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Anomalous waves in gas-liquid mixtures near gas critical point in Gardner equation approximation

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Abstract. The waves in a bubbled incompressible liquid with Van der Waals gas in a bubbles being near critical points is considered in a frame of Gardner equation. It is shown that both coefficients on quadratic and cubic nonlinear terms in Gardner equation change the sign near gas critical point and it results the anomalous waves: negative and limited solitons, kinks, antikinks and breathers. The dynamics and interactions of these waves was studied numerically by high accuracy Fourier methods with periodically boundary conditions. In particular it is revealed that limited solitons always arise from initial distribution with a few identical soliton's pair and stand stable in their form after numerous interactions.

1. Introduction

The existence of rarefaction shock waves in Van der Waals gas was firstly shown by Zeldovich [1] under necessary condition of negative derivation $\partial^2 V/\partial P^2 < 0$. Zones of negative derivation bounded by curves 3,5 and 4,5 under adiabatic and isothermal conditions respectively was calculated in [1] and are shown on figure 1. Adiabatic anomalous zone appears only if adiabatic index $\gamma < 1.049$ is unreal.

The situations radically changes if gas behavior is near isothermal. This condition can be achieved in a small gas bubbles in a liquid under intense heat exchanging during the wave's pass. Since the second derivation near critical point can be vanished, the next term $\partial^3 V/\partial P^3$ of series was considered. In this paper the strict yielding of Gardner equation for nonlinear waves in homogeneous bubbled liquid and its numerical calculations are presented. It appears that anomalous zones of positive third adiabatical and isothermal derivation being shown between curves 2,5 and 4,5 on figure 1 respectively in a frame of Gardner equation are much greater then for ordinary rarefaction shock waves in [1].

The Gardner equation was firstly yielded for inner waves in two layer liquid [2] where anomalous zones of changes the signs of quadratic and cubic terms is too narrow to observe it experimentally, the waves in bubbled liquid near critical point for some gases in bubbles as freons is easy to achieve and thus we predict the future experiments in bubbled liquid near critical point and observing all kind of exotic waves which exist in a frame of Gardner equation with changing signs of nonlinear terms.

2. Derivation of Gardner equation

The homogeneous model of bubbled liquid in acoustic approximation $P_{xx} = \rho_{tt}$ [3] with mixture density definition $\rho = \rho_1(1-\varphi) + \rho_2\varphi$, where $\varphi = 4/3\pi R^3 N$ is gas void fractions, N is bubble numbers in mixture unit volume in incompressible liquid results following wave equation for dimensionless pressure $p = P/P_0$ and dimensionless bubble's volume $v = R^3/R_0^3$

$$p_{xx} = -\gamma v_{tt}/c_0^2 . \tag{1}$$

Here $c_0^2 = \gamma P_0 / \rho_0 \varphi_0$ is sound velocity in a mixture, indexes $1, 2$ and 0 refer to liquid, gas and initial values respectively. For bubble's volume we use dissipationless Reley equation

$$v^{-1/3} v_{tt} - \frac{1}{6} v^{-4/3} v_t^2 = \frac{w_0^2}{\gamma} (p_2 - p), \tag{2}$$

where $w_0^2 = 3\gamma P_0 / \rho_1 R_0^2$ is bubble's resonant frequency, $p_2 = P_2 / P_0$ is dimensionless gas pressure, R_0 is initial bubble's radii. In adiabatic equation of state for Van der Waals gas

$$\left(P_2 + a/V^2\right)(V-b)^\gamma = \left(P_0 + a/V_0^2\right)(V_0 - b)^\gamma,$$

two parameters being expressed through critical values $a = 3P_k V_k^2$, $b = V_k/3$ results its dimensionless equation of state $p_2(v)$

$$p_2(v) = (1+a_0)(1-b_0)^\gamma (v-b_0)^{-\gamma} - \frac{a_0}{v^2}, \tag{3}$$

where $a_0 = 3/\tilde{V}^2 \tilde{P}$, $b_0 = 1/3\tilde{V}$, $\tilde{V} = V_0/V_k$, $\tilde{P} = P_0/P_k$. For weak but finite amplitudes waves nonlinear system (1), (2), (3) can be reduce to Gardner equation if we expand (3) to Taylor's series

$$\begin{aligned}
&p_2(v) = 1 - A(v-1) + B(v-1)^2 - C(v-1)^3, \\
&A = -\left.\frac{dp_2}{dv}\right|_{v=1} = \gamma \frac{1+a_0}{1-b_0} - 2a_0, \\
&B = \frac{1}{2}\left.\frac{d^2 p_2}{dv^2}\right|_{v=1} = \frac{\gamma(\gamma+1)}{2} \frac{1+a_0}{(1-b_0)^2} - 3a_0, \\
&C = -\frac{1}{6}\left.\frac{d^3 p_2}{dv^3}\right|_{v=1} = \frac{\gamma(\gamma+1)(\gamma+2)}{6} \frac{1+a_0}{(1-b_0)^3} - 4a_0.
\end{aligned} \tag{4}$$

