Thermocapillary instability of a liquid layer on interior surface of a rotating cylinder

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Abstract. Stability problem of a liquid, situated on the inner surface of a cylinder, which rotates about its axis with a constant angular velocity, is considered. The flow is assumed to be non-isothermal, and thermocapillary instability is investigated. The temperature at the free boundary satisfies the condition of the third kind with a given Biot number. It is supposed that the free boundary is undeformable. The exact solution of Navier–Stokes and heat conduction equations is obtained. Neutral disturbances of this solution are considered, where Marangoni number is taken as the spectral number. Asymptotics of critical values of Marangoni number for long and short waves are found analytically. Neutral curves are constructed numerically for various values of independent dimensionless parameters. The dependences of critical Marangoni number on Reynolds number, Biot number and the aspect ratio are investigated.

1. Introduction

The isothermal problem of a liquid layer on the surface of the rotating cylinder was studied in [1]–[9].

In [1, 3, 4, 6] the liquid layer on the outside of the cylinder was considered. In [1] principle of change of stability in weightless conditions was investigated. In [3] the existence of the stationary solution of the problem with free boundaries was proven in the exact statement; the evolution equation of the film dynamics was obtained in a thin layer approximation. This equation was studied in [6] later. In [4] a plane problem was considered by the techniques of lubrication theory, the behaviour of liquid rings around the cylinder was described.

The motion on the interior surface of a cylinder was not sufficiently studied. In [2] the bifurcation of rotationally symmetric motions was investigated; a branch condition in terms of critical Weber number was obtained. In [5] the plane problem was considered; steady-state liquid-film profiles were found. In [7] the effect of surface tension on the stability of the film was examined in the case of the plane problem. Three-dimensional coating and rimming flow was studied in [8]. In [9] inertial instability of flows on the inside or outside of a rotating horizontal cylinder was investigated.

In the present work the flow is assumed to be non-isothermal. Reasoning by analogy with [1], axisymmetric disturbances are considered as more dangerous. Thermocapillary instability of a liquid layer on the interior surface of a rotating cylinder is studied.

2. Mathematical model

Let the viscous incompressible thermo-conducting liquid partially filling the space between two neighboring cylindrical surfaces of radii r_1 and $r_2 < r_1$. Surface $r = r_2$ is the free one. The outer surface is rigid and rotates about its axis with a constant angular velocity ω. We will discuss the motion in a cylindrical coordinate system r, θ, z. It is assumed that gravity force is absent. Let us

suppose that liquid density ρ, kinematic coefficient of viscosity ν and thermodiffusion coefficient χ are constant while the surface tension coefficient σ is a linear function of temperature T:

$$\sigma = \sigma_0 - \kappa(T - T_0),$$

where σ_0, κ and T_0 are positive constants. The temperature at the free boundary satisfies the condition

$$\frac{\partial T}{\partial \boldsymbol{n}} = q(T_\Gamma - T),$$

where q is the interfacial heat transfer coefficient, T_Γ is the given temperature of the environment, $\boldsymbol{n}$ is the unit normal vector to the free boundary. The temperature at the solid wall satisfies the condition

$$T = T_S,$$

where $T_S \neq T_\Gamma$ is the given positive constant.

The basic solution to the Navier–Stokes and heat conduction equations has the form

$$\boldsymbol{V} = (0, \omega r, 0), \quad P = \frac{\rho\omega^2 r^2}{2} + \widetilde{p}_0, \quad \widetilde{T} = \frac{\mathrm{Bi}(T_S - T_\Gamma)}{1 + \mathrm{Bi}\ln a}\ln\frac{r}{r_2} + \frac{T_S + T_\Gamma \mathrm{Bi}\ln a}{1 + \mathrm{Bi}\ln a},$$

where $\widetilde{p}_0$ is the constant, $a = \dfrac{r_1}{r_2} > 1$ is the aspect ratio, $\mathrm{Bi} = qr_2$ is Biot number.

We introduce dimensionless parameters, choosing as scales of length, velocity, pressure and temperature the quantities r_2, ωr_2, $\rho\omega^2 r_2^2$ and $\delta T = \dfrac{\mathrm{Bi}(T_S - T_\Gamma)}{1 + \mathrm{Bi}\ln a}$. The basic solution in terms of dimensionless variables has the form

$$\boldsymbol{V} = (0, r, 0), \quad P = \frac{r^2}{2} + p_0, \quad \widetilde{T} = \ln r + c, \tag{1}$$

where p_0 and $c = \dfrac{T_s + T_\Gamma \mathrm{Bi}\ln a}{\mathrm{Bi}(T_s - T_\Gamma)}$ are constants.

