On the undamped vibration absorber with cubic stiffness characteristics

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Abstract. In order to improve the performance of a vibration absorber, a nonlinear spring can be used on purpose. This paper presents an analytical insight on the characteristics of an undamped nonlinear vibration absorber when it is attached to a linear spring-mass-damper oscillator. In particular, the nonlinear attachment is modelled as a Duffing's oscillator with a spring characteristics having a linear positive stiffness term plus a cubic stiffness term. The effects of the nonlinearity, mass ratio and frequency ratio are investigated based on an approximate analytical formulation of the amplitude-frequency equation. Comparisons to the linear case are shown in terms of the frequency response curves. The nonlinear absorber seems to show an improved robustness to mistuning respect to the corresponding linear device. However, such a better robustness may be limited by some instability of the expected harmonic response.

1. Introduction

Rather than avoiding nonlinear effects in the dynamics of mechanical systems, nonlinear elements, such as stiffness and damping, can be added on purpose to improve the performance of mechanical oscillators. One of this application is the vibration absorber, which consists of an auxiliary mass mounted on primary mass through a stiffness and a damper element [1]. The aim is to reduce the vibration of the primary mass. In its basic configuration, the combined system may be treated as a two degree-of-freedom oscillating system, where the primary mass is excited by a harmonic force and the auxiliary mass is suspended by a nonlinear spring.

The dynamics occurring in a two degree-of-freedom system with a cubic stiffness spring between the primary system and the absorber was studied in [2, 3] as a way to absorb the energy from the main structure. The complicated dynamics involved in such a coupled system were explored in [4] and the effect of nonlinearity in either the spring and damper element was investigated in [5].

Explorations on the performance of the nonlinear absorber [6] showed the presence of quasi-periodic oscillations and detached resonance curves limiting the effectiveness of the device. For a small mass ratio between the mass of the absorber and that of the main structure, the effect of the key parameters was investigated in [7], and the characteristics of a vibration neutralizer with cubic stiffness nonlinearity were theoretically studied and experimentally validated in [8]. A recent paper on some interesting characteristics of the nonlinear vibration absorber and its tuning is presented in [9].

In this paper, the author presents a theoretical study whose aim is to investigate the fundamental phenomena involved in the dynamics of a coupled linear and nonlinear oscillator used to mitigating the vibration at a particular frequency. Analytical expressions for the amplitude-frequency equations are used to allow for an insight on the dynamic behavior of the system. The performance of the nonlinear system are compared to those of the equivalent linear system in terms of the frequency response curves.

2. Description of the system and equations of motion

The system of interest in this work is shown in figure 1, which is the non-dimensional undamped scheme of the system studied in [7]. A unit primary mass is suspended through a unit linear spring and excited by a harmonic force of unit amplitude and non-dimensional frequency Ω, as a function of the non-dimensional time τ. The absorber has a mass of μ and is attached to the primary mass through a nonlinear spring, whose restoring force consists of linear stiffness term $\mu\Omega_0^2$ and a cubic stiffness term $\mu\gamma$. For the physical interpretation of the linear and nonlinear stiffness term per unit mass ratio, the interested reader is redirected to [7].The non-dimensional absolute displacement of the primary mass is denoted by y_s and the non-dimensional relative displacement between the primary mass and that of the absorber is denoted by w.

The non-dimensional equations of motion of the system depicted in figure 1 are reported below as

$$y_s'' + y_s + \mu\Omega_0^2 w + \mu\gamma w^3 = \cos(\Omega\tau) \tag{1}$$

$$\mu y_s'' - \mu w'' - \mu\Omega_0^2 w - \mu\gamma w^3 = 0 \tag{2}$$

which can be also conveniently put in matrix form as

$$\begin{bmatrix} 1+\mu & -\mu \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_s \\ w \end{bmatrix}'' + \begin{bmatrix} 1 & 0 \\ 0 & \Omega_0^2 \end{bmatrix}\begin{bmatrix} y_s \\ w \end{bmatrix} = \begin{bmatrix} \cos(\Omega\tau) \\ -\gamma w^3 \end{bmatrix} \tag{3}$$

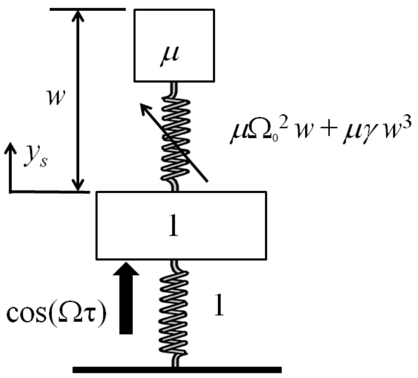

Figure 1. Non-dimensional form of the nonlinear system considered in [7].

