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A mean-field analysis of the simple model of evolving open systems

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Abstract. A recently reported mechanism of letting evolving systems grow only if the interactions in it is moderately sparse is reviewed and examined. It is shown that the mean field analysis, which is known to give a good simple understanding for this transition from growing to non-growing phase but with about 30% difference in its position, is well improved by taking only mean field type information from the simulation of the original model. This supports the validity of the understanding for the mechanism of the transition, obtained from the mean field analysis. The transition stems from an essential balance of two effects: although having more interactions makes each node robust, it also increases the impact of the loss of a node.

1. Introduction

An important and universal feature of real complex systems, such as social, economic, engineering, ecological, and biological systems, is that those are evolving and open: in those systems, constituting elements are not fixed and the complexity emerges (at least persist) under successive introductions of new elements. Those systems sometimes grow, but also sometimes collapse. Therefore why and when, in general, we can have such evolving complex systems is a fundamental question.

In the previous works, we revisited this classical problem using our very simple model. It was found that the model gives either continuous growth or stagnation in system size, depending on the model's unique parameter: the average number of interactions per element m. By introducing a mean-field assumption, it was also found that this transition originates from an essential balance of two effects: although having more interactions makes each element robust, it also increases the impact of the loss of an element [1, 2]. This novel relation might be a origin of the moderately sparse network structure (~ 10 of average degree, independent of the size of the system [3]) ubiquitously found in real systems. In this paper, we verify the mean field model to figure out the origin of the difference between the previously estimated transition point $m \sim 13$ and the real transition point found in simulation, $m \sim 18.5$.

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2. Transition in a simple model of evolving open systems

2.1. The simple model of evolving open systems

In the model, the entire system consists of featureless nodes, connected by directed and weighted links. What the nodes represent can be chemical, gene, animal, individual, or other species, or other diverse kind of elements. The links can be also represent various kinds of interactions or influences among them. The influence of species j on species i is denoted by the weight of the link from node j to node i, a_{ij}. Each species has only one property, "fitness", which is determined by the simple sum of interactions incoming from other species in the system:

$$f_i = \sum_{j}^{incoming} a_{ij}. \tag{1}$$

Each species can survive as long as its fitness is larger than zero, otherwise that go extinct. This extinction process represents the system's intrinsic fast dynamics.

When all the species have positive fitness, we regard that the system is in a stable (persistent) state. In such situation, nothing will happen according to the system's intrinsic dynamics. So we can wait for arbitrary long time, until a new node is introduced by some other slow process. This time scale may corresponds to the evolutionary time scale for ecosystems and biological systems, developmental/learning time scale for coupled neurons, the time scale in which a new individual joins to social communities, etc. And to focus on the most fundamental aspect of evolving systems, we give each new node m random interactions with the resident species. Note that m, the number of links we give for each new species, is the only one model parameter. The following shows the complete description of the model process.

The Evolving Open Systems (EOS) Model

(0) (Create an initial state, for example, randomly.)

(1) Calculate the fitness for each nodes: $f_i = \sum_{j}^{incoming} a_{ij}$

(2) If the fitness of the nodes are all positive, go to (3). If not, delete the species of minimum (and therefore negative) fitness and then re-evaluate the persistence of the system:
   - (i) Delete the nodes
   - (ii) Delete the links connecting to and from that nodes.
   - (iii) Re-evaluate the extinction: go back to (1).

(3) A new node is added to the system.
   - (i) Add a new node, and give m new links from or to it.
   - (ii) The connecting nodes are chosen randomly from the resident nodes, with equal probability $1/N(t)$.
   - (iii) The direction of the link is also determined randomly with equal probability 0.5 for each direction.
   - (iv) The weights of the links are drawn from the standard normal distribution.

(4) Go back to (1) to simulate the (fast) dynamics of the new community.

2.2. The transition in growing behavior

In this model, both the introduction rule and the survival condition are neutral. Therefore it is not easy to foresee whether the system can grow under such process. Simulation results indeed show a fascinating answer: both can happen, depending on the only one parameter m, the number of interactions per species. The system can grow to infinitely large if the number of the interactions per species is in a moderate range ($5 \leq m \leq 18$), and, if not, it stays in a finite size (Table 1).

3. A mean-field model of the EOS model and its verification

3.1. A mean-field model

The structure of the emergent network is found to have no strong deviation from that of Erdős-Rényi random graph with an average number of links $\langle m \rangle \approx m$. Taking such a correlation-less structure as a support, the following mean field approximation was made to better understand the mechanism of the transition.

In our mean field treatment, we describe the system by the distribution function of the fitnesses of the species in it. The change in the fitness of each species in the original model arises from the addition or deletion of an incoming link. Although each such event changes both the network structure and the fitnesses, we here treat only the fitness distribution. Under the mean field treatment, such link change events act as a diffusion process to the fitness distribution function. More precisely, the diffusion is neutral for link addition events but that involves negative drift for link deletion events. Therefore, the process of inclusion of new species is represented by the diffusion process with negative drift and an addition of initial fitness distribution (positive half of a Gauss distribution). The extinction process of the species with non-positive fitness is modeled by making a cut of the fitness distribution function under zero after applying the random walk. The equilibrium fitness distribution is obtained as a fixed distribution against these processes. And once we get the equilibrium distribution numerically, the area in the negative fitness part of the distribution after applying the random walk process gives the probability E of a resident species going extinct during one link addition/deletion event. The average number of nodes that go extinct directly because of the introduction of the new node is $mE/2$. And these extinctions may also trigger further extinctions. If we approximate the graph by a loop-less random network, the expectation value of the total number of extinctions per addition of one node into a system in which all species have m interactions can be calculated as an infinite geometric series as,

$$N_E = \sum_n \left(\frac{mE}{2}\right)^n = \frac{mE}{2 - mE}. \tag{2}$$

Because $N_E = 1$ means that the number of extinctions is equal to the number of additions in the long-term average, $mE = 1$ corresponds to the transition point. This criteria give us the correct transition behavior, i. e. finite phase to divergent phase as m increases. However, the predicted transition point from this mean field model, $m_c \sim 13$, is smaller than that obtained from the simulation by about 30%.