Then we impose small axisymmetric disturbances on the basic solution. We seek velocity, pressure and temperature fields of the form

$$\boldsymbol{V}^1 = (u, \upsilon + r, w), \quad p^1 = P + \frac{p}{\mathrm{Re}}, \quad T^1 = \widetilde{T} + \mathrm{Pr}\,T,$$

where $\mathrm{Re} = \dfrac{\omega r_2^2}{\nu}$ is Reynolds number, $\mathrm{Pr} = \dfrac{\nu}{\chi}$ is Prandtl number. Navier–Stokes and heat conduction equations and boundary conditions are linearized. Reasoning by analogy with [10], we assume that the stability loss of the basic solution leads to an appearance of a new stationary solution. In this case Marangoni number $\mathrm{Ma} = \dfrac{\kappa\delta T r_2}{\rho\nu\chi}$ is taken as the spectral parameter. It is assumed that the free boundary is undeformable. The boundary value problem with a fixed domain is obtained:

$$\Delta u - \frac{u}{r^2} - p_r = -2\,\mathrm{Re}\,\upsilon, \quad \Delta\upsilon - \frac{\upsilon}{r^2} = 2\,\mathrm{Re}\,u, \quad \Delta w - p_z = 0, \quad \frac{1}{r}(ru)_r + w_z = 0, \quad \Delta T = \mathrm{Re}\frac{u}{r}, \tag{2}$$

$$r = a: \ u = \upsilon = w = T = 0, \tag{3}$$

$$r = 1: \ u = \upsilon_r - \upsilon = -T_r + \mathrm{Bi}T = \mathrm{Re}(w_r + u_z) - \mathrm{Ma}T_z = 0, \tag{4}$$

where $\Delta = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{\partial^2}{\partial z^2}$.

Independent dimensionless parameters of the problem are Re, Bi and *a.* It should be noted that in the full statement (a deformable free surface), $\mathrm{Cr}=\frac{\kappa\delta T}{\sigma_0}$ (crispation number) is also the key dimensionless parameter, which characterizes the degree of deformation of the free surface by thermocapillary forces. Cr is small and in analogy with [10, 11] it is neglected.

We seek the solution of the problem (2) – (4) in the form

$$u = A(r)\cos(kz),\ \upsilon = B(r)\cos(kz),\ w = C(r)\sin(kz),\ p = \xi(r)\cos(kz),\ T = \zeta(r)\cos(kz),$$

and obtain the spectral problem

$$\frac{1}{r}(rA')' - (k^2 + \frac{1}{r^2})A + 2\,\mathrm{Re}\,B - \xi' = 0,\ \frac{1}{r}(rB')' - (k^2 + \frac{1}{r^2})B - 2\,\mathrm{Re}\,A = 0,$$
$$\frac{1}{r}(rC')' - k^2 C + \xi k = 0,\ \frac{1}{r}(rA)' + kC = 0,\ \frac{1}{r}(r\zeta')' - k^2\zeta - \mathrm{Re}\frac{A}{r} = 0,$$

$$r = a:\ A = B = C = \zeta = 0,$$

$$r = 1:\ A = B' - B = -\zeta' + \mathrm{Bi}\,\zeta = \mathrm{Re}\,C' + \mathrm{Ma}\,\zeta k = 0.$$

After elimination of *B, C* and ξ, the system and boundary conditions take the form

$$(L^3 - 4\,\mathrm{Re}\,k^2)A = 0, \tag{5}$$

$$\frac{1}{r}(r\zeta')' - k^2\zeta - \mathrm{Re}\frac{A}{r} = 0, \tag{6}$$

$$A(a) = A'(a) = L^2 A(a) = A(1) = ((\frac{\partial}{\partial r} - 1)L^2)A(1) = 0, \tag{7}$$

$$\zeta(a) = -\zeta'(1) + \mathrm{Bi}\,\zeta(1) = 0, \tag{8}$$

$$\mathrm{Re}(A''(1) + A'(1)) = \mathrm{Ma}\,\zeta(1)k^2, \tag{9}$$

where $L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - (\frac{1}{r^2} + k^2)$.

