

Multi-dimensional Fokker-Planck equation analysis using the modified finite element method

J Náprstek and R Král

Institute of Theoretical and Applied Mechanics, v.v.i., Prosecká 76, CZ-190 00 Prague 9, Czech Republic

E-mail: naprstek@itam.cas.cz, kral@itam.cas.cz

Abstract. The Fokker-Planck equation (FPE) is a frequently used tool for the solution of cross probability density function (PDF) of a dynamic system response excited by a vector of random processes. FEM represents a very effective solution possibility, particularly when transition processes are investigated or a more detailed solution is needed. Actual papers deal with single degree of freedom (SDOF) systems only. So the respective FPE includes two independent space variables only. Stepping over this limit into MDOF systems a number of specific problems related to a true multi-dimensionality must be overcome. Unlike earlier studies, multi-dimensional simplex elements in any arbitrary dimension should be deployed and rectangular (multi-brick) elements abandoned. Simple closed formulae of integration in multi-dimension domain have been derived. Another specific problem represents the generation of multi-dimensional finite element mesh. Assembling of system global matrices should be subjected to newly composed algorithms due to multi-dimensionality. The system matrices are quite full and no advantages following from their sparse character can be profited from, as is commonly used in conventional FEM applications in 2D/3D problems. After verification of partial algorithms, an illustrative example dealing with a 2DOF non-linear aeroelastic system in combination with random and deterministic excitations is discussed.

1. Introduction

Let us consider a non-linear multi-degree of freedom (MDOF) dynamic system with random excitation. The respective stochastic differential system can be written as follows:

$$\frac{dx_j(t)}{dt} = f_j(\mathbf{x}, t) + g_{jr}(\mathbf{x}, t)w_r(t), \quad j = 1, \dots, 2n, \quad n - \text{dynamic degrees of freedom} \quad (1)$$

$\mathbf{x} = [x_1, x_2, \dots, x_{2n}]^T$ - response components (hereafter space variables): (i) x_{2j-1} - displacements; (ii) x_{2j} - velocities,

$w_r(t)$ - Gaussian white noises with constant cross-density in a meaning of stochastic moments $K_{rs} = \mathcal{E}\{w_r \cdot w_s\}$; $r, s = 1, m$, m - number of acting noises

$\mathcal{E}\{\cdot\}$ - mathematical mean value operator in the Gaussian meaning,

$f_j(\mathbf{x}, t), g_{jr}(\mathbf{x}, t)$ - continuous deterministic functions of state variables $\mathbf{x}$ and time t ; $j = 1, 2n$.

Provided the input random processes can be regarded as Gaussian white noises either stationary or non-stationary of the Markov type, then the Fokker-Planck equation (FPE) is a very effective tool for finding the cross probability density function (PDF) of the system response. A large number of monographs and papers have been published concerning this widely known partial differential equation (PDE) which has a mostly the linear character, although more complex definitions exist as well. For a comprehensive explanation, see e.g. [1], [2].

The relevant FPE for an unknown PDF in variables $\mathbf{x}, t$ associated to Eq. (1) has the form:

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = - \frac{\partial}{\partial x_j} (\kappa_j(\mathbf{x}, t) \cdot p(\mathbf{x}, t)) + \frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} (\kappa_{jk}(\mathbf{x}, t) \cdot p(\mathbf{x}, t)) \quad (2)$$



$$\kappa_j(\mathbf{x}, t) = f_j(\mathbf{x}, t) + \frac{1}{2} K_{rs} \cdot g_{ls}(\mathbf{x}, t) \frac{\partial g_{jr}(\mathbf{x}, t)}{\partial x_l}; \quad \kappa_{jk}(\mathbf{x}, t) = K_{rs} \cdot g_{jr}(\mathbf{x}, t) g_{ks}(\mathbf{x}, t) \quad (3)$$

$\kappa_j(\mathbf{x}, t)$ - drift coefficients; $\kappa_{jk}(\mathbf{x}, t)$ - diffusion coefficients.

An overview of analytical and semi-analytical methods of FPE solving is given in various resources. One of the most important monographs of this type seems to be [3]. A number of special oriented papers have appeared dealing with analytical or partly analytical FPE solution procedures, e.g. [4], [5].

Numerical methods applied to the analysis of FPE appeared a bit later due to high demands on the CPU performance. The first attempts at the FEM application in numerical treatment of the FPE date back to the early '70s. The publications by Bergman, Spencer and co-authors [6], [7] may be considered as the first systematic studies devoted to the Galerkin-Petrov type formulation of FEM. Some more knowledge to the field has also been contributed by the authors of this paper, see [8]. Regarding numerical procedures, the comprehensive state of the art concerning applications of numerical methods for analysis of the FPE has been published in JPEM by a team of 20 authors under the editorship of G.I. Schuëller, see [9]. Another review focused on FEM has been published by Johnson et al. [10].

Although numerical approaches and namely FEM application have provided many results generally inaccessible by analytical methods, some obstacles have been encountered as well. Papers published to date deal with single degree of freedom (SDOF) systems only, despite also other than Gaussian inputs being discussed. So the respective FPE includes two independent space variables only (x_1, x_2) . However, in general cases, the system Eq. (1) includes $2n$ space variables or, in other words, the vector $\mathbf{x}$ in the respective FPE, Eq. (2), includes $2n$ independent space variables. It is a high number in comparison with the conventional problems of FEM analysis applied to continuum mechanics where it is two or three. Therefore, the Multi-Dimensional (MD) character of FPE implies the need to reconsider all the principle steps to be taken on the way to solution by means of FEM.

2. Finite Element theoretical background

First of all, the basic type of the element should be selected. Despite many earlier experiences, it is necessary to once again realise the type and specific properties of FPE, the character of the analysis and expectations that are put on the results from a physical point of view.

2.1. General considerations

Regarding the concept following (2), Finite Elements (FE) have a local character free of any relation with other elements except those being in the immediate geometric neighborhood. This means that they enable the formulation of relevant local matrices independently of each other. Moreover, FPE is a scalar equation and so it is also an unknown PDF. Hence adequate matrices of individual elements can be directly uploaded into the global matrices without any transformation in space once they are assembled.

