Journal of Physics: Conference Series **754** (2016) 032001 doi:10.1088/1742-6596/754/3/032001

Linear waves on a surface of vertical rivulet

S P Aktershev[1,2], S V Alekseenko[1,2] and D G Arkhipov[1,2]

[1] Kutateladze Institute of Thermophysics, Lavrentiev Ave. 1, Novosibirsk 630090, Russia
[2] Novosibirsk State University, Pirogova St. 2, Novosibirsk 630090, Russia
E-mail: sergey-aktershev@mail.ru

Abstract. The type of film flow whereby the fluid flows in the form of many streamlets is typically called a rivulet flow. Whereas an individual streamlet bounded by two contact lines is called a rivulet. Special attention has been paid to rivulet flows because of their practical value for a variety of devices in power engineering and chemical technology, such as absorbers, distillation columns, evaporators, and heat exchangers for the liquefaction of natural gas. In the present paper the waves in vertical rivulet are investigated analytically. The Kapitza-Shkadov model is used to describe the wavy rivulet flow since it was well proven in the study of nonlinear waves in falling liquid films over a wide range of Reynolds numbers. The equations of the wavy rivulet flow are derived on the basis of the weighed residual method. These equations turn out to be the projections of the Shkadov's model equations on system of basis functions, constructed in special way. Linearizing these equations results in the dispersion relations for plane waves. The stability criterion for rivulet flows is deduced, and the analysis of dispersion relations depending on dimensionless parameters is carried out.

1. Introduction

Compared to uniform liquid film flows, rivulet dynamics exhibit several interesting features, such as the presence of movable contact lines and wetting angle hysteresis. Most theoretical studies, beginning with the pioneering study of Towell and Rothfeld [1], are devoted to stationary and smooth (without waves) rivulets flowing down an inclined plane as well as curved surfaces [2, 3]. Profiles of smooth rivulets and steady regimes of rivulet flows were calculated using both analytical methods in a lubrication approximation [2, 3] and numerical methods based on Navier-Stokes equations [4, 5]. In the simplest case of a thin rivulet flowing down a vertical flat surface, the rivulet profile was shown to represent an arc of a circle.

The stability of a moving rivulet has been studied in a limited number of works. A linear stability analysis was performed in [6-10] based on Navier-Stokes equations for the normal mode (plane waves). In these theoretical studies the stability calculations of rivulet are executed in lubrication approximation, i.e. at very small Reynolds numbers and for plane modes of perturbations only. Alekseenko, Markovich and Shtork [11] performed the first detailed experimental study of the wave motion of a rectilinear rivulet for the case of liquid flow along the lower outer wall of an inclined tube. With application of the imposed periodic pulsations of the flow rate the periodic non-linear waves have been studied, new wave modes are found, and comparison to known two-dimensional film flow is made. In the papers [12, 13] method of PIV was applied in the first time to measure instantaneous velocity field in wavy rivulet. Alekseenko *et al.* [14] were the only researchers to present the results of experimental investigation on periodic, nonlinear waves on a rectilinear rivulet flowing down a vertical wall. Waves in rivulet flow were studied in a wide range of a Reynolds number and forcing frequency for various values of angles of wetting. In [15] the results of numerical simulations of three-dimensional waves on the surface of a rivulet flowing down a vertical plate are presented. Various characteristics of linear and nonlinear regular waves in the rivulet are obtained through numerical calculations as a function of the forcing frequency at different Reynolds numbers and contact wetting angles. Calculations were executed only for those conditions which were implemented in experiments [14] and comparison with experimental data was made. The comparison shows that the applied model adequately describes the shape of the wave surface of a rivulet.

In the present paper the waves in vertical rivulet are investigated theoretically. On the basis of a weighed residual method the approach allowing simplify a problem is developed and dispersion relations for linear waves are deduced analytically at moderate Reynolds numbers.

