

A numerical analysis to evaluate Betz's Law for vertical axis wind turbines

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Abstract. The upper limit for the energy conversion rate of horizontal axis wind turbines (HAWT) is known as the Betz limit. Often this limit is also applied to vertical axis wind turbines (VAWT). However, a literature review reveals that early analytical and recent numerical approaches predicted values for the maximum power output of VAWTs close to or even higher than the Betz limit. Thus, it can be questioned whether the application of Betz's Law to VAWTs is justified.

To answer this question, the current approach combines a free vortex model with a 2D inviscid panel code to represent the flow field of a generic VAWT. To ensure the validity of the model, an active blade pitch control system is used to avoid flow separation. An optimal pitch curve avoiding flow separation is determined for one specific turbine configuration by applying an evolutionary algorithm. The analysis yields a net power output that is slightly ($\approx 6\%$) above the Betz limit. Besides the numerical result of an increased energy conversion rate, especially the identification of two physical power increasing mechanisms shows, that the application of Betz's Law to VAWTs is not justified.

1. Introduction

Increased environmental awareness and the limitation of fossil fuels has led to intensive research in the field of renewable energy. In the sector of wind energy, current systems can be classified into horizontal (HAWTs) and vertical axis wind turbines (VAWTs). For HAWTs the physical upper limit for the energy conversion rate is known as the Betz limit [1]. Often, this limit is also applied to VAWTs. However, a closer look on the flow physics of both systems raises the question whether this application is justified.

The Betz limit is derived on the basis of a uniform steady axial inviscid incompressible flow through an ideally thin rotor with an infinite number of blades. Energy extraction is assumed to take place in a single plane. For HAWTs this approximation is appropriate. For VAWTs the axis of rotation is perpendicular to the inflow such that energy conversion takes place along a curved surface and therefore is more spatially pronounced. Consequently, a meaningful representation of a VAWT rotor by an infinitely thin rotor plane is to be discussed.

Moreover, this assumption requires the comparison of the wake structure of both systems. For HAWTs the wake is convected downstream and dominated by its helical character. Hence, the direct interaction between the shed vortices and the rotor blades is comparably small. Therefore, neglecting this direct blade-wake interaction is justified in the derivation of Betz's Law. However, for VAWTs this effect should be taken into account. Especially on the rotor leeside



there is a strong blade-wake interaction. The resulting non-uniform loading of the rotor is in contrast to the assumptions of Betz's law. Consequently, both physical differences due to the spatial character of the rotor indicate that Betz's Law might not be valid for VAWTs.

Besides these theoretical considerations, previous numerical studies support this hypothesis. In case of VAWTs, first approaches to estimate the power output were presented by Shankar [2] and Strickland [3]. In comparison to experimental data, their momentum based models suffered from an overprediction of the power output. Slight improvements could be identified by Strickland [3] using a multiple streamtube approach, whereby Lapin [4] proposed to represent the rotor as two halves. Using an analytical double actuator disk approach, Newman [5] determined a maximum power coefficient of 0.64, which is 8% above the classical Betz value. For a very large number of discs he derived that the maximum power coefficient tends to a value of 0.66, i.e. 12% above the Betz limit [6]. Loth and McCoy [7] extended the analytical double actuator disk approach by replacing the two straight actuator disks by two semi-cylindrical actuators. For the velocity distribution along the cylinder halves a cosine-type formulation was selected. This closed-form approach led to a maximum power coefficient of 0.617, hence 4% above the Betz limit. Furthermore, considering the blade incident angle in each position to be influenced by streamline curvature, blade camber, and the attachment point of the blade Loth and McCoy derived a value of 0.610 as the maximum power coefficient, which is 3% above the Betz limit. To overcome the drawbacks of streamtube models, e.g. their assumption of a one dimensional flow Madsen et al. [8] used an extended actuator cylinder flow model. For the ideal case of an uniform rotor loading they determined a power coefficient, which is 5% above the classical Betz value.