3. Insight into the frequency response curve

By assuming that the system responds predominantly at the frequency of excitation Ω, the non-dimensional absolute displacement of the primary mass and the non-dimensional relative displacement between the two masses may be set to $y_s = Y_s \cos(\Omega\tau + \varphi_s)$ and $w = W\cos(\Omega\tau + \varphi)$, respectively. If these expressions are substituted back into equations (1) and (2), and the coefficients of the terms $\cos(\Omega\tau)$ and $\sin(\Omega\tau)$ are equated correspondingly, equations in the four unknowns Y_s, φ_s, W and φ may be obtained. After some mathematical manipulation, it turns out that the equations for the amplitudes are independent to those of the phases, so that the following two amplitude-frequency equations can be determined

$$a\gamma^2 W^6 + b\gamma W^4 + cW^2 + d = 0 \tag{4}$$

$$Y_s^2 = \frac{1 - W^4 g - W^2 h}{e} \tag{5}$$

where a, b, c, d, e, g, h are functions of the system parameters and excitation frequency as reported below

$$a = \frac{9}{16} \tag{6}$$

$$b = \frac{3}{2}\left(\Omega_0^2 - \Omega^2\right) + \Omega^4 \frac{g}{e} \tag{7}$$

$$c = \left(\Omega_0^2 - \Omega^2\right)^2 + \Omega^4 \frac{h}{e} \tag{8}$$

$$d = -\frac{\Omega^4}{e} \tag{9}$$

$$e = \left(1 - \Omega^2\left(1 + \mu\right)\right)^2 \tag{10}$$

$$g = -\frac{3}{2}\mu\left(1 - \Omega^2\left(1 + \mu\right)\right) \tag{11}$$

$$h = \Omega^4 \mu^2 - 2\mu\left(1 - \Omega^2\left(1 + \mu\right)\right)\left(\Omega_0^2 - \Omega^2\right) \tag{12}$$

Since the amplitude-frequency relation given by equation (4) is cubic in W^2, a multi-valued response is expected in the frequency response curve at particular frequencies.

The frequency response curve of the non-dimensional amplitude displacement of the primary mass is determined by solving for the roots of the polynomial equation (4) and then substituting the solutions into equation (5). This is shown in figures 2-7 for different values of the mass ratio μ, frequency ratio Ω_0 and nonlinearity γ. The stability of the harmonic solution is determined by applying the approach described in [4]. Unstable solutions are denoted by dashed lines in the frequency response curve reported in this paper. The frequency response function of the corresponding linear system is also plotted as a thin line for reference in figures 2-7.

The undamped resonance frequencies of the linear system, which correspond to the natural frequencies ω_n of the system given in equation (3) with $\gamma = 0$, can be determined by solving the corresponding eigenvalue problem, yielding to calculate the solutions of the following equation

$$\det\left(\begin{bmatrix} 1+\mu & -\mu \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \Omega_0^2 \end{bmatrix} - \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \tag{13}$$

which gives

$$\omega_n^2 = \frac{1}{2}\left(1 + \Omega_0^2\left(1+\mu\right) \pm \left(\left(1 + \Omega_0^2\left(1+\mu\right)\right)^2 - 4\Omega_0^2\right)^{1/2}\right) \tag{14}$$

They are indicated in figures 2-7 as thin dotted vertical lines.

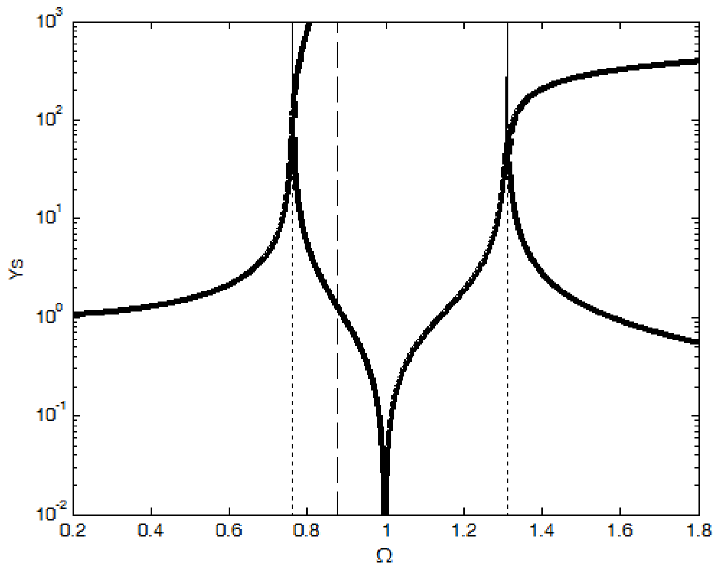

Figure 2. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.3, Ω_0 = 1 and γ = 10^{-6} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