The character of the FE is uncommon, being given by a large MD on one side and by the structure of the FP operator Eq. (2) on the other side. Hence the choice of a suitable FE is a crucial step, which has never been treated in the literature until now. The discussion of this problem was been complicated and has not been uniquely completed to date. Nevertheless, with respect to adequate literature, the simplex element in the most simple form with $2n + 1$ nodes in the space of $2n$ dimension seems to be the most convenient. Only linear interpolation between two neighboring nodes are considered. Roughly speaking, the simplex element appears to be the most flexible from the viewpoint of the mesh generation in MD conditions, enabling integration into very simple close-form formulae; their form is always convex; only one universal element is needed to cover the entire any domain consistently. For a detailed explanation, see e.g. [11]. Many additional beneficial properties of the simplex element can be found and proved. For these reasons, the MD simplex element has been chosen for further analysis.

Let us briefly outline the basic properties of the simplex element and a structure of its system matrices. It should be remembered at the outset that FPE is linear and of the second order in space coordinates $\mathbf{x}$. Therefore, the simplex element can be addressed by nodes in vertices only. Intermediate nodes on edges are not attached. So that linear approximation within the element in all space coordinates is considered. In general, the simplex in $2n$ space is always a convex set S defined by $2n + 1$ nodes $N_1, N_2, \dots, N_{2n+1}$

which do not lie in one hyper-plane and any two lines connecting arbitrary two points N_k, N_l must not be parallel. In other words, provided global coordinates of nodes are $x_{ij}, i \in \langle 1, 2n+1 \rangle, j \in \langle 1, 2n \rangle$, the following should hold:

$$V_{2n} = \frac{1}{(2n)!} \begin{vmatrix} x_{1,1}, & x_{1,2}, & \dots, & x_{1,2n}, & 1 \\ x_{2,1}, & x_{2,2}, & \dots, & x_{2,2n}, & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{2n+1,1}, & x_{2n+1,2}, & \dots, & x_{2n+1,2n}, & 1 \end{vmatrix} > 0 \quad (4)$$

where V_{2n} is the volume of the $2n$ -dimensional simplex element. The volume of this hyper-brick element with respect to Eq. (4) is $V_{2n}^b = (2n)!V_{2n}$.

The original field $p^e(\xi, t)$ within the simplex element is approximated by a set of $2n+1$ linear functions or hyper-planes:

$$p(\xi, t) \approx p^e(\xi, t) = \sum_{j=1}^{2n+1} p_j^e(t) \cdot \psi_j(\xi), \quad \psi_j(\xi) = \sum_{i=1}^{2n} a_{ij} \xi_i + a_{2n+1,j} \cdot 1, \quad (5)$$

where $p_j^e(t)$ are values of the field $p^e(\xi, t)$ in j -th node and ξ is the vector of local centered coordinates:

$$\xi = [\xi_1, \dots, \xi_{2n}, 1]^T, \quad \text{where} \quad \xi_i = x_i - x_{C,i}, \quad x_{C,i} = \sum_{j=1}^{2n+1} x_{j,i} / (2n+1) \quad (6)$$

The $\psi_j(\xi)$ is a shape function corresponding to j -th unit state. They can be arranged into the vector $\psi(\xi)$. So that the expressions appearing in Eq. (5) can be rewritten as follows:

$$p^e(\xi, t) = \mathbf{p}^{Ne,T}(t) \cdot \psi(\xi), \quad \mathbf{p}^{Ne,T}(t) = [p_1^e(t), \dots, p_{2n+1}^e(t)], \quad \psi(\xi) = [\psi_1(\xi), \dots, \psi_{2n+1}(\xi)]^T, \quad (7)$$

where the superscript Ne at $\mathbf{p}^{Ne,T}(t)$ means the vector of PDF values in nodes of the simplex elements. Parameters $(a_{1j}, a_{2j}, \dots, a_{2n,j}, a_{2n+1,j})$ in Eq. (5) are $(2n+1)^2$ unknown constants. They are arranged into the matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{2n+1, 2n+1}$. So the vector of shape functions can be rewritten in the form:

$$\psi(\xi) = \mathbf{A} \cdot \xi \quad (8)$$

Taking into account the matrix in Eq. (4), we can define the matrix $\mathbf{X}^c \in \mathbb{R}^{2n+1, 2n+1}$

$$\mathbf{X}^c = \begin{bmatrix} \xi_{N_1,1}, & \xi_{N_1,2}, & \dots, & \xi_{N_1,2n}, & 1 \\ \xi_{N_2,1}, & \xi_{N_2,2}, & \dots, & \xi_{N_2,2n}, & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{N_{2n+1},1}, & \xi_{N_{2n+1},2}, & \dots, & \xi_{N_{2n+1},2n}, & 1 \end{bmatrix} \quad (9)$$

Hence we can write:

$$\mathbf{X}^c \cdot \mathbf{A}^T = \mathbf{I} \implies \mathbf{A}^T = (\mathbf{X}^c)^{-1}, \quad \mathbf{I} - \text{unit matrix.} \quad (10)$$

The inverse matrix $(\mathbf{X}^c)^{-1}$ always exists with respect to Eq. (4) and uniquely determines constants a_{ij} and hence also the shape functions of interpolating expressions Eq. (5) or (7).

2.2. Simplex element local matrices

For convenience and better clarity, we rewrite FPE previously introduced by Eq. (2) in the form:

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = - \nabla_x^T (\kappa(\mathbf{x}, t) p(\mathbf{x}, t)) + \frac{1}{2} (\nabla_x^T (\Upsilon(\mathbf{x}, t) p(\mathbf{x}, t)) \nabla_x) \quad (11)$$

where ∇_x^T is the vector of differential operators and $\kappa(\mathbf{x}, t)$, $\Upsilon(\mathbf{x}, t)$ fields of drift and diffusion coefficients following Eq. (3) :

$$\begin{aligned}\nabla_x^T &= [\partial/\partial x_1, \dots, \partial/\partial x_{2n}], & \kappa(\mathbf{x}, t) &= [\kappa_j(\mathbf{x}, t)]^T \in \mathbb{R}^{2n} \\ \Upsilon(\mathbf{x}, t) &= [\kappa_{jk}(\mathbf{x}, t)] \in \mathbb{R}^{2n, 2n}\end{aligned}\quad (12)$$

Note that matrix $\Upsilon(\mathbf{x}, t)$ is always symmetric.