All of these numerical approaches have in common that the predicted maximum power output for VAWTs exceeds the Betz limit. However, will a numerical analysis based on a more detailed representation of the flow physics confirm these results? The fact that corresponding numerical models like U2DiVA or CACTUS predict values close to the Betz limit for specific turbine configurations [9] gives a hint regarding this question. The approach presented in this paper clearly proves that the maximum power output of VAWTs is capable of exceeding the Betz limit.

2. Numerical Method

To determine the flow field, a free vortex model is combined with a 2D inviscid panel code, which is used for blade representation. Therefore, three-dimensional or viscous effects will be neglected. Note that the same assumptions are valid for Betz's Law. Though, special attention is needed regarding flow separation. Initially, the optimal turbine configuration in terms of maximum energy conversion is unknown. Since the inviscid approach cannot account for separation an overprediction of power output might be possible. To actively avoid flow separation an active blade pitch control is used. It enables to control the physical angle of attack for each blade as a function of the azimuthal angle. Furthermore, active blade pitch is intended to increase the energy conversion rate. To avoid flow separation and simultaneously increase the energy conversion rate, the optimal pitch angle at each azimuthal blade position has to be determined. Selecting an optimal pitch angle curve is challenging due to the fact that a VAWTs exhibits a strong blade wake interaction. To tackle this problem, a multiobjective evolutionary optimization algorithm (MOEA) introduced by Marneft et al. [10] is used. Its high robustness in terms of optimization of complex and noisy functions makes it an effective choice for this problem. The maximization of the mean power coefficient \bar{c}_p defines the main objective. To ensure the integrity of the utilized models, an additional objective concerning the angle of attack is applied. At each azimuthal blade position this second objective complies with the constraint that the physical angle of attack has to be in the range of $\pm 10^\circ$, such that the flow can be assumed attached.

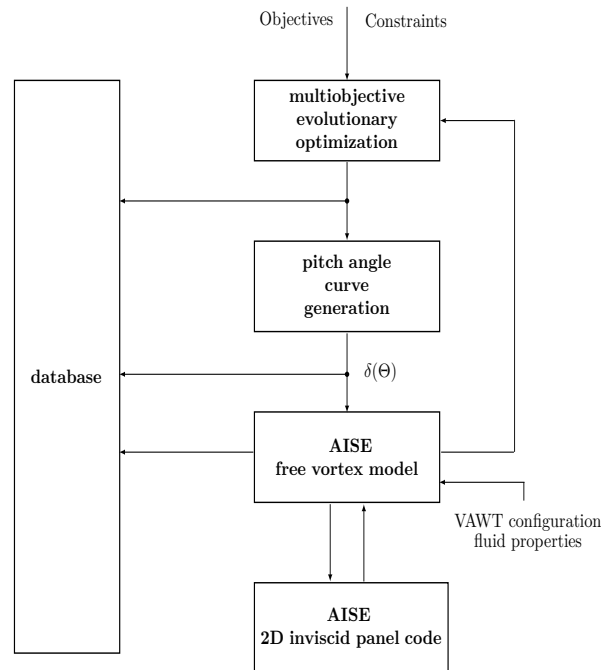


Figure 1. Numerical setup

The structure of the coupling between MOEA and the flow solver, which is named AISE (Aeroelastic Integrated Simulation Environment) is shown in Figure 1. The module used for the generation of a spline based pitch angle curve function $\delta(\Theta)$ represents the link between MOEA and AISE. For each pitch angle curve the mean power coefficient \bar{c}_p is evaluated using AISE. Finally, the mean power coefficient and the distribution of the angle of attack $\alpha(\Theta)$ are inserted into MOEA.

Since the presented approach makes use of an active blade pitch control system, the auxiliary power required for the blade actuation, which lowers the net power output should be taken into account. In the current approach, only the part that is due to the adjustment against the aerodynamic pitching moment of the blade is considered. The part that results from the blade inertia highly depends on the structural blade concept and mechanical supporting point. Thus, this part is neglected since it is unknown until the construction of the turbine.