The spectral problem (5) – (9) for the parameter Ma is degenerate. Ma can be found uniquely for a given set of determinative parameters. The solution of the problem is determined up to an arbitrary multiplicative constant. So it can be supposed that

$$A''(1) + A'(1) = 1, \tag{10}$$

and A can be found from the boundary value problem (5), (7), (10). Then ζ is found from the boundary value problem (6), (8). It can be shown using variation of parameters that $\zeta(1) \neq 0$. So Ma is defined uniquely from (9).

3. Asymptotics and numerical results.

Asymptotics of neutral curves for the problem (5) – (9) are found analytically.

Asymptotics for $k \to 0$ (long waves) have the form

$$A_0(r) = \frac{2(a^4 - r^2)(r^2 - 1)\ln a - (a^2 - 1)((a^2 - r^2)(r^2 - 1) + 2(a^2 - 1)r^2 \ln r)}{4r(a^4 - 1 - 4\ln a)},$$

$$\zeta_0(r)=\frac{\mathrm{Re}}{64(a^4-1-4\ln a)(1+\mathrm{Bi}\ln a)}(-16\mathrm{a}^4\ln^3 a(1+\mathrm{Bi}\ln r)+(a^2-1)(5a^4-r^2(r^2-12)-$$
$$4a^2(r^2+3)+(\mathrm{Bi}(11-16a^2+5a^4)-12+8r^2(a^2-1))\ln r-8a^2\ln^2 r)-2\ln^2 a(4+4a^2-3\mathrm{Bi}+$$
$$4\mathrm{Bi}\,r^2-\mathrm{Bi}\,r^4+4a^4(1+\mathrm{Bi}\,(r^2-1))-4a^2(a^2-1)\mathrm{Bi}\ln r-8a^4\mathrm{Bi}\ln^2 r)+\ln a(11\mathrm{Bi}-12-8r^2-$$
$$12\mathrm{Bi}\,r^2+2r^4+\mathrm{Bi}\,r^4+a^4(14-8\,r^2-4\mathrm{Bi}(r^2-1))+a^2(12-\mathrm{Bi}(15-16r^2+r^4))+2(4-3\mathrm{Bi}+$$
$$4\mathrm{Bi}\,r^2-8a^2\mathrm{Bi}r^2+a^4(8+3\mathrm{Bi}+4\mathrm{Bi}\,r^2))\ln \mathrm{r}+8a^2\ln^2 r(\mathrm{Bi}-a^2(\mathrm{Bi}-2)))),$$
$$\mathrm{Ma}=\frac{64\left(3-4a^2+a^4+4\ln a\right)(1+\mathrm{Bi}\ln a)}{k^2\left((a^2\text{-}1)^2(5a^2-11)+2\ln a(3(a^4+2a^2-3)-4\ln a(2a^4\ln a+a^4+a^2+1))\right)}+O(1),\ k\to 0.$$

Asymptotics for $k\to\infty$ (short waves) have the form

$$A_\infty(r)=-\frac{1}{2}k(r-1)e^{-k(r-1)},\ \ \zeta_\infty(r)=-\frac{\mathrm{Re}}{8k^2}(1+k(r-1)+k^2(r-1))e^{-k(r-1)},$$
$$\mathrm{Ma}=8k^2+O(1),\ k\to\infty.$$

It should be noted that asymptotics for critical Ma values coincide with Pearson's short-wave asymptotics for Ma [10].

Critical values of Marangoni number Ma^* and wave number k^* are obtained numerically using Runge–Kutta method of the fourth order for various values of independent dimensionless parameters; the neutral curves on k, Ma plane are constructed.

The figure 1 shows the neutral curves for different values of Bi when a = 1.1, Re = 1. Critical values are $k^*=19.94$, $\mathrm{Ma}^*=8793.2$ for $\mathrm{Bi}=1$; $k^*=22.58$, $\mathrm{Ma}^*=12107.78$ for $\mathrm{Bi}=10$; $k^*=23.74$, $\mathrm{Ma}^*=15653.48$ for $\mathrm{Bi}=20$. Thus, with increasing Bi, Ma^* and k^* also increase.

It should be noted that for water, when $\mathrm{Ma}^*=8793.2$ and $r_2=3\cdot10^{-2}m$, then $\mathrm{Cr}=5\cdot10^{-4}$. Thus, the neglect of Cr is justified.