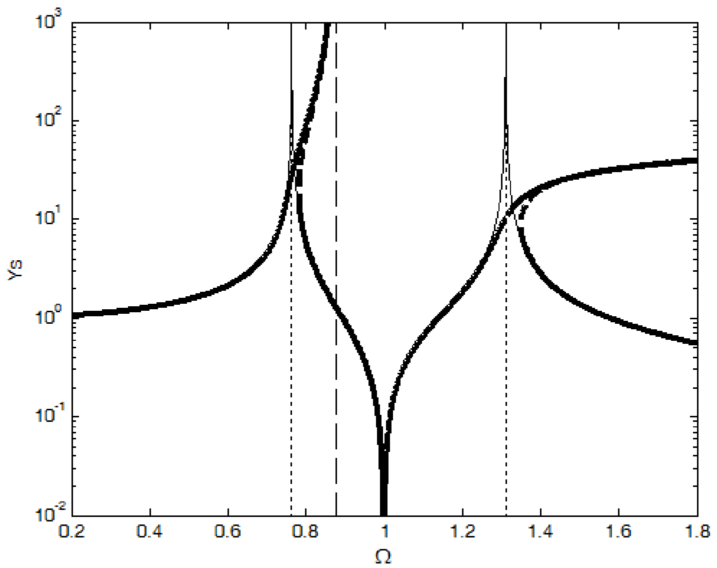

Figure 3. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.3, Ω_0 = 1 and γ = 10^{-4} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

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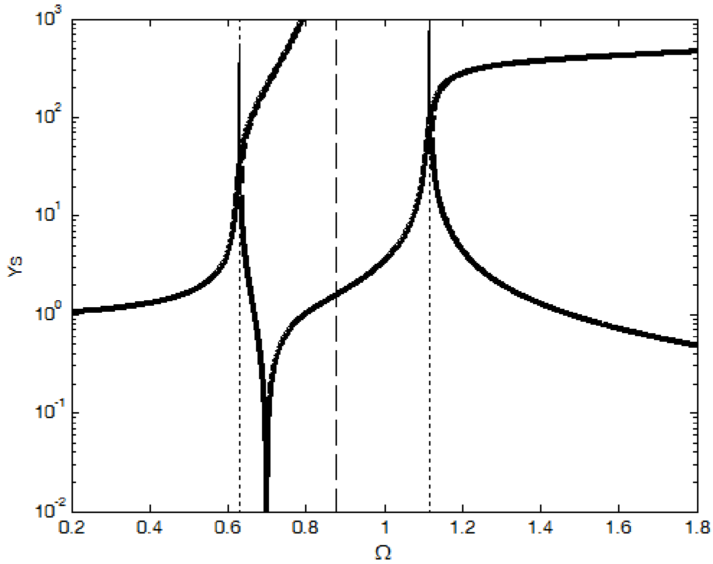

Figure 4. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.3, Ω_0 = 0.7 and γ = 10^{-6} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

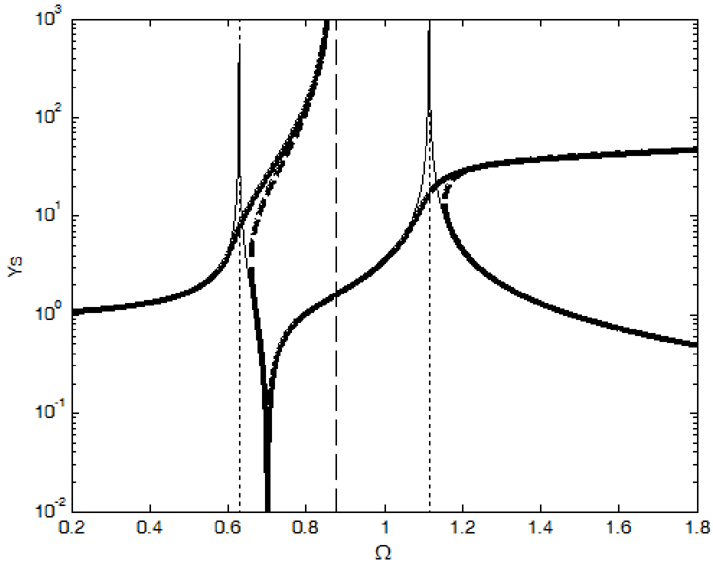

Figure 5. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.3, Ω_0 = 0.7 and γ = 10^{-4} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