3. Results

The following section starts with the validation of the used 2D inviscid flow solver AISE. Subsequently, the results of the applied optimization strategy will be presented and discussed. Finally, the identification of the characteristic power output increasing flow mechanisms provides an indication that the application of Betz's Law to VAWTs is not justified.

3.1. Code validation

First, the code was tested against selected scenarios to verify AISE's capability of reproducing the major 2D airfoil flow physics. Each test case verifies the functionality of a different part of the model and evaluates its accuracy. In addition, this verification strategy is characterized by an increasing flow field complexity for each further test case. Validated numerical and experimental results from the literature are used as reference data.

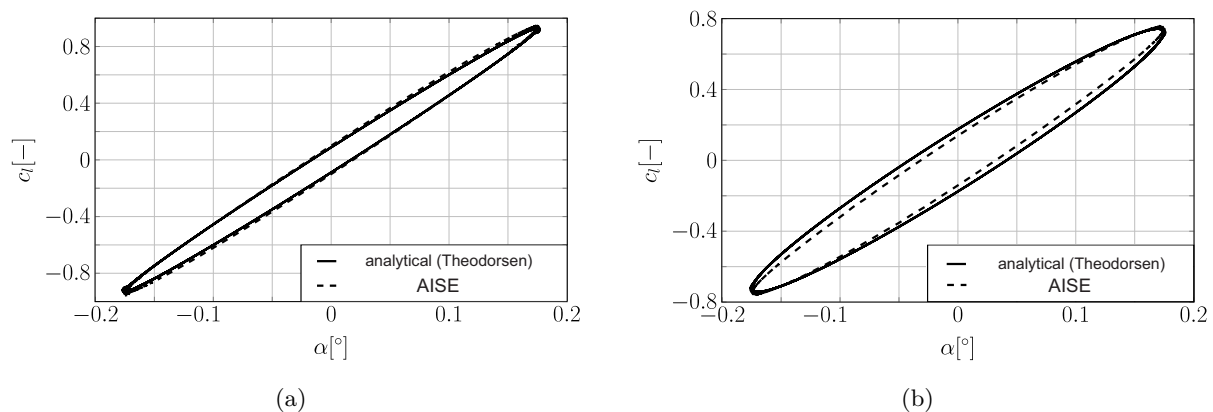


Figure 2. Lift hysteresis computed with 2D-AISE, comparison with analytical results of Theodorsen, where a) $k=0.2$, b) $k=0.4$

The first test case considers the numerical prediction of the inviscid static lift polar of a NACA0015 profile. The lift polar predicted by AISE matches the analytical solution with a very good agreement. Hence, the correct implementation of the panel method is verified. The correct representation of the blade-vortex interaction and the formulation of the time dependent potential has been verified using the analytical results of Theodorsen [11] for different generic blade pitching motions. For two reduced frequencies ($k=[0.2, 0.4]$) Figure 2 shows the predicted lift slope of AISE and the analytical solution of Theodorsen.

In the field of VAWT simulation the numerical results of a 2D inviscid flow of Ferreira et al. [9] for a two-bladed turbine (NACA0015, $\sigma=0.1$, $\lambda=4.5$) were used as a reference. Considering the distribution of the normal (c_n) and tangential force coefficient (c_t) in Figure 3 the code prediction matches the reference with a convincing agreement. This result verifies the code's capability in the field of 2D inviscid VAWT simulations.

3.2. Results of the applied optimization strategy

For the current approach of an optimal energy conversion rate, an inviscid flow over a three-bladed turbine configuration (NACA0015, $\sigma=0.08$, $\lambda=7$) is considered. This configuration was chosen at random as the presented work aims not at the definition of an overall optimal turbine energy conversion rate. The predicted power output for the unpitched turbine ($\delta(\Theta=[0, 2\pi])=0$) is clearly ($\approx 18\%$) below the classical Betz limit. While the rotors windward side yields a significant amount of the overall torque production (Figure 4b) the rotors leeside barely contributes to it. The power drop on the rotors leeside is a result of a non-optimal angle of attack of the blades (Figure 4a). Therefore, the pitch angle control system has to modify the angle of attack in terms of an optimal energy conversion rate.