2. Theoretical model

Let us consider a thin liquid layer flow over a vertical flat plate as a rivulet of constant width 2*b*. We introduce a Cartesian coordinate system *Oxyz* with the *Ox*-axis directed along gravity and the *Oy*-axis directed along the normal to the plate (figure 1*b*). We assume that the layer thickness *h* is substantially smaller than the wavelength λ (long-wave approximation) and the rivulet semi-width *b*. All of these assumptions are in good agreement with experimental observations [14]. As a first approximation with respect to the small parameters $\varepsilon_\lambda = h/\lambda$ and $\varepsilon_b = h/b$, the waves in a liquid layer may be described by the integral boundary layer (IBL) model [15], which was developed for three-dimensional long waves in a falling liquid film. The corresponding momentum and mass balance equations in dimensionless variables x/H_0, z/H_0, h/H_0, q/q_m, m/q_m, t/t_m are as follows

$$\begin{aligned} &\frac{\partial q}{\partial t} + \frac{6}{5}\left(\frac{\partial}{\partial x}\frac{q^2}{h} + \frac{\partial}{\partial z}\frac{qm}{h}\right) = \frac{3}{\mathrm{Re}_m}\left(h - \frac{q}{h^2}\right) + hWe\frac{\partial \Delta h}{\partial x}, \\ &\frac{\partial m}{\partial t} + \frac{6}{5}\left(\frac{\partial}{\partial z}\frac{m^2}{h} + \frac{\partial}{\partial x}\frac{qm}{h}\right) = hWe\frac{\partial \Delta h}{\partial z} - \frac{3m}{\mathrm{Re}_m h^2}, \\ &\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial m}{\partial z} = 0. \end{aligned} \tag{1}$$

Here, $q(x,z,t) = \int_0^h u dy$ and $m(x,z,t) = \int_0^h w dy$ are the liquid flow rates per unit width along the *Ox* and *Oz* axes, respectively, $\Delta h = \partial^2 h/\partial x^2 + \partial^2 h/\partial z^2$. We choose the unperturbed film thickness H_0 at the symmetry axis of the rivulet as the scale and introduce the scales of velocity $gH_0^2/3\nu$, time $t_m = 3\nu/gH_0$, and flow rate $q_m = gH_0^3/3\nu$. Here, $\mathrm{Re}_m = gH_0^3/3\nu^2$ is the Reynolds number,

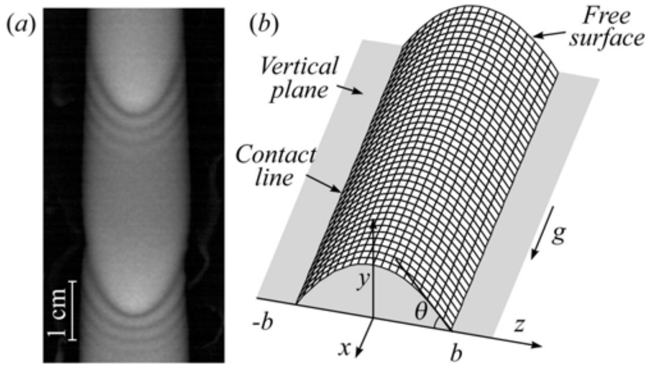

Figure 1. (*a*) Visualization of wavy rivulet flow using the LIF technique (experimental setup of Alekseenko *et al*. [22], forcing frequency: 17 Hz) and (*b*) rivulet flow diagram.

$We = \left(3Fi / \mathrm{Re}_m^5\right)^{1/3}$ is the Weber number, $Fi = \sigma^3 / \rho^3 g \nu^4$ is Kapitza number, σ is the surface tension, ρ is the density, and ν is the kinematic viscosity of the fluid. With regard to the rivulet flow, these equations must be supplemented by the follows boundary conditions at the contact line (fixed contact line): $h\big|_{z=\pm b} = 0$, $q\big|_{z=\pm b} = 0$, $m\big|_{z=\pm b} = 0$. It is follows from (1), that *h* and *q* are even functions of coordinate *z*, and *m* is odd function of coordinate *z*. In case of a undisturbed flow (without waves) the partial derivatives on *x* and *t* in the equations (1) and also the flow rate *m* are equal to zero, thus a cross-section of rivulet is a parabola $h_0(z) = 1 - z^2 / b^2$ and $q_0(z) = h_0^3(z)$.