Table 1. The phase diagram of the EOS model. The transition at between $m = 18$ and 19 The transition at between $m = 4$ and 5 simply stems from the sparseness of the network (i.e. percolation) and hence is not re-examined in this study.

$m:$	≤ 4	$5 \sim 18$	≥ 19
$N(t):$	finite	diverging	finite

3.2. Verification of the mean-field model

The possible cause of the difference between the real transition point and that of mean field analysis are two: neglecting any structural correlation and taking the tree structure for the calculation of N_E, and the error in the estimation of the fitness function and E. To verify which of these two contributions is dominant, we here evaluate the extinction probability directly from the simulation. The extinction probability in an emergent network can be calculated as

$$\begin{aligned} E_i &= \frac{1}{N}\sum_i^N \Phi(-f_i), \\ E_e &= \frac{\#_{link}(a_{ij} > f_i)}{\#_{link}}, \end{aligned} \tag{3}$$

where E_i and E_e are the extinction probability during the link addition and deletion events. f_i and a_{ij} are the fitness of the ith node and the weight of the link from jth node to ith node in the emergent network, and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\, dt \tag{4}$$

is the cumulative distribution of the standard normal distribution. Substituting E by these "empirically" obtained extinction probabilities, the same mean-field treatment gives the average number of extinctions as

$$\langle N_E\rangle = \frac{mE_i}{2}\left[1 + \frac{\langle m\rangle E_e}{2} + \left(\frac{\langle m\rangle E_e}{2}\right)^2 + \cdots\right] = \frac{mE_i}{2 - \langle m\rangle E_e}, \tag{5}$$

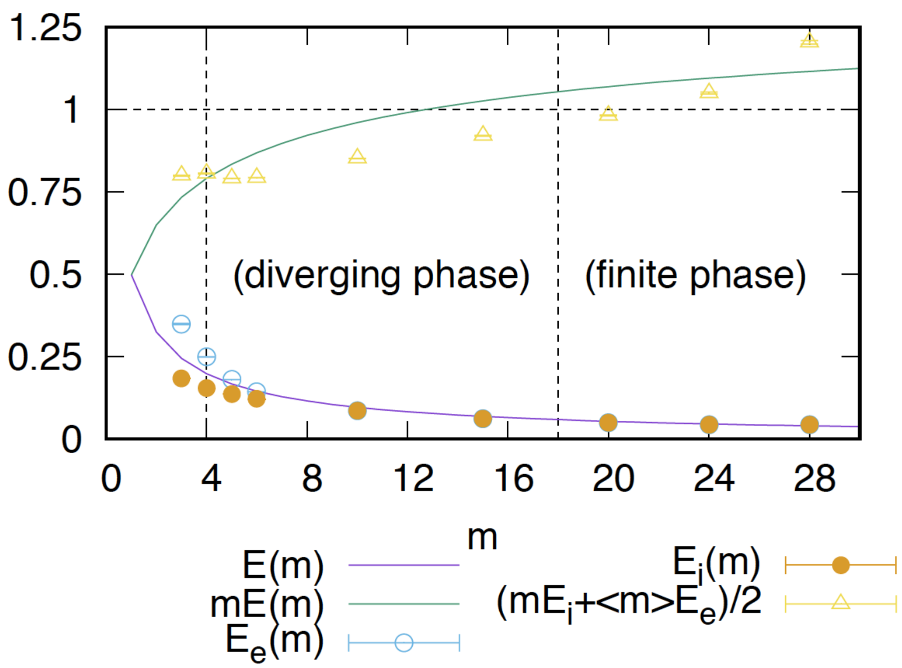

Figure 1. Estimation of the transition point, using the semi-analytical mean field theory, ($mE = 1$) [2], and the mean field treatment with empirically obtained extinction probabilities (Eq. (5)).

where $\langle m \rangle$ is the average degree of the emergent network. Therefore this empirical test predicts the transition at

$$\frac{mE_i + \langle m \rangle E_e}{2} = 1. \tag{6}$$

As shown in Figure 1, while the extinction probability in the original self-consistent mean field theory is over estimated, the empirically obtained average extinction probability predicts the transition point almost perfectly, which means that the correction due to the correlation among the interaction coefficients, the loop structure (i.e. non-zero clustering coefficient), and so on of the emergent network is very small. This result suggests the validity of basic understanding of the transition mechanism obtained from the mean field picture: having more links makes each node more robust, but the net impact of loosing a node increases hence in total the robustness of the entire system decreases.

4. Conclusions

We verified the mean field approximation for the simple EOS model, using the information of emergent networks in the original model. The good agreement in the transition point, without taking any structural information into account, support the validity of the mean field treatment. This confirms that the transition essentially stems from an essential balance of two effects: although having more interactions makes each node robust, it also increases the impact of the loss of a node.

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