Using the optimized pitch angle curve depicted in Figure 4a a converged power output, which is $\approx 12\%$ higher than that of the Betz limit, is determined.

This result, however, needs to be revised carefully regarding uncertainties and errors inherent to numerical methods. If the lower and upper bound of the overall error are known, it is possible to determine the quality of the numerical result. For the current case, however, it is appropriate to consider the worst case. That is, only the values of the upper bounds are required since their sum accounts for the maximum possible deviation due to the error of the numerical method. Reducing the predicted power coefficient by the overall maximum error yields the minimal power coefficient, derived in terms of an error correction. If this error-corrected power

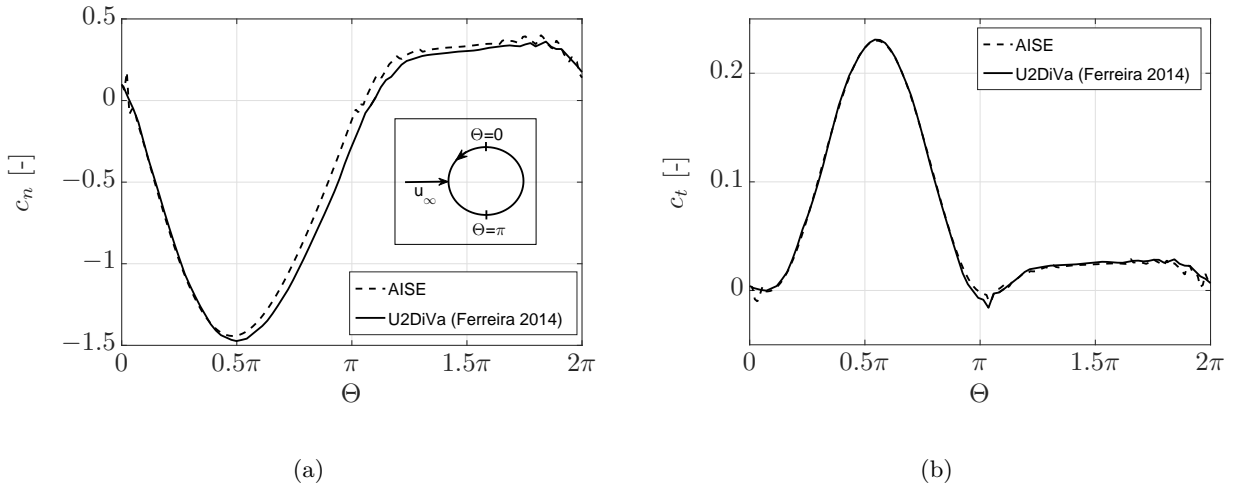


Figure 3. Comparison of normal (a) and tangential (b) force coefficient for a 2D inviscid VAWT flow (NACA0015, $\sigma=0.1$, $\lambda=4.5$)