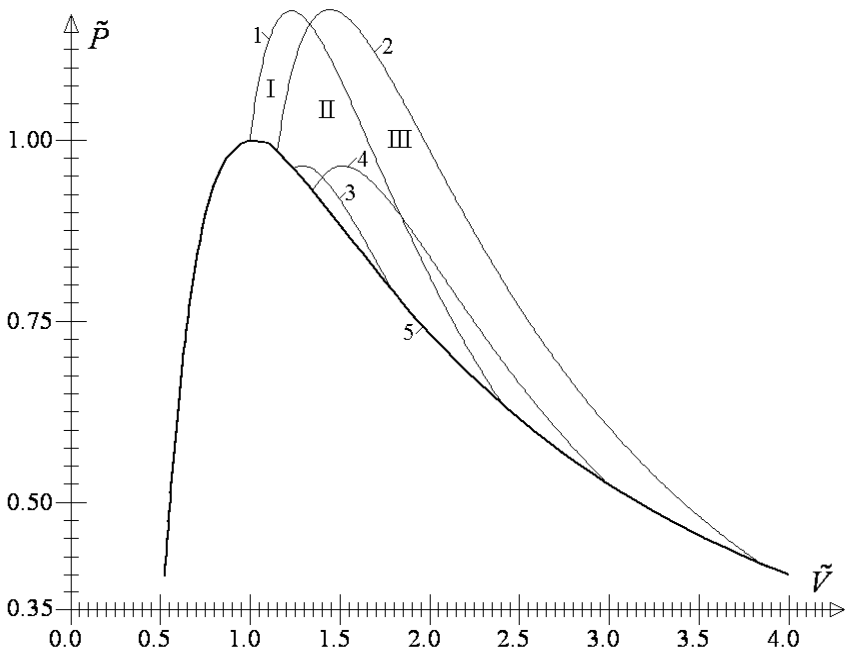


Figure 1. The borders of anomalous zones : 1 – $\alpha_T = 0$, 2 – $\beta_T = 0$, 3 – $\alpha_S = 0$, 4 – $\beta_S = 0$, 5 – binodal.

Substitute (4) into linearized Reley equation, twice differentiate it by x and take into account (1) we come to Boussinesq equation for volume perturbation $\tilde{v} = v - 1$

$$\frac{A}{\gamma}\tilde{v}_{xx} - \frac{1}{c_0^2}\tilde{v}_{tt} - \frac{B}{\gamma}\tilde{v}_{xx}^2 + \frac{C}{\gamma}\tilde{v}_{xx}^3 + \frac{1}{w_0^2}\tilde{v}_{xxtt} = 0. \tag{5}$$

Transition from (5) to Boussinesq equation for dimensionless pressure perturbation $\tilde{p} = p - 1$ can be done with linear link $\tilde{v} = -A^{-1}\tilde{p}$ in (4)

$$\tilde{p}_{xx} - \frac{1}{c^2}\tilde{p}_{tt} - \frac{B}{\gamma A^2}\tilde{p}_{xx}^2 + \frac{C}{\gamma A^3}\tilde{p}_{xx}^3 + \frac{1}{w^2}\tilde{p}_{xxtt} = 0, \tag{6}$$

where $c^2 = c_0^2 A/\gamma$, $w^2 = w_0^2 A$. For the waves propagating in one direction (6) reduce to Gardner equation, which preferable to write in dimensionless variables $x' = xk$, $t' = ckt$, $k = \sqrt{2}w/c$, $u = \tilde{p}(\gamma+1)/12\gamma$ in coordinate system moving with c velocity (all primes further are omitted)