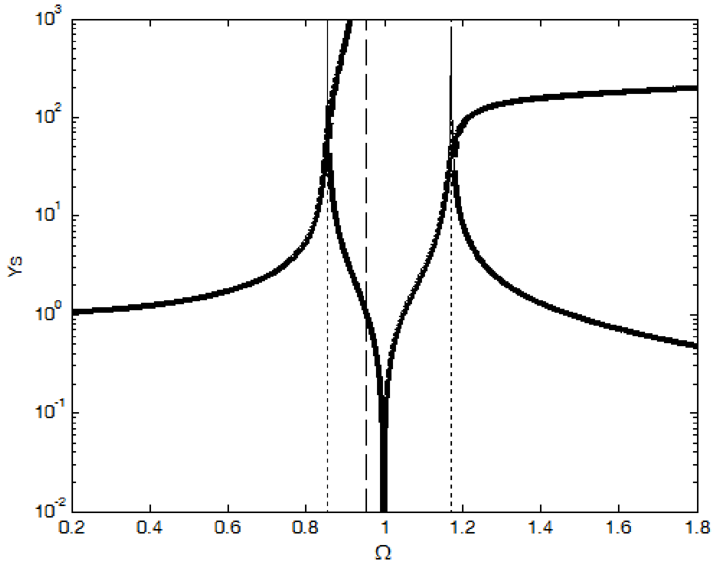

Figure 6. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.1, Ω_0 = 1 and γ = 10^{-6} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

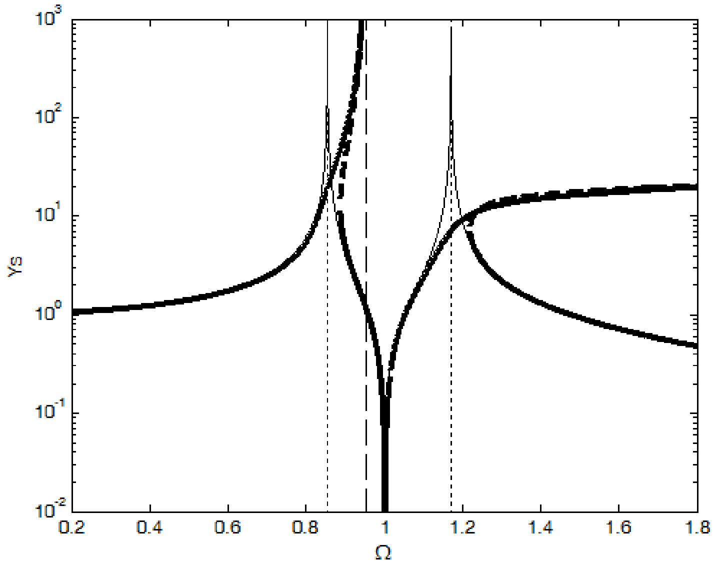

Figure 7. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.1, Ω_0 = 1 and γ = 10^{-4} with stable (thick solid line) and unstable (thick dashed line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

4. Effect of nonlinearity, frequency ratio and mass ratio

It is also interesting to determine which is the limiting case for the natural frequencies in equation (14) for an infinitely stiff absorber suspension, i.e.

$$\lim\Big|_{\Omega_0\to\infty} \omega_n = \begin{bmatrix} \dfrac{1}{\sqrt{1+\mu}} \\ \infty \end{bmatrix} \tag{15}$$

It results that as the frequency ratio Ω_0 increases (which is related to the ratio between the stiffness of the absorber suspension and that of the primary mass), one of the resonance frequency indefinitely increases, while the other tends to the finite expression of $(1+\mu)^{-1/2}$. This frequency is indicated in figures 2-7 by a thin dashed vertical line.