The Galerkin-Petrov procedure will be used now to deduce local matrices of the simplex finite element. Basically, it means establishing the orthogonality of FPE with $p(\mathbf{x}, t)$ approximated in the space of shape functions following Eq. (7) and the shape functions themselves. In particular, FPE including approximation $\mathbf{p}^{Ne, T}(t)\psi(\xi)$ should be multiplied by column vector $\psi(\xi)$ and subsequently integrated over the whole volume of the simplex element in $2n$ -dimensional space. This procedure provides a linear differential system in the normal form for PDF time functions $\mathbf{p}^{Ne}(t)$ at nodes of the corresponding finite element.

Getting through the integration phase, a certain simplification can be afforded. In general, the size of the particular finite element is small and therefore also the change of variable coefficients of the FPE is small as well. So we can approximate the influence of coefficients $\kappa(\mathbf{x}, t)$, $\Upsilon(\mathbf{x}, t)$ by the values and the values of their space derivatives in the element center, e.g. in the point $\mathbf{x}_C$. Hence possible coefficients are not subdued to the integration. Their influence is applied as constants $\kappa(\mathbf{x}_C, t)$, $\Upsilon(\mathbf{x}_C, t)$ (or functions of time) fixed after differentiation by the operator ∇_x but before the integration is applied. Then the integration makes for easier manipulating only with shape functions and their derivatives which result in constants due to the linear character of shape functions. This enables the separation of the step of integration from the specific character of individual dynamic equations Eq. (1), in particular, of functions $f_j(\mathbf{x}, t)$, $g_{jr}(\mathbf{x}, t)$. Therefore the integration remains in the closed form formulae and no Gauss points should be involved. Let us be aware that the formal step of differentiation within the element, i.e. with respect to ξ , is identical with that following global coordinates $\mathbf{x}$. So that, in principal, for purposes of differentiation within the element, we can put $\nabla_x^T \equiv \nabla_\xi^T$.

Results of the Galerkin-Petrov procedure concerning one term on the left side and two terms on the right side of Eq. (11) are outlined as follows:

(A) Left side. With respect to Eqs. (5), (7), (10) it can be deduced:

$$\begin{aligned}\int_V \psi(\xi) \frac{d\mathbf{p}^{Ne, T}(t)}{dt} \psi(\xi) \cdot dV &= \mathbf{E}^e \frac{d\mathbf{p}^{Ne}(t)}{dt}, \\ \text{where: } \mathbf{E}^e &= \mathbf{A}^e \mathbf{J}^e \mathbf{A}^{e, T}, \quad \mathbf{J}^e = \int_V \xi \cdot \xi^T \cdot dV, \quad \mathbf{E}^e, \mathbf{J}^e \in \mathbb{R}^{2n+1, 2n+1}\end{aligned}\quad (13)$$

where a superscript e at matrices symbolises a relevance with a particular finite element. Integrals in Eq. (13) are considered as multiple in $2n$ dimensions over volume V filling up the simplex element and $dV = \prod_{i=1}^{2n} d\xi_i$ is a relevant multiple differential of $2n$ -space.

(B) Right side - the first term. In the first step:

$$-\nabla_x^T(\kappa(\mathbf{x}, t)p(\mathbf{x}, t)) = -(\nabla_x^T \kappa(\mathbf{x}, t))p(\mathbf{x}, t) - (\nabla_x^T p(\mathbf{x}, t))\kappa(\mathbf{x}, t) \quad (14)$$

Now, a similar procedure to the treatment of the left side in Eq. (13) is applied to the first part above:

$$\begin{aligned}-\int_V \mathbf{A}\xi \nabla_x^T(\kappa(\mathbf{x}, t)) \mathbf{p}^{Ne, T}(t) \mathbf{A}\xi \cdot dV &= -(\nabla_x^T \kappa(\mathbf{x}_C, t)) \left(\int_V \mathbf{A}\xi \xi^T \mathbf{A}^T dV \right) \mathbf{p}^{Ne}(t) = \\ &= -\kappa_C(t) \cdot \mathbf{E} \cdot \mathbf{p}^{Ne}(t), \quad \kappa_C(t) = -\nabla_x^T(\kappa(\mathbf{x}_C, t))\end{aligned}\quad (15)$$

where κ_C is a scalar arising from the divergence $-\nabla_x^T(\kappa(\mathbf{x}_C, t))$. This results from a summation of the first derivatives of $\kappa(\mathbf{x}_C, t)$ with respect to global coordinates x_j evaluated in the element center x_C . The second part of Eq. (14) vanishes and therefore the above result represents the whole first term on the right side of Eq. (11).

(C) Right side - the second term. Expanding the double differentiation, we obtain four parts:

$$\begin{aligned} \frac{1}{2} \nabla_x^T (\Upsilon(\mathbf{x}, t) p(\mathbf{x}, t)) \nabla_x &= \frac{1}{2} (\nabla_x^T (\Upsilon(\mathbf{x}, t) \nabla_x) p(\mathbf{x}, t) + \frac{1}{2} \nabla_x^T (\nabla_x^T p(\mathbf{x}, t) \Upsilon(\mathbf{x}, t))^T \\ &+ \frac{1}{2} (\nabla_x^T \Upsilon(\mathbf{x}, t)) \nabla_x^T p(\mathbf{x}, t) + \frac{1}{2} (\nabla_x^T p(\mathbf{x}, t)) (\Upsilon(\mathbf{x}, t) \nabla_x) \end{aligned} \quad (16)$$

The 3rd and 4th parts vanish. Let us proceed now to the first part:

$$\begin{aligned} \frac{1}{2} \int_V \mathbf{A} \xi \left(\nabla_x^T \Upsilon(\mathbf{x}_C, t) \nabla_x \right) \mathbf{p}^{Ne, T}(t) \mathbf{A} \xi dV &= \frac{1}{2} \left(\nabla_x^T \Upsilon(\mathbf{x}_C, t) \nabla_x \right) \left(\int_V \mathbf{A} \xi \xi^T \mathbf{A}^T dV \right) \mathbf{p}^{Ne, T}(t) = \\ &= \Upsilon_C(t) \cdot \mathbf{E} \cdot \mathbf{p}^{Ne, T}(t), \quad \Upsilon_C(t) = \frac{1}{2} \nabla_x^T \Upsilon(\mathbf{x}_C, t) \nabla_x \end{aligned} \quad (17)$$

where $\Upsilon_C(t)$ is evaluated from the second derivatives of $\Upsilon(\mathbf{x}_C, t)$ in the center x_C similarly to evaluating $\kappa(\mathbf{x}_C, t)$, see Eq. (15).