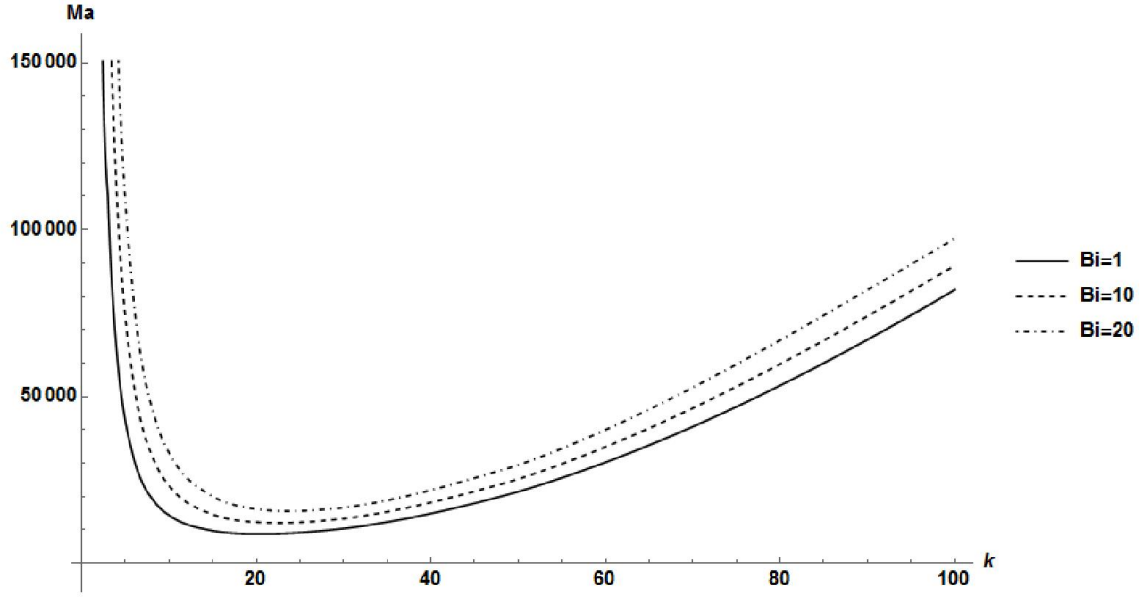

Figure 1. The neutral curves for a = 1.1, Re = 1 and various values of Bi.

The figure 2 shows the neutral curves for different values of a when Bi = 1, Re = 1. Critical values are $k^* = 19.94$ $\mathrm{Ma}^* = 8793.2$ for $a = 1.1$; $k^* = 13.72$, $\mathrm{Ma}^* = 4107.37$ for $a = 1.15$; $k^* = 10.57$, $\mathrm{Ma}^* = 2406.07$ for $a = 1.2$ Thus, with increasing a, Ma^* and k^* decrease.

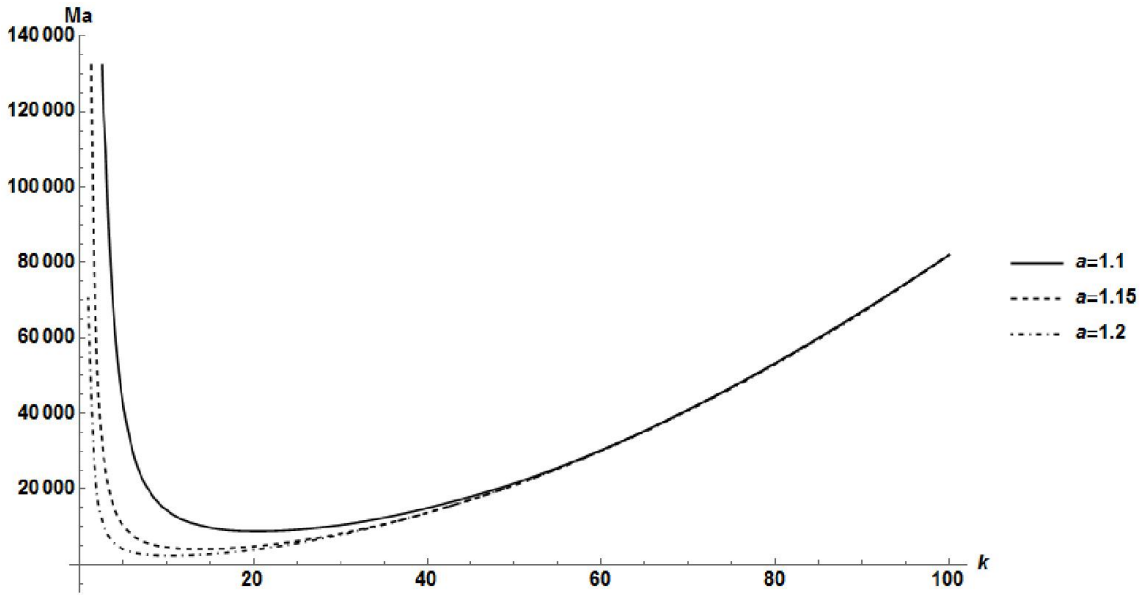

Figure 2. The neutral curves for Bi = 1, Re = 1 and various values of a.