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0, \tag{7}$$

where equation coefficients

$$\alpha = \frac{(\mathrm{d}^2 p_2/\mathrm{d}v^2)_S}{12(\mathrm{d}p_2/\mathrm{d}v)_S^2}, \quad \beta = -\frac{(\mathrm{d}^3 p_2/\mathrm{d}v^3)_S}{24(\mathrm{d}p_2/\mathrm{d}v)_S^3} \tag{8}$$

can be expressed through coefficients A, B, C from (4). The equations of curves $\alpha = 0$ and $\beta = 0$ result to follow equations on dimensionless $\tilde{P}\tilde{V}$ diagram

$$\alpha = 0: \quad \tilde{P} = \frac{2(3\tilde{V}-1)^2}{\gamma(\gamma+1)\tilde{V}^4} - \frac{3}{\tilde{V}^3}, \quad \beta = 0: \quad \tilde{P} = \frac{8(3\tilde{V}-1)^3}{3\gamma(\gamma+1)(\gamma+2)\tilde{V}^5} - \frac{3}{\tilde{V}^3}., \tag{9}$$

The curves of zero derivations (9) are shown in figure 1 as α_T, β_T under isothermal conditions ($\gamma = 1$) and as α_S, β_S under adiabatic conditions ($\gamma = 1.049$). Curves $\alpha_\mathrm{T} = 0$, $\beta_\mathrm{T} = 0$ and bimodal curve divide areas on $\tilde{P}\tilde{V}$ diagram near critical point onto three isothermal anomalous zones: zone I ($\alpha_\mathrm{T}, \beta_\mathrm{T} < 0$), zone II ($\alpha_\mathrm{T} < 0,\ \beta_\mathrm{T} > 0$) and zone III ($\alpha_\mathrm{T}, \beta_\mathrm{T} > 0$). Analogous adiabatic zones linked with curves α_S, β_S are much less and so do not marked on figure 1. Beyond these three zones the cubic nonlinearity looses his sense and instead (7) we must take ordinary KdV equation ($\alpha = 1,\ \beta = 0$).

3. Analytical and numerical solutions of Gardner equation

The Gardner equation (7) which was firstly derived for inner waves in two layer liquid [2] is now well studied since it is can be transferred to modified Korteveg-de-Vries equation (mKdV)

$$Q_t - 3\alpha^2/2\beta \cdot Q_x + 6\beta Q^2 Q_x + Q_{xxx} = 0, \tag{10}$$

by Gardner substitution $u = Q - \alpha/2\beta$ and thus becomes fully integrable by Inverse Scattering Transform Method [4]. In turn the KdV equation

$$Q_t + 6QQ_x + Q_{xxx} = 0,$$

can be transform to Gardner equation (7) by generalized Miura transform $Q = u/\alpha + u^2\beta/\alpha^3 + u_x(-\beta)^{1/2}/\alpha^2$, but reverse transform from Gardner to KdV is possible only if $\beta < 0$ when differential relation between u and Q becomes real.

Equation (7) has the stationary solution in a form of positive u_+ and negative u_- solitons

$$u_{\pm}(\xi) = \frac{U}{\alpha \pm \sqrt{\alpha^2 + \beta U}\,\mathrm{ch}(\xi\sqrt{U})}, \tag{11}$$

where $\xi = x - Ut$, which transform to limited soliton with amplitude $u_{\max} = \alpha/\beta$ if $\beta < 0$ and U coming to $-\alpha^2/\beta$. Solitons of both polarities $u_{\pm}$, which exist in zones II, III are shown in figure 2. Transition of negative solitons to limited soliton in zone I if $U \Rightarrow -\alpha^2/\beta$ are shown on figure 3.

Limited soliton presents itself superposition of kink ($u_{\mathrm{m}} > 0$) and antikink ($u_{\mathrm{m}} < 0$) step waves

$$u(\xi) = u_{\mathrm{m}}\mathrm{th}\left(u_{\mathrm{m}}\sqrt{-\beta}\xi\right) - \frac{\alpha}{2\beta}, \quad U = 2u_{\mathrm{m}}^2\beta - \frac{3\alpha^2}{2\beta}. \tag{12}$$

with shifted phase and amplitudes $u_{\mathrm{m}} = \pm\sqrt{\alpha}/2\beta$. Another valuable amplitude of kink and antikink (12) is $u_{\mathrm{m}} = \pm\sqrt{3}\alpha/2\beta$ when their propagation velocities U are zero.