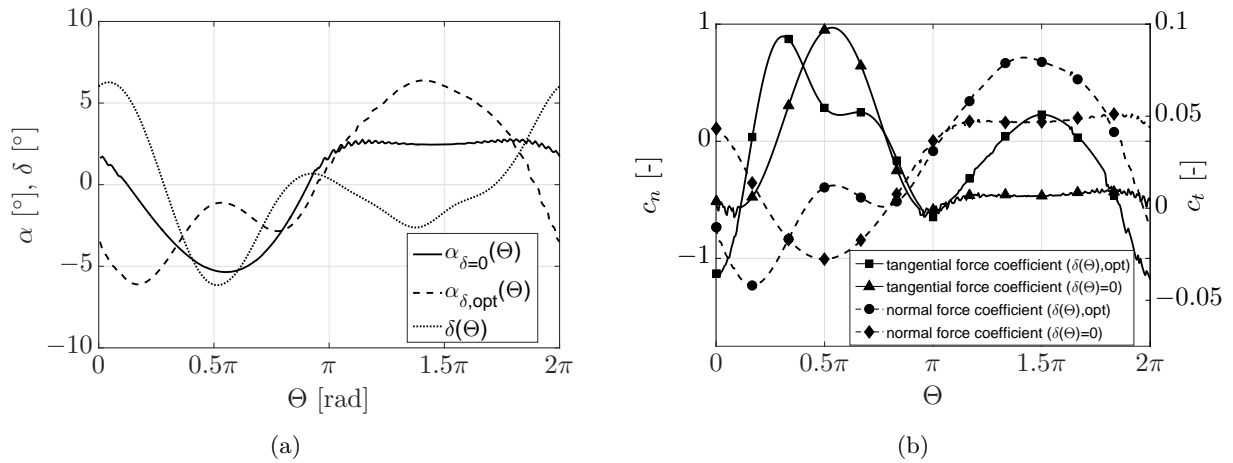


Figure 4. Blade angle of attack for unpitched $\alpha_{\delta=0}(\Theta)$ and optimized $\alpha_{\delta,opt}(\Theta)$ case, as well as pitch angle curve $\delta(\Theta)$ (a); normal and the tangential force coefficient for one blade (b)

coefficient $\overline{c_{p,\delta,opt,corr}}$ is still higher than the Betz limit, a statement on a higher power coefficient of a VAWT than the Betz Limit can be made. Therefore, as shown below the sources of errors and uncertainties of the numerical approach have to be discussed and quantified before a conclusion on the upper energy conversion rate using the optimal pitch curve can be drawn.

The following section presents the results of the conducted error and uncertainty analysis. After a short outline of the different error sources the focus is on the vortex convection model. In Figure 5a different sources of errors and uncertainties are listed, sorted by their upper error bound. Each error is expressed in terms of a percentage reduction of the original mean power coefficient $\overline{c_{p,\delta,opt}}$. Here, minor errors result from the time stepping scheme, the temporal discretization and the standard deviation of the numerical models in case of noisy input. The major part of the error in the mean power coefficient is due to the used viscous core radius model [12].

Another major source of error is linked to the free vortex model, which is utilized for the convection of the shed vortices. It ensures a high level of freedom concerning the wakes formation

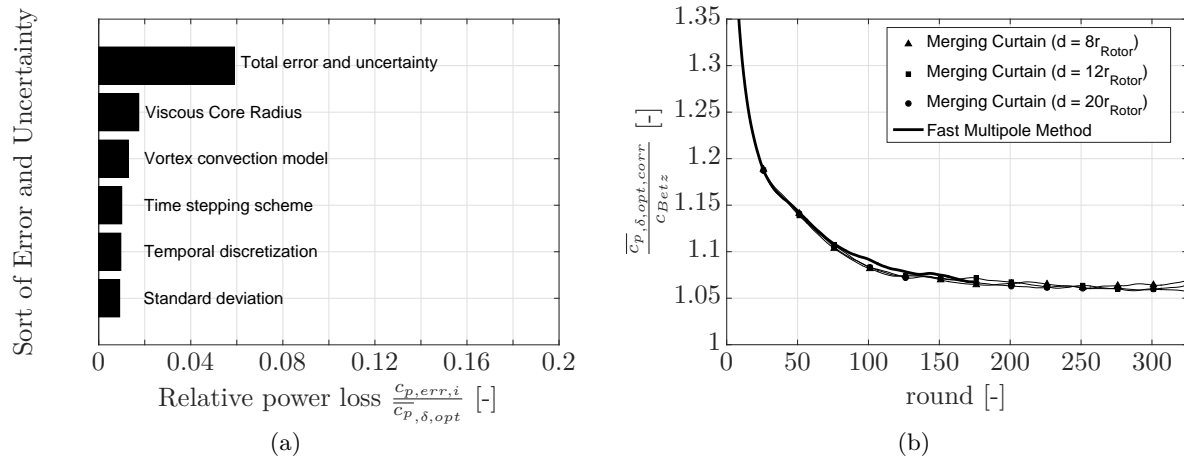


Figure 5. Sources of error and uncertainty (a) and development of mean power coefficient over simulation rounds (b)

