates (Re ~ 1), due to the film thickness smallness compared to the wavelength, the solution of the system is represented as a series on small parameter. As a result, restricting the expansion to the first two terms, one evolutionary equation for the film thickness h was obtained:

$$h_t + \frac{\mathrm{Re}}{\mathrm{Fr}} h^2 h_x + \mathrm{Re}\,\tau_0 h h_x$$

$$+\varepsilon\frac{\partial}{\partial x}\left(\frac{1}{3}\varepsilon\mathrm{ReW}h^3h_{xxx}+\frac{2}{15}\frac{\mathrm{Re}^3}{\mathrm{Fr}^2}h^5h_x(h+\tau_0\mathrm{Fr})+\frac{1}{2}\mathrm{Re}h^2\tau_0\int\hat{h}_k k\tau(k)e^{ikx}\mathrm{d}k\right)=0. \qquad (1.2)$$

Here dimensionless parameters are introduced: Reynolds number $\mathrm{Re}=\rho h_0u_0/\mu$, Froude number $\mathrm{Fr}=u_0^2/gh_0$, parameter $\mathrm{W}=\sigma/\rho l_0u_0^2$, and $\varepsilon=h_0/l_0$ is relation of specific film thickness h_0 to specific wave length l_0. Direction of coordinate x coincides with the direction of gravity vector. In addition, in the equation (1.2) and in the dimensionless complexes we use the characteristic scales of velocity u_0 and time l_0/u_0. Here p is the pressure, σ is the surface tension, ρ is the density, μ is the dynamic viscosity of liquid, g is the acceleration of gravity, τ_0 is undisturbed component of gas shear stress on the film surface, $\tau(k)=\tau_r(k)+i\tau_{im}(k)$ are the Fourier components of tangential stresses of gas due to the boundary curvilinearity, and $\hat{h}(k,t)$ are the Fourier components of the surface form expansion:

$$\hat{h}(k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}h(x,t)e^{-ikx}\mathrm{d}x.$$

We should emphasize that the approximation of small Reynolds numbers (Re ~ 1) is used to derive the equation (1.2), and it is assumed that the Weber number is large: $\mathrm{W}\varepsilon\sim1$. In the case of spatial periodic solutions of equation (1.2), the integral term is replaced by the respective Fourier series.

Restricting ourselves to perturbations of small but finite amplitude, introducing slow and fast times into consideration and using transformation $h=1+\varepsilon h_1,\quad t_0=t,\quad t_1=\varepsilon t$, from equation (1.2) we obtain:

$$\frac{\partial h_1}{\partial t_0}+\frac{\mathrm{Re}}{\mathrm{Fr}}\left(1+\mathrm{Fr}\,\tau_0\right)\frac{\partial h_1}{\partial x}=0, \qquad (1.3)$$

$$\begin{aligned}&\frac{\partial h_1}{\partial t_1}+\frac{\mathrm{Re}}{\mathrm{Fr}}\left(2+\mathrm{Fr}\,\tau_0\right)h_1\frac{\partial h_1}{\partial x}+\frac{\mathrm{W\,Re}\,\varepsilon}{3}\frac{\partial^4h_1}{\partial x^4}\\&+\frac{2}{15}\frac{\mathrm{Re}^3}{\mathrm{Fr}^2}\frac{\partial^2h_1}{\partial x^2}(1+\tau_0\mathrm{Fr})+\frac{1}{2}\mathrm{Re}\,\tau_0\int i\hat{h}_{1k}k^2\tau(k)e^{ikx}\mathrm{d}k=0.\end{aligned} \qquad (1.4)$$

Equation (1.3) implies that in the first approximation (fast times), perturbations of small but finite amplitude propagate with a characteristic constant velocity $c_0=\mathrm{Re}/\mathrm{Fr}\left(1+\mathrm{Fr}\,\tau_0\right)$. In this approximation, the motion occurs without changing the initial form of perturbations.

Equation (1.4) describes the nonlinear evolution of perturbations on large (slow) times. The characteristic longitudinal scale l_0 is defined so that the coefficients for the second and fourth derivatives in equation (1.4) are the same. This implies that the ratio for ε will take the form: $\varepsilon=2\mathrm{Re}^2/5\mathrm{W}\,\mathrm{Fr}^2(1+\mathrm{Fr}\,\tau_0)$. Considering this choice after replacement $t=bt_1,\quad h_1=AH,$ $b=\mathrm{WRe}\varepsilon/3,\ A=2\mathrm{Fr}b/\left(\mathrm{Re}(2-\mathrm{Fr}\tau_0)\right)$, the equation (1.4) will be rewritten in the form:

$$\frac{\partial H}{\partial t}+2H\frac{\partial H}{\partial x}+\frac{\partial^2H}{\partial x^2}+\frac{\partial^4H}{\partial x^4}+B\int_{-\infty}^{\infty}ik^2\,\tau(k)\,\hat{H}(k,t)e^{ikx}\mathrm{d}k=0. \qquad (1.5)$$

Here $B=\dfrac{\mathrm{Re}\,\tau_0}{2b}\equiv\dfrac{3\tau_0}{2\,\mathrm{W}\,\varepsilon}$.