It may then be noted that as the nonlinearity increases, which means that the absorber suspension becomes harder and harder, the lower resonance frequency of the nonlinear system approaches the asymptotic value of $(1+\mu)^{-1/2}$, which is visible in the frequency response curve illustrated in figures 2-7.

The effect of the nonlinearity for a hardening absorber is then to bend both the resonance frequencies of the coupled two degree-of-freedom system to the higher frequencies. In particular, the higher resonance frequency is pushed away and the corresponding peak is pushed down, which is beneficial since it increases the bandwidth of the device and its robustness to mistuning, as shown in figure 7. On the other hand, the lower resonance frequency is limited by the asymptote, which depends on the mass ratio only. This can be set accordingly in order for the lower resonance frequency not to interact within the frequency range of interest for the device. For the values of nonlinearity used to plot figures 2-7, it is also noted that the anti-resonance frequency does not change sensibly respect to the linear case.

The effect of the mass ratio on the nonlinear system is similar to that of the linear system, i.e. an increase of the mass ratio is beneficial since the two resonance peaks are pushed apart, improving the robustness to mistuning. This is observed by comparing figures 2, 6, and figures 3, 7, which refer to different values of the mass ratio. However the practical increase of the mass ratio is somehow limited by the application, since we do not usually wish to increase much the mass of the absorber respect to the mass of the structure to which it is attached.

The effect of the frequency ratio is related to the location of the anti-resonance dip. For the values of nonlinearity and mass ratio used to plot figures 2-7, the frequency ratio appears to correspond to the anti-resonance frequency.

It is interesting to note that if the anti-resonance frequency is higher than the limiting value of $(1+\mu)^{-1/2}$, as in figures 2, 3 and in figures 6, 7, a single-valued response is granted at and around the anti-resonance frequency. On the other hand, if the anti-resonance frequency is lower than the limiting value of $(1+\mu)^{-1/2}$, as in figures 4, 5, the system response at and around the anti-resonance frequency is characterized by multi-valuedness. This condition should be avoided in practical applications since the response at a particular frequency may exhibit uncontrolled and dangerous amplitude changes.

5. Anti-resonance frequency

In the assumption of an undamped system, an approximate analytical expression for the anti-resonance frequency as a function of the system parameters may be derived. This is obtained by setting $Y_s = 0$ into equation (5) and then substituting the result for W into equation (4) together with the expressions given in equations (6)-(12). After some algebraic manipulation the following expression is obtained

$$\Omega_0^2 = \Omega_A^2 - \frac{3\gamma}{4\mu^2\Omega_A^4} \tag{16}$$

which relates the anti-resonance frequency Ω_A, the frequency ratio Ω_0, the mass ratio μ and nonlinearity γ. It is also noted that when the nonlinearity is small, equation (16) reduces to $\Omega_A \approx \Omega_0$, which is observed in figures 2-7. However, when the nonlinearity of the system increases, it is $\Omega_A > \Omega_0$. This means that if we want to keep the anti-resonance dip at 1, the frequency ratio has to be set so that is $\Omega_0^2 = 1 - 3\gamma\left(4\mu^2\right)^{-1}$**.** Since the frequency ratio is a positive real number, for each values of the mass ratio, a limiting value for the nonlinearity exists so that $\gamma < 4/3\mu^2$. At such limiting value for nonlinearity, the system reduces to a quasi-zero stiffness system [10].

Figures 8 and 9 show the frequency response curve of the primary mass displacement amplitude for two different values of the mass ratio, when the nonlinear coefficient is set to be close to the limiting value. In particular, in figure 8 the mass ratio is set to 0.1, the nonlinear coefficient is set to 0.01, where the limiting value gives $4/3\mu^2 = 0.0133$, and the frequency ratio is set to $\left(1 - 3\gamma\left(4\mu^2\right)^{-1}\right)^{1/2} = 0.50$. In figure 9 the mass ratio is set to 0.05, the nonlinear coefficient is set to 0.003, where the limiting value gives $4/3\mu^2 = 0.0033$, and the frequency ratio is set to $\left(1 - 3\gamma\left(4\mu^2\right)^{-1}\right)^{1/2} = 0.32$.

In figure 8, it is observed that a new region of unstable harmonic solutions appears in the frequency response curve which is denoted by a thick dotted line, and this is indeed a region of quasi-periodic motion [11]. The frequency range of such unstable region is very close to the anti-resonance frequency so that the potential benefit of bending the higher resonance peak to the higher frequencies is counter effected by an instability, which is detrimental to the device robustness to mistuning.