Supposing $h = h_0 \tilde{h}$, $q = q_0 \tilde{q}$, $m = q_0 \tilde{m}$ we introduce a new functions $\tilde{h}$, $\tilde{q}$, $\tilde{m}$. Instead of coordinate *z* we use the variable $\xi = z / b$. The function $\tilde{h}$, $\tilde{q}$, $\tilde{m}$ should have the finite values at a contact line, thus boundary conditions will be automatically fulfilled, since $h_0\big|_{\xi=\pm 1} = q_0\big|_{\xi=\pm 1} = 0$. Using $\tilde{h}$, $\tilde{q}$, $\tilde{m}$, we rewrite the equations (1) as follows

$$\begin{gathered}
h_0^3 \frac{\partial \tilde{q}}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{\tilde{q}^2}{\tilde{h}} h_0^5\right) + \frac{6}{5b}\frac{\partial}{\partial \xi}\left(\frac{\tilde{q}\tilde{m}}{\tilde{h}} h_0^5\right) = \frac{3h_0}{\mathrm{Re}_m}\left(\tilde{h} - \frac{\tilde{q}}{\tilde{h}^2}\right) + We h_0 \tilde{h} \frac{\partial L}{\partial x}, \\
h_0^3 \frac{\partial \tilde{m}}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{\tilde{q}\tilde{m}}{\tilde{h}} h_0^5\right) + \frac{6}{5b}\frac{\partial}{\partial \xi}\left(\frac{\tilde{m}^2}{\tilde{h}} h_0^5\right) = \frac{We}{b} h_0 \tilde{h} \frac{\partial L}{\partial \xi} - \frac{3h_0}{\mathrm{Re}_m \tilde{h}^2}\tilde{m}, \qquad (2) \\
h_0 \frac{\partial \tilde{h}}{\partial t} + h_0^3 \frac{\partial \tilde{q}}{\partial x} + \frac{1}{b}\frac{\partial}{\partial \xi}\left(\tilde{m} h_0^3\right) = 0.
\end{gathered}$$

Here, $h_0 = 1 - \xi^2$, $L \equiv \Delta\left(h_0 \tilde{h}\right) = \left(1 - \xi^2\right)\frac{\partial^2 \tilde{h}}{\partial x^2} + \frac{1}{b^2}\left(\left(1 - \xi^2\right)\frac{\partial^2 \tilde{h}}{\partial \xi^2} - 4\xi \frac{\partial \tilde{h}}{\partial \xi} - 2\tilde{h}\right)$.

3. The application of a weighed residual method for wavy rivulet flow model

On interval $-1 < \xi < 1$ we define two different systems of orthogonal basis polynomials: even polynomials $\Phi_j(\xi)$ and odd polynomials $\Psi_j(\xi)$, $j = 1,2,3,\ldots$ Let's spread out even functions $\tilde{h}$ and $\tilde{q}$ abreast on $\Phi_j(\xi)$, and odd function $\tilde{m}$ abreast on $\Psi_j(\xi)$:

$$\tilde{h} = \sum_{j=1}^{\infty} H_j(x,t)\Phi_j(\xi), \quad \tilde{q} = \sum_{j=1}^{\infty} Q_j(x,t)\Phi_j(\xi), \quad \tilde{m} = \sum_{j=1}^{\infty} M_j(x,t)\Psi_j(\xi). \qquad (3)$$

As a whole, relations (3) do not satisfy (2) (substitution in the equations gives the residual). According to a weighed residual method, the projections of a residual to basis functions should be equated to zero, as a result we obtain the system of differential partial equations for H_j, Q_j, M_j. Coefficients of these equations are calculated through integrals from product of basis polynomials on function from ξ. in the equations (2).

We define a scalar product of polynomials $\Phi_k(\xi)$ and $\Phi_j(\xi)$ by means of weight function $g(\xi) = \left(1 - \xi^2\right)^3$ as follows: $\left\langle \Phi_k \middle| \Phi_j \right\rangle = \int_{-1}^{1} g(\xi)\Phi_j(\xi)\Phi_k(\xi) d\xi$.