Concerning the second part, the Gauss-Ostrogradski theorem should be used in order to remove the second derivative which would bring the integrand implicitly to zero due to first power of ξ in $\psi(\xi)$. Performing adequate operations, we obtain:

$$\begin{aligned} \frac{1}{2} \int_V \mathbf{A} \xi \nabla_x^T \left(\nabla_x^T \mathbf{p}^{Ne, T}(t) \mathbf{A} \xi \Upsilon(\mathbf{x}_C, t) \right)^T dV &= -\frac{1}{2} \int_V \mathbf{A}^* \Upsilon(\mathbf{x}_C, t) \mathbf{A}^{*T} \mathbf{p}^{Ne}(t) dV = \mathbf{F}(t) \mathbf{p}^{Ne}(t), \\ \mathbf{F}(t) &= \frac{1}{2} \mathbf{A}^* \Upsilon(\mathbf{x}_C, t) \mathbf{A}^{*T} \cdot K, \quad \mathbf{F}(t) \in \mathbb{R}^{2n+1, 2n+1}, \quad K = \int_V dV = V_{2n}, \end{aligned} \quad (18)$$

where K is the element volume V_{2n} . Summarising the previous results Eqs. (13), (15), (17), (18), the system of ODEs for PDF time history can now be written in the form:

$$\mathbf{E}^e \frac{d\mathbf{p}^{Ne}(t)}{dt} = (-\kappa_C(t) \cdot \mathbf{E}^e + \Upsilon_C(t) \cdot \mathbf{E}^e + \mathbf{F}^e(t)) \mathbf{p}^{Ne}(t) \quad (19)$$

Matrices $\mathbf{E}^e$, $\mathbf{F}^e(t)$ have a local character. Unlike problems of elasticity, the PDF investigated is a scalar function and therefore no special vector transformation is necessary when uploading matrices $\mathbf{E}^e$, $\mathbf{F}^e(t)$ of individual elements into the global matrices of the whole system. Only appropriate incident vectors should be respected. Finally, we obtain the ODE system as follows:

$$\begin{aligned} \mathbf{E} \frac{d\mathbf{p}(t)}{dt} &= \mathbf{S}(t) \cdot \mathbf{p}(t), \quad \mathbf{S}(t) = -\kappa_C(t) \cdot \mathbf{E} + \Upsilon_C(t) \cdot \mathbf{E} + \mathbf{F}(t), \\ \mathbf{p}(t) &\in \mathbb{R}^\nu, \quad \mathbf{E}, \mathbf{F}(t), \mathbf{S}(t) \in \mathbb{R}^{\nu, \nu} \end{aligned} \quad (20)$$

where ν is the number of nodes in the FE mesh. Vector $\mathbf{p}(t)$ represents unknown PDF values in nodes of the FE mesh. Unlike previous papers dealing with random vibration of SDOF systems only, regarding the FEM solution of FPE for general MDOF systems, the authors see it as inconsistent to keep denoting "stiffness" and "mass" matrices for $\mathbf{S}$ and $\mathbf{E}$, refiring some analogy with governing equations of dynamics (mass, damping and stiffness matrices). Therefore we suggest the denotations: $\mathbf{S}(t)$ - System Matrix (SM), $\mathbf{E}$ - Evolution matrix (EM).

Let us remember that the matrix $\mathbf{S}(t)$ is generally a function of time, provided any white noise intensity entering into the system Eq. (1) is variable in time in any meaning of the term, or if a deterministic time variable function is contained in additive excitation or in a system coefficient representing either a deterministic multiplicative excitation or system parameter variability. In such a case, the stationary solution of FPE does not exist and the full differential system Eq. (20) should be investigated together with adequate initial and boundary conditions for a general time dependent solution or for a quasi-periodic solution, see for instance [12].

3. Computer implementation

To implement the above algorithm into the computer, a couple of steps should be taken. It is necessary (i) to compile the relevant mesh generator; (ii) to develop an efficient matrix assembler; (iii) to employ a suitable solver for a large ODE system and, last but not least, (iv) to identify and eliminate specific problems related to a large ODE system and its very unconventional attributes.

3.1. Multi-dimensional mesh generation

For a dynamical system with n -DOFs investigated by the Fokker-Planck equation using FEM, the mesh spatial dimensionality of $2n$ should be generated. Various tools exist for mesh generation. However, most of them are adapted for space dimensionality $N \leq 3$. Moreover, many of them are not available as open source due to the integration into commercial software packages. Nevertheless, the high dimensionality, which should not be limited in advance, creates hitherto unknown complications. The main problems arise from the need of a fully abstract derivation process without any possibility of geometric imagination, in contrast to problems with 2D/3D domains. Therefore, the new mesh generator environment has been developed, especially suited to the needs of FP equations.

Unlike earlier studies with problems of low spatial dimensions, it has come to light that the simplex elements are the most appropriate and rectangular (multi-brick) elements should be abandoned. For many reasons, the multi-dimensional simplex elements are selected that fulfill the strict requirements for mesh adaptivity, a low number of nodal unknowns and also easy implementation of the hyper-volume integration. The generator of non-uniform adaptive mesh in multidimensional space is based on the iterative method, where the triangulation Delaunay procedure for higher dimensions is applied to the randomly distributed points and iteratively adapted with respect to the predefined distance functions $h(\mathbf{x})$, [13]. Basically, this means that the points are relocated in each iterative step towards satisfying the demanded mesh density. Within this process, a number of issues has to be solved, including the elimination of redundant points, detection and correction of generated elements due to the coplanar points or unsatisfactory shape quality. With regard to the mesh quality inspection, the geometric mean and the arithmetic mean ratio have been used that provide information about badly conditioned elements. In order to improve the shape quality, an optimal relocation of the degenerate vertex points is performed, based on the smart QL smoothing method within each iterative step, [14].