in time. However, the computational costs of this approach rapidly increase because of the underlying n-body problem characteristic. Its computational costs are proportional to the squared number of vortices $n_{Vortices}^2$. In terms of optimization, it is necessary to limit the computational costs since a high number of pitch angle curves have to be evaluated. Hence, the merging curtain approach of McIntosh and Babinsky [13] has been used. Here, vortices passing a virtual line downstream of the turbine are merged into a new vortex, while nearly preserving the overall vorticity. Consequently, by using this technique the computational costs are reduced since it depends on the number of vortices.

However, the position of the merging curtain influences the wake formation. Hence, the upstream influence of the wake depends on the position of the merging curtain. Therefore, the influence of the merging curtain's axial distance on the power coefficient has been investigated in Figure 5b. To quantify the error the results are compared to those, when a Fast Multipole Method (FMM) [14] is used to calculate the wake formation. This method renders the need for the merging of vortices redundant due to its reduced computational costs ($\sim n_{vortices}$). Therefore, its findings were used as a reference. However, despite the linear dependence its efficiency is limited in the sense that many vortices could exceed the available computational resources. Consequently, its application in the optimization scheme is not advantageous.

It is shown in Figure 5b that three axial distances of the merging curtain ($d=[8, 12, 20] \cdot r_{Rotor}$) were used to calculate the mean power coefficients temporal development. The results agree well with the findings of the FMM. In case of smaller axial distances of the merging curtain an overprediction of the power coefficient has been observed. For the three cases shown in Figure 5b the error of the merging curtains application is approximately 1% in terms of the original predicted power coefficient $\bar{C}_{p,\delta,opt}$.

Considering the total overall error, the determined original mean power coefficient $\bar{C}_{p,\delta,opt}$ has to be reduced by about 6%. This value is the worst case, since only the upper bound of each error was used. In this case, the application of the optimal pitch angle curve yields a converged predicted power output $\bar{C}_{p,\delta,opt,corr}$, which is approximately 6% above the Betz limit (Figure 5b). This power coefficient is the minimal power coefficient derived in terms of an error correction. Since this minimal value exceeds the Betz limit it can be stated that the application of Betz's Law to VAWTs is not justified.