Let's take the first even polynomial $\Phi_1 = 1$. The second even polynomial we take as $\Phi_2 = \Phi_1 + a_2 \xi^2$, where the constant a_2 is calculated by means of a condition of orthogonality $\left\langle \Phi_1 \middle| \Phi_2 \right\rangle = 0$. Thus, we obtain $\Phi_2 = 1 - 9\xi^2$. Each subsequent even polynomial Φ_{n+1} is defined by recurrence formula $\Phi_{n+1} = \Phi_1 + a_2\Phi_2 + a_3\Phi_3 + \ldots a_n\Phi_n + a_{n+1}\xi^{2n}$, where constants $a_2, a_3, \ldots a_{n+1}$ are calculated from conditions of orthogonality of Φ_{n+1} to previous polynomials $\Phi_1, \Phi_2, \ldots \Phi_n$. Similarly, for odd polynomials we take $\Psi_1 = \xi$, and each subsequent polynomial Ψ_{n+1} is defined as $\Psi_{n+1} = \Psi_1 + a_2\Psi_2 + a_3\Psi_3 + \ldots a_n\Psi_n + a_{n+1}\xi^{2n+1}$, where constants $a_2, a_3, \ldots a_{n+1}$ are calculated by means of conditions of orthogonality of Ψ_{n+1} to previous polynomials $\Psi_1, \Psi_2, \ldots \Psi_n$. Thus, we obtain the second odd polynomial $\Psi_2(\xi) = \xi - 11\xi^3/3$. Note the important property of the proposed basis, namely, near to a contact line the profiles of a thickness and flow rate are similar geometrically to corresponding profiles for a undisturbed flow, that is $h \sim (1-\xi^2)$ и $q \sim (1-\xi^2)^3$. Thus the contact angle varies according to a wave phase.

4. Plane waves

In the most simple case each basis consist of one element: $\Phi = 1$, $\Psi = \xi$, that is $\tilde{h} = H(x,t)$, $\tilde{q} = Q(x,t)$, $\tilde{m} = \xi M(x,t)$. It corresponds to a case of plane waves for which profile of wavy rivulet remains a parabola in any section. Let's project the equations (2) on correspondent polynomials. For this purpose the first and third equations (2) multiply on $\Phi_1 = 1$, and the second equation (2) multiply on $\Psi_1 = \xi$. Then the equations are integrated on a variable ξ (integrals are calculated analytically), and we obtain

$$\begin{aligned}
&\frac{\partial Q}{\partial t} + \frac{32}{33}\cdot\frac{\partial}{\partial x}\left(\frac{Q^2}{H}\right) = \frac{35}{8\,\mathrm{Re}_m}\left(H - \frac{Q}{H^2}\right) + \frac{We}{6}H\left(7\frac{\partial^3 H}{\partial x^3} - \frac{35}{2b^2}\frac{\partial H}{\partial x}\right),\\
&\frac{\partial M}{\partial t} + \frac{96}{143}\cdot\frac{\partial}{\partial x}\left(\frac{QM}{H}\right) = \frac{96}{143}\cdot\frac{M^2}{bH} - \frac{63}{8\,\mathrm{Re}_m}\frac{M}{H^2} - \frac{21}{4b}WeH\frac{\partial^2 H}{\partial x^2},\\
&\frac{\partial H}{\partial t} + \frac{24}{35}\frac{\partial Q}{\partial x} = 0.
\end{aligned} \tag{4}$$

The first and third equations (4) are very similar to the equations of model [15] for two-dimensional waves in a film, but differ in coefficients. Unlike a film flow, there are terms containing additional parameter b in (4).

Let's linearize the equations (4) concerning small perturbations, supposing $H = 1 + \tilde{H}$, $Q = 1 + \tilde{Q}$, where $\tilde{H} << 1$, $\tilde{Q} << 1$, $M << 1$. As a result we obtain

$$\begin{aligned}
&\frac{\partial \tilde{Q}}{\partial t} + \frac{32}{33}\left(2\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{H}}{\partial x}\right) = \frac{35}{8\,\mathrm{Re}_m}\left(3\tilde{H} - \tilde{Q}\right) + \frac{We}{6}\left(7\frac{\partial^3 \tilde{H}}{\partial x^3} - \frac{35}{2b^2}\frac{\partial \tilde{H}}{\partial x}\right),\\
&\frac{\partial M}{\partial t} + \frac{96}{143}\frac{\partial M}{\partial x} = -\frac{21\cdot We}{4b}\frac{\partial^2 \tilde{H}}{\partial x^2} - \frac{63}{8\,\mathrm{Re}_m}M,\\
&\frac{\partial \tilde{H}}{\partial t} + \frac{24}{35}\frac{\partial \tilde{Q}}{\partial x} = 0.
\end{aligned} \tag{5}$$