In this sense, the computational domain in the shape of a hyper-block is generated in the multi-dimensional subspace concurrently with a final number of computational sub-tasks and consequently mirrored on to the entire domain. This method of meshing fully benefits the symmetry of the problem and considerably reduces the computational time. The code has been tested in several ways. First of all, a comparison with results obtained for 2D/3D domains using conventional generators was made in several regimes. Concerning 4D, various aspects were evaluated with regard to the consistency of elements, detection of "nearly degenerated" elements and possibilities of a recovery and patches in the mesh. Numerical stability was also verified.

3.2. Assembling of global differential and algebraic systems, numerical integration of ODE system

Several important properties of the FPE should be remembered before we start the assembling process. As was already mentioned, the FPE is a scalar equation. Supposing Gaussian inputs only and synchronous processing (no delayed control interventions, optimal filtering, etc.), it remains linear and differential only. Also the role of any element is local. This means that the element is related to the neighboring elements only. These attributes mean a significant facilitation of the assembling process. On the other hand, the FPE is not self-adjoint, which leads to non-symmetric global matrices.

As a computational environment for executing the finite element assembly together with a time step solution of the given non-stationary problem, MATLAB software is used. Vectorised environment for matrix assembling has been created in order to enable an effective parallel computing in 8 processors (limited by license agreement). The main attributes of this environment can be characterised as follows:

- for-loop syntax vectorised using matrix-array operations; if used, then the loops run over low indices such as dimensions or number of element vertices;
- Single Program-Multiple Data (SPMD) parallel computing is adopted, enabling work with the distributed data array simultaneously using one single code;
- optional hyper-volume integrations – closed form vs. Gauss quadrature;
- code fully and automatically extended to n -dimensional boundary value problems of the second order (inverse Jacobian, determinant, shape functions, etc.);
- global matrices reordering based on the reverse Cuthill-McKee permutation method;
- time discretisation performed using two-level time-stepping scheme with adaptive time step control.

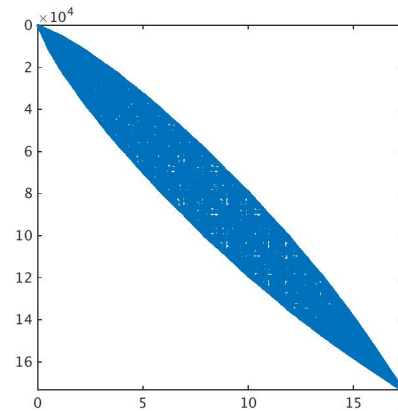


Figure 1. Plot of the sparsity pattern of global matrices **E** and **S** for 4D problem.

Performance of the software chain has been carefully evaluated, testing the assembly of global matrices using P1 triangular (2D) and hyperhedral elements (4D). The parameters of the tests together with respective elapsed time are shown in Table 1. A significant reduction in assembly time is evident, inspecting the last column of the Table. In order to ensure faster convergence in the solution of linear equations, reordering of the nonsymmetric system matrices **S** and **E** is carried out by applying permutation such that nonzero elements were closer to the diagonal. Fig. 1 shows the sparsity pattern of the global matrix for a 4D problem, generated by using the reverse Cuthill-McKee ordering method. If compared to the 2D Fokker-Planck solution problem, the matrix bandwidth pronouncedly expands as the spatial dimensionality increases. Note that in the 2D case the tridiagonal matrix with very low system matrix bandwidth emerges after the reordering.

Table 1. Assembly of global matrices using P1 triangular (2D) and hyperhedral elements (4D).

Dimension	No. of processors	No. of elements ($\times 10^3$)	Size of S and E ($\times 10^3$)	Total time of assembling S and E [s]
4	1	5900	221	223.4
4	2	5900	221	152.4
4	4	5900	221	124.7
4	8	5900	221	91.2
2	8	55	27	0.157
2	8	588	295	2.08

Time discretisation of Eq. (20) is based on the two-level method, where a combination of the simple explicit scheme and robustness of implicit one ensures the second-order accuracy. The forward Euler method as the predictor gives an estimation $\bar{\mathbf{p}}^{n+1}$ for the implicit Crank-Nicolson numerical integration algorithm as the corrector as follows:

$$\begin{aligned}\bar{\mathbf{p}}^{n+1} &= \mathbf{p}^n + f(t^n, \mathbf{p}^n) \Delta t \\ \mathbf{p}^{n+1} &= \mathbf{p}^n + \frac{1}{2} [f(t^n, \mathbf{p}^n) + f(t^{n+1}, \bar{\mathbf{p}}^{n+1})] \Delta t\end{aligned}\quad (21)$$

where $\mathbf{p}(t^n) = \mathbf{p}^n, \forall n = 0, 1, \dots, M-1$ is the solution at time $t^n = n\Delta t$, Δt is the time step and $f(t^n, \mathbf{p}^n) = \mathbf{E}^{-1} \mathbf{S} \mathbf{p}^n$. Naturally, the inverse matrix $\mathbf{E}^{-1}$ is very computational expensive. Therefore, the system of linear equations

$$\mathbf{E} \frac{d\mathbf{p}(t^n)}{dt} = \mathbf{r}^n \quad (22)$$

with $\mathbf{r}^n = \mathbf{S} \mathbf{p}^n$ is solved for derivative $d\mathbf{p}(t^n)/dt$ by an iterative biconjugate gradients method (BICG). For higher and faster convergence, the initial vector starting the iterative procedure of BICG is estimated

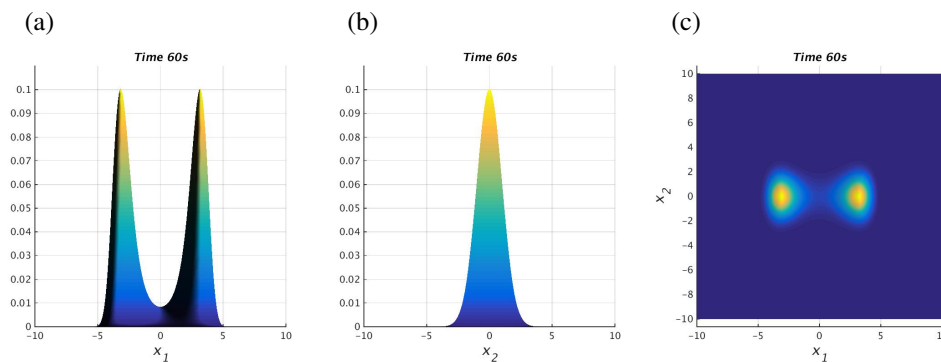


Figure 2. PDF of Duffing type SDOF system using simplex elements.

using the Adams-Bashforth methods. The adaptive time-stepping method with automatic time step control proved to be a very useful tool for the solver acceleration. For this purpose, an evolutionary PID controller is implemented that takes into account the relative changes in the solution using simple mechanism capturing the dynamics of the flow.