The figure 3 shows the neutral curves for different values of Re when Bi = 1, a = 1.2. Critical values are $k^* = 10.57$, $\mathrm{Ma}^* = 2406.07$ for $\mathrm{Re} = 1$; $k^* = 12.88$, $\mathrm{Ma}^* = 6441.86$ for Re=500. Thus, with increasing Re, Ma^* and k^* also increase.

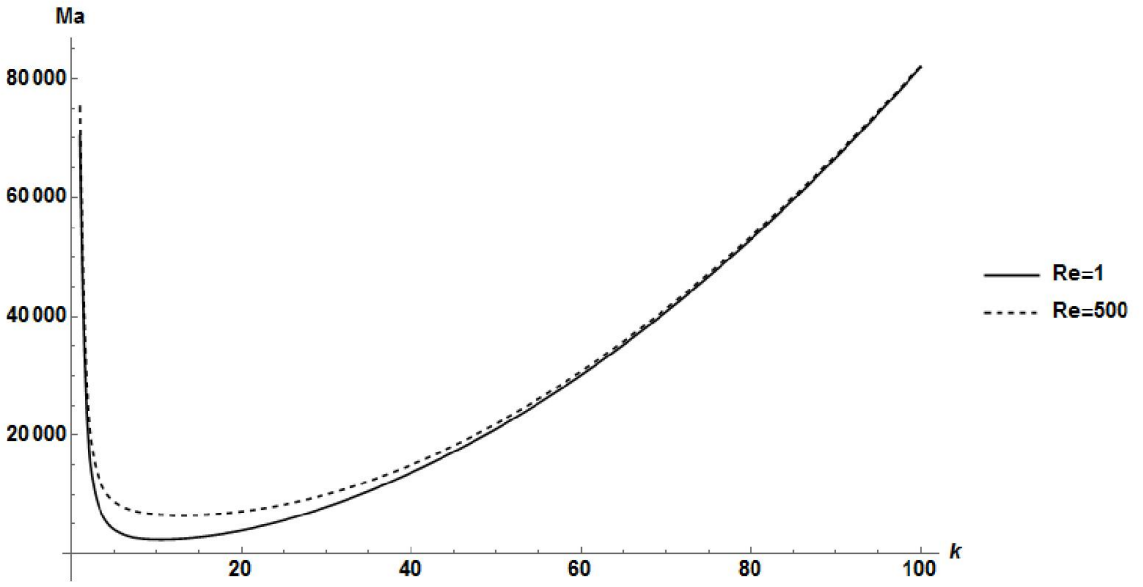

Figure 3. The neutral curves for Bi = 1, a = 1.2 and various values of Re.

4. Conclusion.

For non-isothermal liquid, situated on the inner surface of the rotating cylinder, the exact solution of Navier–Stokes and heat conduction equations is obtained. The basic solution has only the azimuthal component of the velocity as non-zero, which is proportional to radial coordinate *r*. The pressure is the quadratic function of *r* and the temperature is the logarithmic function of *r*. Small disturbances are considered, where Marangoni number is chosen as the spectral parameter.

Asymptotics of neutral curves for long and short waves are found analytically. The main term of asymptotics has the order of $O(k^{-2})$ when $k \to 0$ and $O(k^2)$ when $k \to \infty$. Critical values of Marangoni number and wave number are obtained numerically for various values of independent dimensionless parameters. The dependences of critical Marangoni number on Reynolds number, Biot number and the aspect ratio are investigated. It is obtained that the critical Marangoni number and the critical wave number increase with increasing Re and Bi, and decrease with increasing *a*.

It should be noted that critical values of Rayleigh number Ra^* in the terms of the temperature gradient is proportional to $(a-1)^4$ (see [12]) when Ma is not proportional to any degree of $a-1$. Thereby, it could be concluded that for thin layers ($a-1<<1$) thermocapillary mechanism of instability dominates and the neglect of convection induced by buoyancy force is justified.

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