More information about stationary waves gives the analyses of phase plane (u, u_{ξ}), searching out separatrix, limit cycles and singularity points, where $\mathrm{d}u_{\xi}/\mathrm{d}u$ does not determined. According to first integral of (7) $u_{\xi}(u) = \pm u(U - 2\alpha u - \beta u^2/2)^{1/2}$ there are always three singular points: $u_{\xi} = 0$, $u_1 = 0$, $u_{2,3} = [-2\alpha \pm (4\alpha^2 + 2\beta U)^{1/2}]/\beta$. First point is always a saddle. Second and third are centre points if $\beta > 0$. Separatrix which refer to solitons begin and end at saddle, passing around the centers where there are numerous closed curves referring to conoidal waves. If $\beta < 0$ first and second points are the same, but third one is saddle and thus exists only one polarity solitons and conoidal waves. In special case $U = -\alpha/\beta$ two separatrix referring to kink and antikink connect both saddles. Phase plane for the case $\alpha = \beta = -1$, $U = 0.9$ are shown on figure 4. Bold lines are the separatrix.

Second class of analytical solutions of Gardner equation are breathers. They exist only for positive $\beta > 0$ and represent itself the envelop waves on substrate. The breather of mKdV at $\beta = 1$

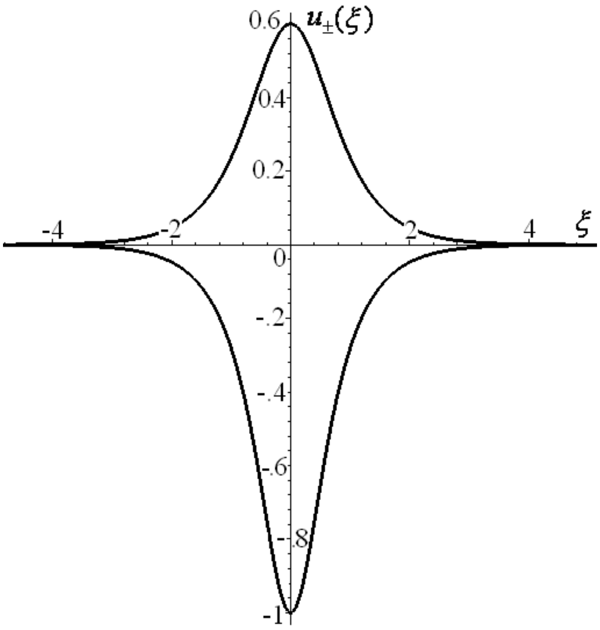


Figure 2. Positive u_+ and negative u_- solitons (11) at zones II, III at $\alpha = 1, \beta = 5$, $V = 3$.

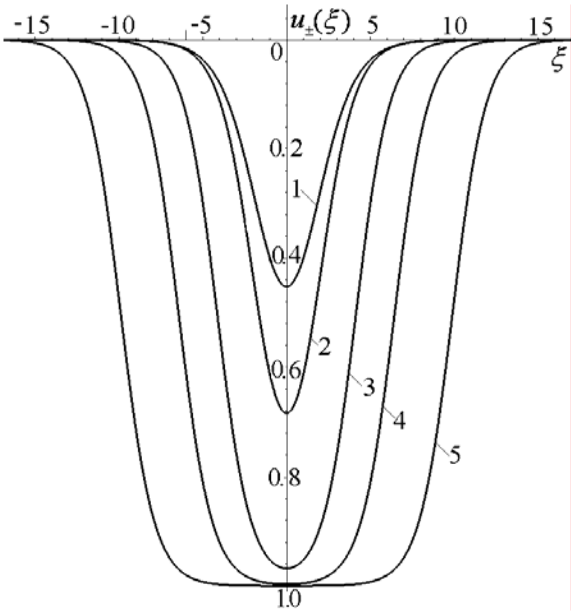


Figure 3. Normal and limited solitons (11) at zone I: $\alpha = 1, \beta = -5$, 1 – $U = 0.7$, 2 – $U = 0.9$, 3 – $U = 0.999$, 4 – $U = 1 - 10^{-5}$, 5 – $U = 1 - 10^{-7}$.