4. Numerical experiments

Three application examples of the above background will now be presented. Systems were selected where either analytical solution is known, e.g. Boltzman solution, or a similar system was investigated using semi-analytical procedures. A large variety of system parameters were considered, together with a large interval of variance of the input processes. It was revealed that high or low input noise intensities required serious interventions in the solution algorithms regarding the final ODE system. For this reason, special adaptations were necessary in order to achieve a balanced dosing of input processes with respect to the phase of the solution process, particularly during the transition process starting at the Dirac type initial conditions.

4.1. Duffing equation

The Duffing equation is a frequently used mathematical model of a SDOF nonlinear system in Physics and Engineering. The equation in basic or normal form under white noise additive and multiplicative excitations can be written as follows:

$$\begin{aligned} \ddot{x} + 2\omega_b\dot{x} - \omega_0^2 \cdot x(1 + w_s(t) - \alpha^2 x^2) &= w_a(t) \Rightarrow \\ \Rightarrow \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \omega_0^2 x_1(1 - \alpha^2 x_1^2) - 2\omega_b x_2 + \omega_0^2 x_1 w_s(t) + w_a(t) \end{aligned} \quad (23)$$

The relevant FP equation can be evolved:

$$\frac{\partial p}{\partial t} = - \frac{\partial (x_2 p)}{\partial x_1} - \frac{\partial [(\omega_0^2 x_1(1 - \alpha^2 x_1^2) - 2\omega_b x_2)p]}{\partial x_2} + \frac{1}{2} \frac{\partial^2 [(K_{ss} \cdot \omega_0^4 x_1^2 + 2 \cdot K_{as} \cdot \omega_0^2 x_1 + K_{aa})p]}{\partial x_2^2} \quad (24)$$

Eq. (23) describes the Miesess truss movement under white noise excitation. The stiffness linear part is negative and consequently the system includes an unstable stationary point in the origin $(0, 0)$. Two stable stationary points have position $(\pm 1/\alpha, 0)$. The repulsivity level in the origin depends on a relation of both stiffness parts and on the multiplicative noise w_s density. For more details, see [8], where also various nonsymmetric cases were discussed. The dynamical and excitation parameters are: $\omega_0^2 = 1.0$, $\omega_b = 0.05$, non-linearity ratio $\alpha^2 = 0.1$ and $K_{aa} = 0.2$. Let us deal thoroughly with the case of the additive excitation only.

Some selected results are plotted in Fig. 2. They represent the stationary state when the initial conditions have subsided $p(\mathbf{x}, 0) = (0, 0)$ (60s elapsed time). The image (a) is the front view demonstrating x_1 coordinate (displacement), the side view (b) shows x_2 coordinate (velocity) and (c) is the plane view of PDF in x_1, x_2 plane. Note that in [8], rectangular elements have been used, while Fig. 2 shows results obtained using simplex elements. The coincidence of both sets was perfect.

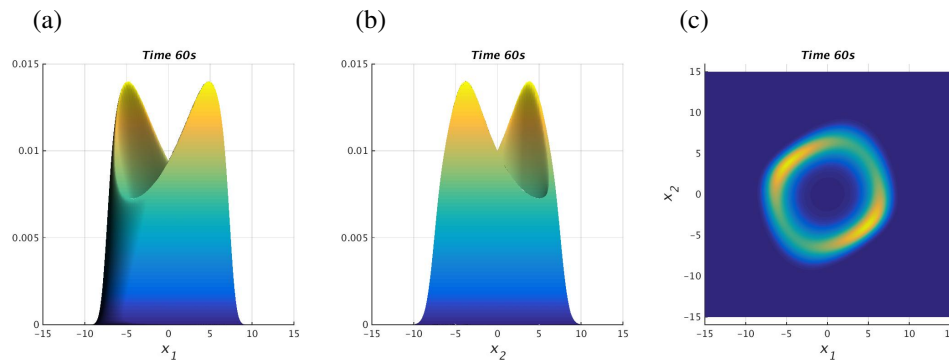


Figure 3. PDF of Van der Pol type SDOF system using simplex elements.

4.2. Nonlinear system of Van der Pol type

The Van der Pol equation being excited by additive and multiplicative random noises has the form:

$$\begin{aligned} \ddot{x} - 2\omega_b(1 + w_b(t) - \beta^2 x^2)\dot{x} + \omega_0^2 \cdot x &= w_a(t) \Rightarrow \\ \Rightarrow \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_0^2 x_1 + 2\omega_b(1 - \beta^2 x_1^2)x_2 + 2\omega_b x_2 w_b(t) + w_a(t) \end{aligned} \quad (25)$$

FP equation corresponding to the Van der Pol system (25) reads:

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{\partial (x_2 p)}{\partial x_1} + \frac{\partial [(\omega_0^2 x_1 - 2\omega_b x_2(1 - \beta^2 x_1^2 + \omega_b K_{bb}) - \omega_b K_{ab})p]}{\partial x_2} \\ &+ \frac{1}{2} \frac{\partial^2 [(4 \cdot K_{bb} \cdot \omega_b^2 x_2^2 + 4 \cdot K_{ab} \omega_b x_2 + K_{aa})p]}{\partial x_2^2} \end{aligned} \quad (26)$$

Contrary to the previous case a known simple solution of the stationary version of Eq. (26) analogous to Boltzman solution does not exist. Nevertheless, the result for $t \rightarrow \infty$ can be compared with a number of approximative analytical solutions, see e.g. [2]. The parameters entering Eq. (26) are $\omega_0^2 = 1.0$, $\omega_b = 0.1$, non-linearity ratio $\beta^2 = 0.1$ and $K_{aa} = 0.2$. Let us note that, from the viewpoint of detailed numerical integration, any arbitrary value in the interval $K_{aa} \in (0.2, 6.0)$ did not pose a problem. The division of the integration process into two or more stages as in the Duffing system was not necessary.