Thus, in the case of small Reynolds numbers, the problem of studying perturbations on the surface of the liquid film, flowing under gravity in the known stress field at the interface, is reduced to the analysis of one nonlinear integral-differential equation.

Equation (1.5) is an interesting example of model equations arising in the study of the evolution of perturbations in active-dissipative media. The instability of linear perturbations is determined by its terms with the second derivative and the term containing the integral (conditioned by stress perturbation at the interface of film–gas). Dissipation is provided by the fourth derivative, simulating capillary effects. Indeed, neglecting a nonlinear term in (1.5) and presenting its solution as $H \sim \exp(ik(x-ct)),$ we obtain the following relation for dispersion:

$$c \equiv c_r + ic_i = i(k-k^3) + Bk\tau(k). \qquad (1.6)$$

Disturbances will be unstable if the imaginary part of the phase velocity c_i exceeds zero.

As the second term in the right part of (1.6), responsible for the perturbation stability, decreases faster than, for example, the first one, then the unstable are the long-wave perturbations. Their wave numbers are smaller than the neutral wave number k_n satisfying the equation:

$$1-k_n^2 + B\tau_{im}(k_n) = 0. \qquad (1.7)$$

As it can be seen from (1.7), the above described choice of the characteristic longitudinal scale l_0 implies that the neutral wave number $k_n = 1$ in the case of the freely falling film (*B* = *0*). We will choose the parameters of the undisturbed flow, so that the neutral wave number k_n different from the unit. At the same time demand that this value k_n corresponds to a definite value τ_{im}. The value *B* corresponding to this situation is determined from (1.7):

$$B = \frac{k_n^2 - 1}{\tau_{im}(k_n)}.$$

2. Results of nonlinear wave calculation on the model equation

For finding solutions of equation (1.5) periodic on *x* the function *H* is presented as a spatial Fourier series:

$$H(x,t) = \sum_n H_n(t)\exp(iknx). \qquad (2.1)$$

Since *H* is real function, then $\bar{H}_{-n} = H_n$. The bar indicates the operation of complex conjugation. Steady-state traveling solutions, whose wave numbers within the instability area are far from neutral wave numbers, are found numerically.

Having substituted (2.1) into equation (1.5) we obtain an infinite system of ordinary differential equations for Fourier harmonics $H_n(t)$. Supposing that all $H_n(t)$ with indices $|n| \geq N$ are equal to zero, obtain its finite-dimensional counterpart.

At the points with neutral wave numbers k_n the steady-state traveling linear solutions bifurcate from the trivial solution $H = 0$. As it is seen from (1.6), they have the following phase velocity and frequency, respectively:

$$c_0 = Bk_n\tau_r(k_n), \qquad \omega_0 \equiv k_n c_0 = Bk_n^2\tau_r(k_n).$$

In the vicinity of the neutral wave number k_n, the solutions are represented in the form of a series (2.1). At the same time, we believe that

$$H_n = \delta^{|n|}\tilde{H}_n,$$
$$k = k_n + \delta^2 k_2 + \cdots \qquad (2.2)$$

and introduce the set of different times:

$$t_n = \delta^n t, \qquad n = 0, 1, 2, \cdots. \qquad (2.3)$$

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Here δ is a small parameter.

Using the representations (2.2), (2.3) and substituting the series (2.1) into the equation (1.5), we collect the coefficients of the same degrees of parameter δ and equate them to zero. As a result, after a rather simple but cumbersome calculations we obtain:

$$H = A\exp[i(kx-\omega t)] + A^2 A_{H2}\exp[2i(kx-\omega t)] + C.C. \tag{2.4}$$

Here $A=\delta\left|\tilde{H}_1\right|$, $k = k_n + A_k A^2$, $\omega = \omega_0 + A_\omega A^2$, $C.C.$ is the complex-conjugate expression.

From (2.4) it is clear that in the solution it is possible to take δ equal to 1, and to use the amplitude of the first harmonic A as the small parameter. Coefficients A_{H2}, A_k, A_ω depend only on the values $\tau(k_n)$, $d\tau(k)/dk|_{k_n}$, $\tau(2k_n)$. Due to the bulkiness their explicit forms are not presented here. The phase velocity with an accuracy of A^2 is:

$$c \equiv \frac{\omega}{k} = \frac{\omega_0 + A_\omega A^2}{k_n + A_k A^2} = c_0 + \frac{A^2}{k_n}(A_\omega - c_0 A_k).$$

Calculating the below presented results, we used data on stress pulsations obtained by the model of Benjamin in the work [1].