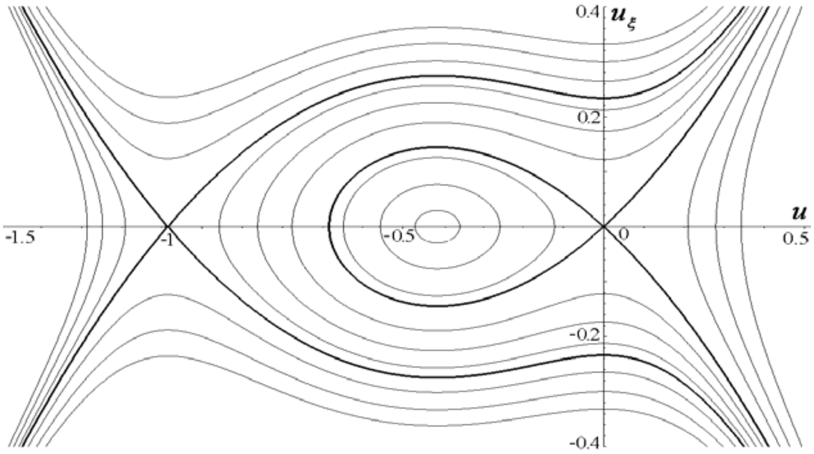
$$u(x,t) = -\frac{2k_1}{\operatorname{ch}\vartheta_1} \cdot \frac{\sin\vartheta_2 + k_1/k_2 \cos\vartheta_2 \operatorname{th}\vartheta_1}{1 + k_1^2/k_2^2 \cos^2\vartheta_2 / \operatorname{ch}^2\vartheta_1}, \tag{13}$$

is also a breather solution of (7) after Gardner substitution. Here $\vartheta_1 = k_1 x - (k_1^3 - 3k_1 k_2^2)t + \ln(k_2/k_1)$, $\vartheta_2 = k_2 x - (3k_2 k_1^2 - k_2^3)t$, $k_{1,2}$ are wave numbers.

Numerical solutions of Gardner equation was found by high accuracy Fourier method with periodical boundary conditions. Due to connection the end and beginning of x integrating interval in Fourier method, wave dynamics and it's interactions can be observed during unlimited time.

The breakup of initial negative Gaussian wave onto four pairs of ordinary soliton and one limited soliton are shown in figure 5. It is well seen that after fifteen round race of limited soliton and more then hundred interactions soliton pairs are stayed stable. Soliton's pair dynamics is observed firstly.

More complicate interactions of three breathers (13) with common $k_1 = 0.5$ and $k_2 = 1$, 2.5, 5 are shown in figure 6. The interactions arise because of different and negative propagating velocities which are proportional to k_2, so the first and the fastest breather with $k_2 = 5$ run down the second and third ones at $t = 0.523$ and $t = 1.866$ respectively. After a long time $t = 17.17$ breathers running in circled integrating interval and nineteen breather's elastic interactions their forms do not changed.



Fifure 4. Phase plane of Gardner equation's stationary solutions.

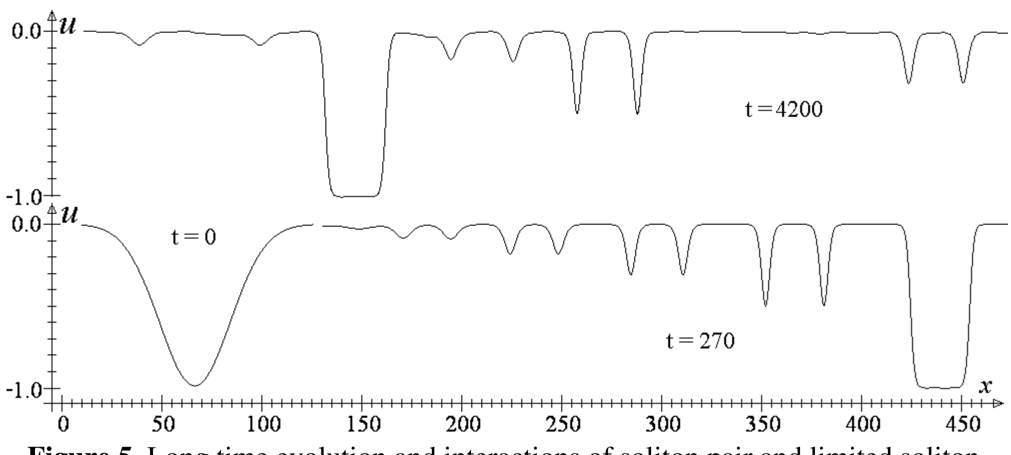


Figure 5. Long time evolution and interactions of soliton pair and limited soliton precipitated from initial Gaussian wave at $\alpha = \beta = -1$.