Similarly to the previous subsection, images (a-c) represent the front, side and plane views of the PDF function. Once again, more details can be found in [8]. A comparison of the analysis using rectangular and simplex elements did not expose any differences. Note that the limit cycle is observed in image (c).

4.3. Two degree of freedom system - analysis using simplex elements

Let us consider the two degree of freedom (TDOF) system depicted in Fig. 4. The system includes non-linear stiffness (Duffing type) and non-linear viscous damping (Van der Pol type). The interaction between the masses is considered to be linear. Both DOFs are subjected to the independent external zero-mean white noise excitation processes $w_{a,j}$, $j = 1, 2$ with correlations $E[w_{a,1}(t)w_{a,2}(t+\tau)] = K_{ij}\delta(\tau)$, where $K_{12} = K_{21} = 0$. The response of the system is four-dimensional and it can be described as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ k_{11}(1 - \alpha^2 x_1^2)x_1 - k_{12}x_3 - c_{11}x_2 - c_{12}x_4 \\ x_4 \\ -k_{21}x_1 - k_{22}x_3 - c_{21}x_2 + c_{22}(1 - \beta^2 x_3^2)x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_{a,1} \\ w_{a,2} \end{bmatrix} \quad (27)$$

where x_1, x_3 are the displacements of the first and the second mass and x_2, x_4 are the corresponding velocities, respectively. The stiffness and damping properties of the system and the non-linearity ratios with noise intensities are chosen as:

$$\mathbf{k} = \begin{bmatrix} 1 & -0.2 \\ -0.2 & 0.8 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0.1 & -0.05 \\ -0.05 & 0.2 \end{bmatrix} \quad \alpha^2 = 0.1 \quad \beta^2 = 0.2 \quad K_{11} = K_{22} = 0.2 \quad (28)$$

The Fokker-Planck equation for the response of the nonlinear transient dynamical system is given by

$$\begin{aligned} \frac{\partial p}{\partial t} = & -\frac{\partial(x_2 p)}{\partial x_1} - \frac{\partial(x_4 p)}{\partial x_3} + \frac{\partial[(k_{11}(1 - \alpha^2 x_1^2)x_1 - k_{12}x_3 - c_{11}x_2 - c_{12}x_4)p]}{\partial [(k_{21}x_1 + k_{22}x_3 + c_{21}x_2 - c_{22}(1 - \beta^2 x_3^2)x_4)p]} \\ & + \frac{1}{2} \left[\frac{\partial^2(K_{11}p)}{\partial x_2^2} + \frac{\partial^2(K_{22}p)}{\partial x_4^2} \right] \end{aligned} \quad (29)$$

Initial conditions at $t = 0$ are introduced in the form of Dirac function of adequate dimensionality, expressed by the multivariate Gaussian function with very small variance and zero co-variance in all coordinates:

$$p(\mathbf{x}, 0) = \frac{1}{\sqrt{(2\pi)^{2n} |\Sigma|}} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) \quad (30)$$

where $\mathbf{x}$ is $2n$ -dimensional column vector with nodal coordinates, $|\Sigma|$ refers to the determinant of covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{2n, 2n}$ with nonzero diagonal terms, $\Sigma_{j,j} = 0.25$, $j = 1, 4$, characterizing stochastic independence of $\mathbf{x}$ components at $t = 0$. Finally $\boldsymbol{\mu}$ is $2n$ -dimensional vector of the mean values, here considered to be a zero vector. With respect to the order of the FP differential equation, only the Dirichlet boundary conditions $p \rightarrow 0$ as $|x_1|, |x_2|, |x_3|, |x_4| \rightarrow \infty$ are satisfactory if the domain is adequately large and boundary conditions, in fact, do not influence the solution itself.

The computational domain discretised into the non-uniform mesh in the adaptive scheme is given by $-10 \text{ m} \leq x_1, x_2, x_3, x_4 \leq 10 \text{ m}$. In places with expected PDF evolution, the relatively fine mesh with the simplex edge size of 0.25 m was applied. With the distance from these locations, the edge size increased linearly up to 0.8 m at the rate of element growth 1.1 . For the finite element solution, a total number of $5,900$ thous. $\approx 50^4$ multi-dimensional polyhedrons are used that gives 22^4 nodal values. By employing the concurrent finite element code working with up to 8 processing units, substantial savings in assembly time were achieved, while maintaining the optimal memory requirements for the total storage. This resulted in an almost 2.5 times reduction in time if 1 against 8 cores is compared, see Table 1.

In Fig. 5, time histories of PDF captured at time sequences: (a) $t = 0 \text{ s}$; (b) $t = 8 \text{ s}$; (c) $t = 32 \text{ s}$ have been depicted. At the top, displacement versus velocity components x_1, x_2 for the first DOF are demonstrated, whereas the bottom images display the respective components x_3, x_4 related to the second DOF. It is obvious that the first degree has the very rapid tendency to pass from the Dirac-like shape into a form with two peaks as it corresponds to the Duffing characteristic of stiffness. The stochastic response of the second DOF initially evolves into the form of being characteristic of the Van der Pol system. If $t > 30 \text{ s}$, linear stiffness and damping coupling start to be more pronounced as indicated in Fig. 6. These images show a stationary state at $t = 50 \text{ s}$. Apparently, the distribution of PDF is no longer symmetrical as attained in the pure Duffing system, see Fig. 2. The effect of the non-diagonal terms is also manifested by the different height at the peak values for both the DOFs, however, still maintaining the key features of the Duffing and Van Der Pol type of non-linearity. The combination of two peaks together with a visible limit cycle is obvious.

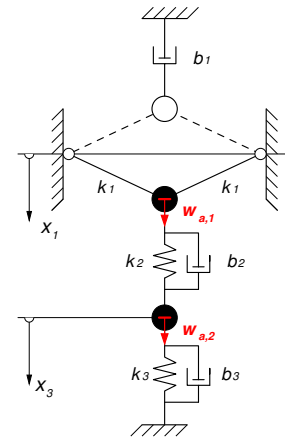


Figure 4. Duffing - Van der Pol 2DOFs system with linear stiffness and damping coupling.