Figures 1and 2 show the dependences of the coefficients A_k, A_ω on the parameter B for $k_n = 1.1$ and $k_n = 1.5$, respectively. As can be seen from these figures, the type of branching here is soft, i.e. the correction to the wave number is negative. In other words, the wave numbers of the steady-state traveling regimes of small but finite amplitude are in the range of linear instability.

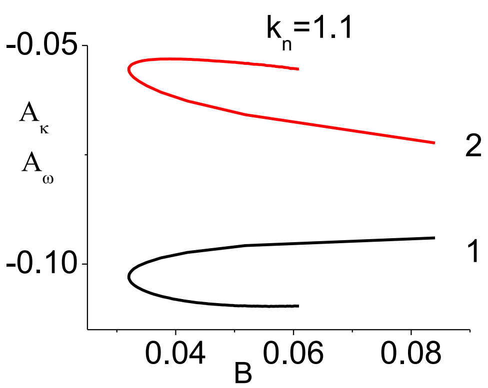

Figure 1. Dependencies of the coefficients A_k (curve 1) and A_ω (curve 2) on the parameter *B*. Neutral wave number $k_n = 1.1$.

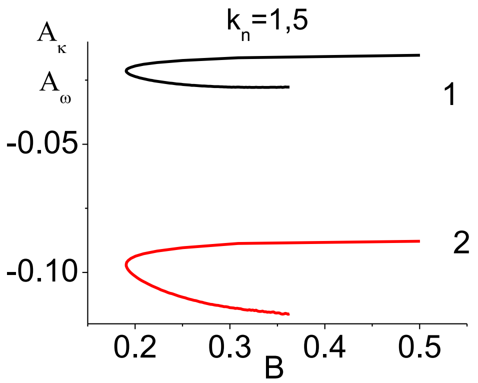

Figure 2. Dependencies of the coefficients A_k (curve 1) and A_ω (curve 2) on the parameter *B*. Neutral wave number $k_n = 1.5$.

To obtain the steady-state traveling solutions of this family (the first family) with wave numbers lying beyond the vicinity of the neutral wave number k_n, the problem was solved numerically. At truncating series (2.1) for values of N from 8 to 15. The evolution of the periodic perturbations was considered. The calculations show that in the region of wave numbers, where there is only the first family of steady-state traveling solutions, all initial perturbations evolve to the solution of this family.

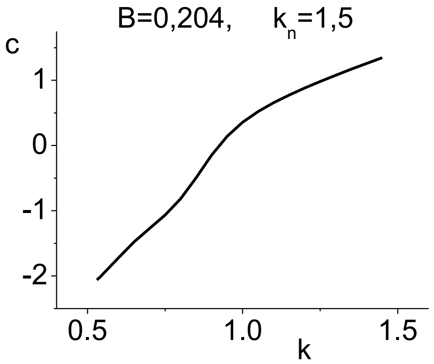

Figure 3. Dependence of the velocity of steady-state traveling solutions of the first family on the wave numbers *k*.

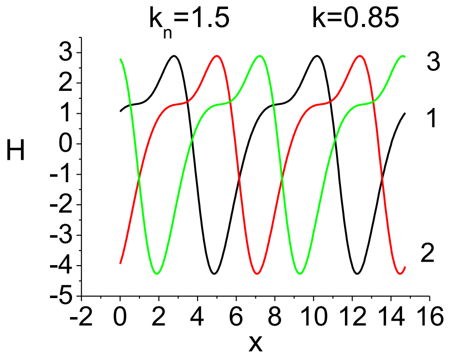

Figure 4. Surface profiles for three different time moments: t = 100 (*1*), 200 (2), 300 (3).

Figure 3 shows the dependence of the velocity of steady-state traveling solutions of the first family obtained at the parameter value $B = 0.204$. In this case, the neutral wave number $k_n = 1.5$. As an example, figure 4 shows the solution of this family with wave number $k = 0.85$. Here wave profiles are shown for three different time points. Two wave lengths $2\lambda\,(\lambda = 2\pi / k)$ are plotted along the abscissa axis. The wave velocity c is equal to -0.495.

3. Conclusion

Thus, nonlinear waves on the film of liquid flowing under gravity in the known stress field at the interface have been considered. In the case of small Reynolds numbers the problem was reduced to considering solutions of the nonlinear integral-differential equation for film thickness deviation from the undisturbed level. In this paper the nature of the wave branching from the undisturbed solution was investigated. The weakly nonlinear steady-state traveling solutions of this equation with wave numbers in the vicinity of neutral wave numbers were constructed analytically The steady-state traveling solutions of the first family, whose wave numbers within the instability area are far from neutral wave numbers, were found numerically. The examples of some periodic solutions to this equation were presented.

Acknowledgements

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References

[1] Vozhakov I S, Arkhipov D G and Tsvelodub O Yu 2015 *Thermophysics and Aeromechanics* **22** 191-202