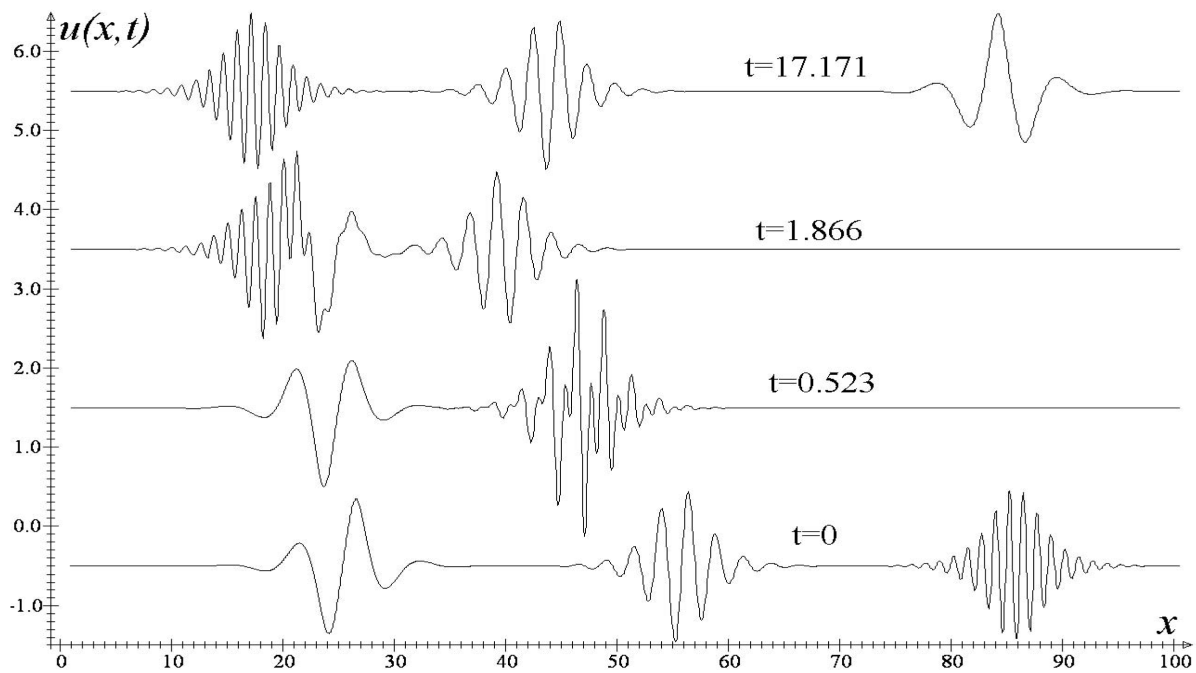


Figure 6. Interactions of three breathers (13) at $\alpha = \beta = 1$.

4. Conclusions

It is shown that in bubbled incompressible liquid with Wan der Waals gas in bubbles pressure waves can be correctly considered in a frame of Gardner equation, with coefficients of quadratic and cubic nonlinear terms being proportional to $\partial^2 V / \partial P^2$ and $\partial^3 V / \partial P^3$ respectively.

The analyze of nonlinear coefficients on P,V diagram showed three ample anomalous zones where their signs are differed from those ones in ideal gas if gas behavior is near isothermal.

Since near isothermal behavior of gas bubbles in a liquid is easy to achieve, all known solutions of Gardner equation like negative and limited solitons, kinks, antikinks and breathers in bubbled liquid near critical point are the anomalous waves and can be experimentally observed instead of rarefaction shock waves predicted by Zeldovich in pure nonideal gas being always adiabatic in shock waves.

As illustrations the dynamics and interactions of these anomalous waves was studied numerically by high accuracy Fourier methods with periodically boundary conditions. In particular it is revealed that limited solitons always arise from negative initial distribution with a few identical soliton's pair and stand stable in their form after numerous interactions.

5. Acknowledgments

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References

[1] Zeldovich Ya B 1946 On possibility of rarefaction shock waves existence *Zh. Eksp. Teor. Fiz.* **16** 363
[2] Kakutani T, Yamasaki N 1978 Solitary waves in two-layer fluid *J. Phys. Soc. Jap.* **45** 674
[3] Van Wingaarden L. 1972 One-dimensional flow of liquids containing small gas bubbles *Ann. Rev. Fluid Mech.* **4** 369
[4] Wiles J W 1981 On internal solitary waves *Tellus* **38** 394
[5] Slunyaev A V, Pelinovsky E N 1999 Soliton dynamics of large amplitude *Zh. Eksp. Teor. Fiz.* **116** 318