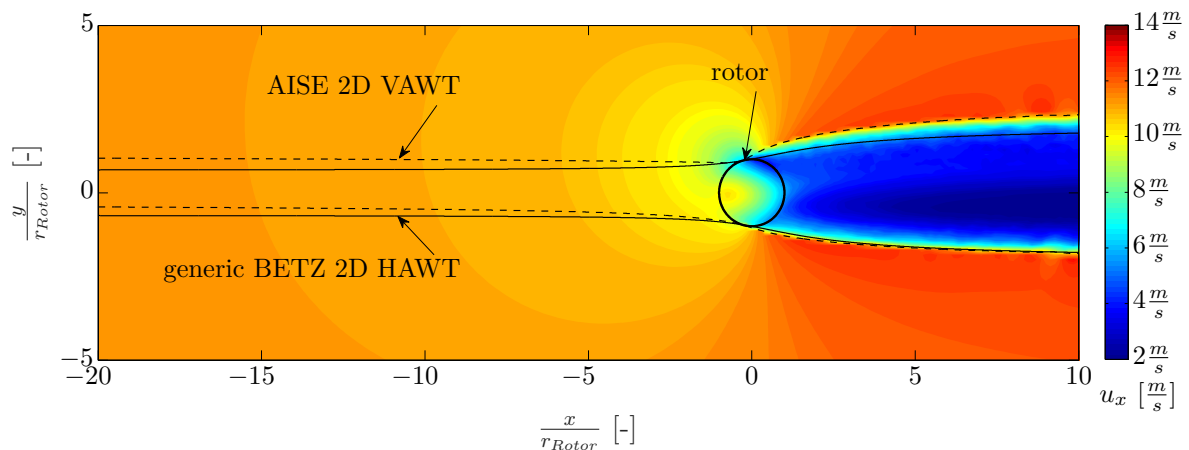


Figure 6. Time-averaged flow field of axial velocity u_x

3.3. Characteristic power output increasing flow mechanisms

However, the derived numerical value is not sufficient to explain the higher energy conversion rate. It is necessary to identify the characteristic flow mechanisms, which yield the increased energy conversion.

Two main mechanisms determine the higher power output. The main portion of the increased torque generation is caused by higher angles of attack. Especially on the rotor's leeside (Figure 4a), but also in the first third of its windward side, the magnitude of the angle of attack is increased compared to the case of the unpitched turbine. Consequently, the tangential force coefficient (Figure 4b) and as such the energy conversion in these sections of the revolution is higher.

Compared to the case of the unpitched turbine, the increased angle of attack on the rotor's leeside is only advantageous if at the same time a higher mass flux \dot{m} that leads to an increased energy flux $\dot{m}u_\infty^2$ can be exploited. To increase the energy flux, especially on the windward side a carefully selected spatial distribution of energy conversion is crucial. In case of the unpitched turbine the high energy conversion in the center of the rotor's windward side creates a deficit in kinetic energy. The deficit's downstream convection prevents energy conversion in the rotor's leeside. It is shown in Figure 6 that the determined pitch angle curve prevents the blockage of the rotor, which arises from the aforementioned deficit. This is achieved by reducing the magnitude of the angle of attack in the center of the rotor's windward side (Figure 4a). In this region nearly no reduction of the axial velocity is observed. Hence, the energy flux entering the rotor is increased compared to the case of the unpitched turbine. To quantify this increase the streamtube encompassing the flow through the rotor is considered. In case of the Betz limit, the inlet area A_{Inlet} of this streamtube equals $2/3$ of the rotor area A_{Rotor} . However, evaluating the time-averaged flow field for the case of the optimized pitch angle curve (Figure 6) yields an area ratio, which is approximately 5% larger compared to that of the Betz limit. This finding shows, that a major part of the increased energy conversion rate results from the increased energy flux. Consequently, the increased energy flux in combination with the increased angle of attack in the rotor's leeside enhances the overall potential in terms of energy conversion inside the turbine.

This discussion shows that energy conversion in case of a VAWT varies from that of a HAWT due to its more spatial character, as it takes place along a curved surface. This fundamental difference and the predicted power output above the Betz limit show that the application of Betz's Law to VAWTs is not valid.

4. Conclusion

To evaluate the applicability of the Betz Law in the field of vertical axis wind turbines (VAWT) an optimization approach has been presented, which is used to increase the energy conversion rate of a generic VAWT. Since the simple numerical computation is not sufficient to explain the higher energy conversion rate, the characteristic flow mechanisms also have been discussed.

The numerical method features a combination of a 2D inviscid panel code and a free vortex model. To ensure the model validity an active blade pitch control system has been applied to avoid flow separation. Using an evolutionary algorithm, an optimal pitch curve has been determined, which yields a power output that is approximately 6% higher than the Betz limit ($\approx 12\%$). This result has been carefully revised regarding uncertainties and errors inherent to the numerical method. The determined original mean power coefficient has to be reduced by about 6% such that the corrected power output is 6% above the Betz limit.

Two main mechanisms can be linked to the higher power output. First, an increase in angle of attack of the rotor's leeward side improves the overall torque production. Second, the observed increase in the energy flux in the leeward rotor half is linked to a reduction in the turbine's blockage, which is due to a decreased blade angle of attack in the center of the windward blade's revolution.

This discussion shows that energy conversion in case of a VAWT is different from that of a HAWT due to its more spatial character. This leads to a power output above the Betz limit for VAWTs.

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