In figure 9, where the mass ratio is lower than that used to plot figure 8, a similar consideration may be drawn. In addition, the appearance of a detached resonance curve [11] is evident on the lower frequency range respect to the anti-resonance frequency.

6. Conclusions

This paper presented a preliminary analysis on some dynamic phenomena involved in the use of a nonlinear vibration absorber to reduce the vibration of a linear host structure. The whole system is modeled as a two degree-of-freedom oscillator, where the primary system is supposed to behave linearly on its own, and a second oscillator, i.e. the vibration absorber, is attached to it through a nonlinear spring with linear plus cubic stiffness coefficients.

An approximate non-dimensional analytical approach is followed in order to derive simple and usable close-form expressions to help investigating the effect of the main parameters on the system dynamics. The frequency response curve of the displacement amplitude of the host structure is considered and the effect of the nonlinear absorber in investigated. The performance of the coupled nonlinear system is compared to that of the corresponding linear system.

It is shown how the effect of the nonlinearity is to bend the higher resonance peak to the higher frequency, which is beneficial to improve the robustness of the device to mistuning, while the bending of the lower resonance peak is limited by an asymptotic values which depends on the mass ratio only. This limitation may be interestingly adopted to better choose the absorber parameters in order to avoid unwanted multi-valued response at a particular frequency. A simple expression relating the anti-resonance frequency, the mass ratio, the frequency ratio and the nonlinearity is derived, and it is show how this is used to tune the absorber appropriately.

Some interesting undesired dynamic effects are shown to appear in the frequency response curve for certain values of the system parameters, such as regions of instability of the harmonic response and inner closed detached resonance curves. These findings motivate for a future deeper investigation.

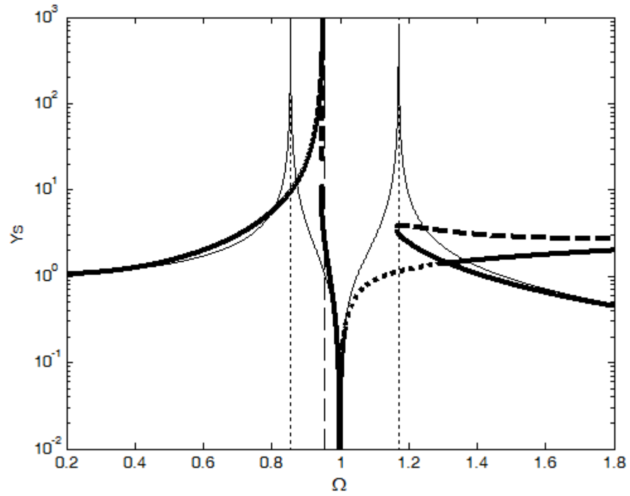

Figure 8. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.1, Ω_0 = 0.50 and γ = 0.01 with stable (thick solid line), unstable (thick dashed line) and quasi-periodic (thick dotted line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

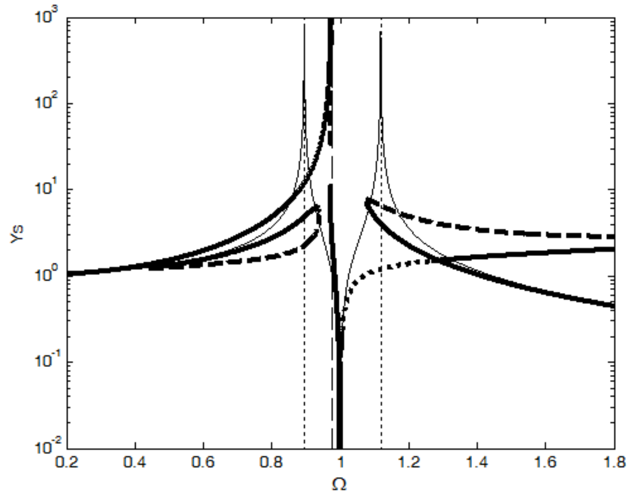

Figure 9. Non-dimensional absolute displacement amplitude of the primary mass for μ = 0.05, Ω_0 = 0.32 and γ = 0.003 with stable (thick solid line), unstable (thick dashed line) and quasi-periodic (thick dotted line) branches. The linear case is represented by a thin solid line. Vertical dotted lines denote the natural frequencies of the linear system as from Eq. (14), and the vertical dashed line denotes the limiting case for an infinitely stiff absorber suspension as from Eq. (15).

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