4.1. Dispersion relations

Let's present perturbations in the form of a traveling wave: $\tilde{H} = H_a \exp(ik(x - Ct) + \beta t)$, $\tilde{Q} = Q_a \exp(ik(x - Ct) + \beta t)$, $M = M_a \exp(ik(x - Ct) + \beta t)$. Here, H_a, Q_a, M_a. are amplitudes, k is a wavenumber, C is a wave phase velocity, β is a time increment of the wave. Substituting it in the equations (5), after some transformations yields system of two dispersion relations for β and C:

$$\left(\beta Re_m+\frac{35}{16}\right)^2=\left(k\,Re_m\right)^2\left(\left(C-\frac{32}{33}\right)^2-E\right)+\left(\frac{35}{16}\right)^2,$$
$$\left(\beta Re_m+\frac{35}{16}\right)\left(C-\frac{32}{33}\right)=\frac{157}{66}. \tag{6}$$

Here, $E=\frac{41}{35}\left(\frac{16}{33}\right)^2+\frac{2We}{b^2}+\frac{4We}{5}k^2$. Distinction between dispersion relations for waves in a rivulet and in a film consists not only in coefficients of (6) but also available additional parameter *b*. The system (6) the same as in case of a film flow, is easily reduced to one quadratic equation with respect to $Y=\left(\beta\,\mathrm{Re}_m+\frac{35}{16}\right)^2$. Having the value of *Y* we obtain $\beta=\frac{1}{\mathrm{Re}_m}\left(-\frac{35}{16}\pm\sqrt{Y}\right)$ $C=\frac{1}{66}\left(64\pm\frac{157}{\sqrt{Y}}\right)$. Here, sign "plus" corresponds to unstable mode, and the sign "minus" corresponds to stable mode (in the same way, as for waves in a film flow). For unstable mode $C > 0$ and the perturbations are growing ($\beta > 0$) in some range of a wavenumber. For stable mode $C < 0$, and the perturbations are damping at any wavenumber ($\beta < 0$).

4.2. Stability criterion of rivulet flow

In an asymptotics $\beta\,\mathrm{Re}_m << 1$ (at $k\to 0$) from the second equation (6) we obtain $C=72/35$. Then from the first equation (6) we find $\beta=\frac{8k^2\,\mathrm{Re}_m}{35}\left(A-\frac{4We}{5}k^2\right)$, where $A=\frac{64\cdot 79}{121\cdot 35}-\frac{2We}{b^2}$. From here it follows that long-wave disturbances can be unstable if $A > 0$. Thus a criterion of instability can be written as a condition for the rivulet half-width $b>\sqrt{\frac{35\cdot 121\cdot We}{32\cdot 79}}\approx\sqrt{1.6752\cdot We}$. This condition been applied to dimensional rivulet half-width is $b_{\dim}>\left(\frac{3\nu^2}{g}\right)^{1/3}\sqrt{\frac{1.6752\cdot(3Fi)^{1/3}}{\mathrm{Re}_m}}$. Thus, at $b<b_{\min}$ the rivulet flow is stable with respect to 2D disturbances of any wavelength.

Calculated dispersion dependences are shown in figure 2 for Re_m = 25 and various values of the dimensionless parameter *b*. From figure it is clear that reducing of parameter *b* leads to decreasing of the increment, and increasing of the phase velocity. Figure 3 shows dependence of increment on the wave number for various values of Reynolds number. From figure it is clear, that the range of instability on a wave number extends with increase of Re_m. The maximum of increment also increases with growth of Re_m. Figure 4 shows the dependence of increment on a wave number for Re_m = 10 and

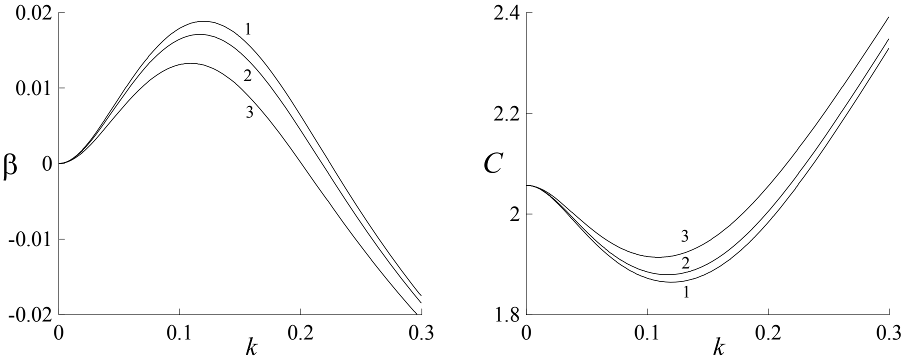

Figure 2. Increment β and phase velocity *C* of plane waves at Re_m = 25 for various values of dimensionless parameter *b*: 1− 50, 2− 25, 3− 15 (water, $Fi^{1/3}$=3270).

various values of Kapitza number. From figure it is clear that area of instability and the maximum of increment decrease with parameter $Fi^{1/3}$ growth.