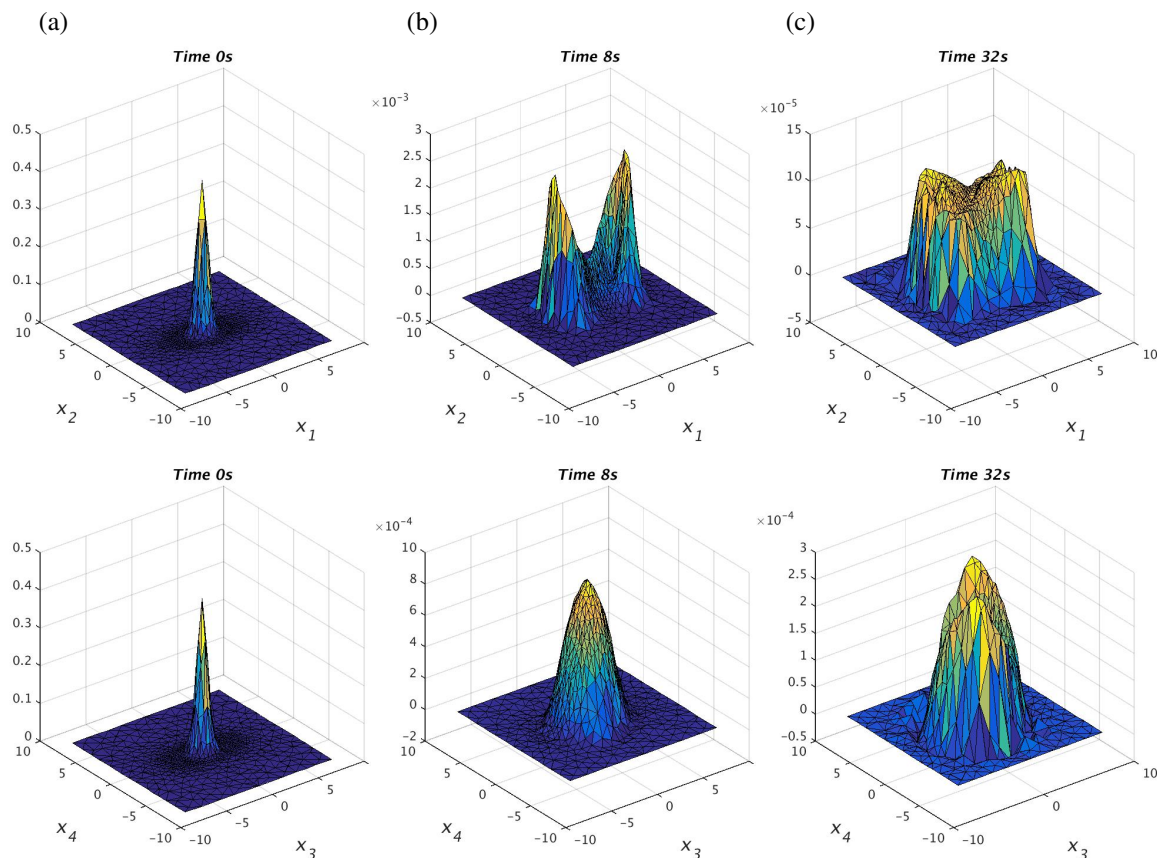


Figure 5. Time evolution of PDF in the transition regime at time instants $t = 0, 8, 32$ s.

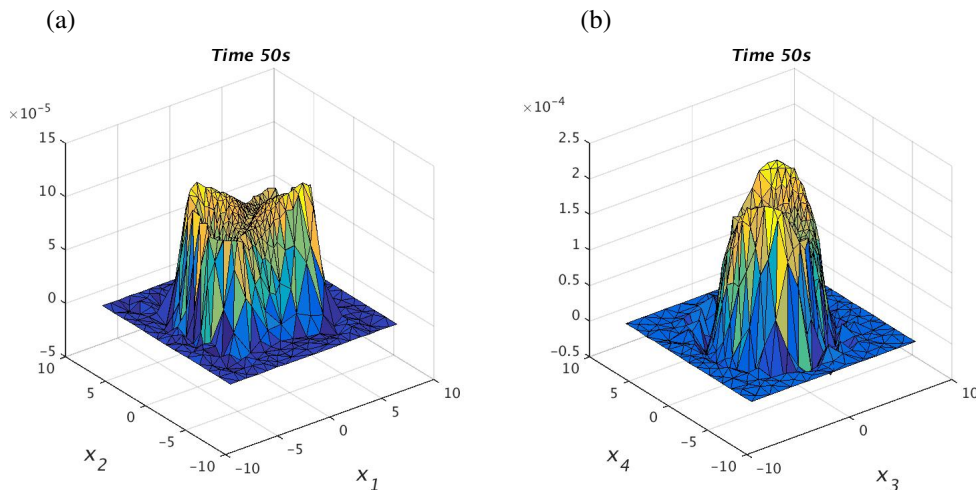


Figure 6. PDF in the stationary state (after 50 s); (a) 1.DOF with x_1, x_2 components, (b) 2.DOF with x_3, x_4 components.

5. Conclusion

The finite element method approved its efficiency in the numerical investigation of FPE, providing the possibility of obtaining details inaccessible to any analytical or semi-analytical procedures. This applies mainly to transition and other non-stationary or quasi-stationary processes.

Comparing hyper-brick and simplex elements, it reveals that the latter are incomparably more effective in element mesh flexibility and need only one type of (simplex) element. Integration is simple in both cases, using closed formulae. Approximate integration by means of Gaussian points can be avoided.

Nevertheless, results provided by closed formulae were validated by those obtained numerically using several procedures. Coincidence was perfect.

Space variability of coefficients in an original stochastic differential system can be neglected within one element if the variability is not extremely high. Due to the small size of the elements, the coefficients can be considered as being constant, reflecting their values in the element center. This enables the evaluation of the element matrices locally and independently on a nonlinear structure of the original nonlinear dynamic system. The variability of coefficients can be included only during uploading the element matrices into the global ones.

If system matrix $\mathbf{S}$ is time independent and well conditioned, then for $t \rightarrow \infty$ the stationary solution independent of initial conditions can be obtained. In such a case, the left hand side of the FPE can be canceled and a homogeneous linear algebraic system with the auxiliary constraint of PDF normalization can be investigated. On the other hand, the iterative method being based on dynamic relaxation strategy can also be used very effectively, if appropriate initial approximation is available.

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