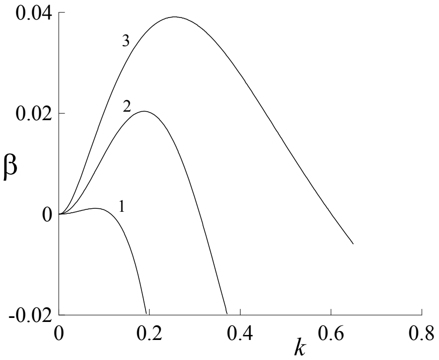

Figure 3. Dependence of increment of plane waves on a wavenumber for b = 10, $Fi^{1/3}$=300 and various values of Reynolds number Re_m: 1− 5, 2− 10, 3− 20.

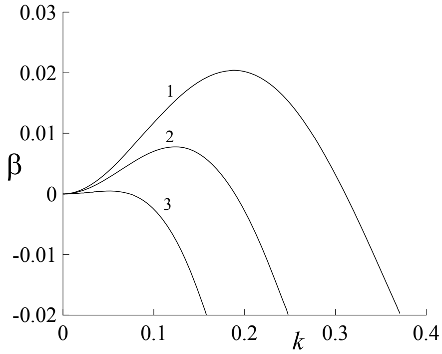

Figure 4. Dependence of increment of plane waves on a wavenumber for Re_m = 10, b = 10 and various values of parameter $Fi^{1/3}$: 1− 300, 2− 600, 3− 1200.

5. Conclusions

The 3D wavy rivulet flow has been studied theoretically. The approach reducing a problem to 2D one has been proposed on the basis of weighted residual method. The equations of the 2D rivulet flow have been deduced using projections of initial equations on the system of orthogonal polynomials. The system of equations for plane waves in the vertical rivulet has been deduced, and dispersion relations have been obtained. The stability criterion of rivulet flow has been deduced, and the analysis of influence of dimensionless parameters on dispersion dependences has been carried out.

Acknowledgments

The work was supported by the Russian Science Foundation (grant No. 14-29-00093).

References

[1] Towell G D and Rothfeld L B 1966 *AIChE J.* **12** 972-980
[2] Wilson S K and Duffy B R 1998 *Phys. Fluids* **10** 13-22
[3] Paterson C, Wilson S K and Duffy B R 2013 *Eur. J. Mech*. B **41** 94-108
[4] Perazzo C A and Gratton J 2004 *J. Fluid Mech*. **507** 367-379
[5] Tanasijczuk A J, Perazzo C A and Gratton J 2010 *Eur. J. Mech.* B **29** 465-471
[6] Davis S H 1980 *J. Fluid Mech*. **98** 225-242.
[7] Weiland R H and Davis S H 1981 *J. Fluid Mech*. **107** 261-280
[8] Young G W and Davis S H 1987 *J. Fluid Mech*. **176** 1-31
[9] Diez J, Gonzalez A and Kondic L 2012 *Phys. Fluids* **24** 032104
[10] Benilov E S 2009 *J. Fluid Mech.* **636** 455-474
[11] Alekseenko S V, Markovich D M and Shtork S I 1996 *Phys. Fluids* **8** 3288-3299
[12] Alekseenko S V, Bobylev A V and Markovich D M 2008 *J. Engng. Thermophys.* **17** 259-272.
[13] Alekseenko S V, Antipin V A, Bobylev A V and Markovich D M 2007 *Exp. Fluids* **43**, 197-207
[14] Alekseenko S V, Bobylev A V, Guzanov V V, Markovich D M and Kharlamov S M 2010 *Thermophys. Aeromech*. **17** 371-384
[15] Alekseenko S V, Aktershev S P, Bobylev A V, Kharlamov S M and Markovich D M 2015 *J. Fluid Mech.* **770** 350-373
[16] Demekhin E A and Shkadov V Ya 1984 *Fluid Dynamics* **19** 689-695