

Response moments of dynamic systems under non-Gaussian random excitation by the equivalent non-Gaussian excitation method

Takahiro Tsuchida and Koji Kimura

Department of Mechanical and Environmental, Informatics, Tokyo Institute of Technology,
Tokyo, 152-8552, Japan

E-mail: ttsuchida@mei.titech.ac.jp

Abstract. Equivalent non-Gaussian excitation method is proposed to obtain the response moments up to the 4th order of dynamic systems under non-Gaussian random excitation. The non-Gaussian excitation is prescribed by the probability density and the power spectrum, and is described by an Itô stochastic differential equation. Generally, moment equations for the response, which are derived from the governing equations for the excitation and the system, are not closed due to the nonlinearity of the diffusion coefficient in the equation for the excitation even though the system is linear. In the equivalent non-Gaussian excitation method, the diffusion coefficient is replaced with the equivalent diffusion coefficient approximately to obtain a closed set of the moment equations. The square of the equivalent diffusion coefficient is expressed by a quadratic polynomial. In numerical examples, a linear system subjected to non-Gaussian excitations with bimodal and Rayleigh distributions is analyzed by using the present method. The results show that the method yields the variance, skewness and kurtosis of the response with high accuracy for non-Gaussian excitation with the widely different probability densities and bandwidth. The statistical moments of the equivalent non-Gaussian excitation are also investigated to describe the feature of the method.

1. Introduction

Response analysis methods for dynamic systems subjected to random excitation have been studied widely for many years[1, 2]. In these studies, the random excitation has often been assumed to be a Gaussian process. One reason for this assumption is due to the fact that many random phenomena have the probability densities similar to the Gaussian distribution. However, in some cases, the random excitation exhibits highly non-Gaussianity. The typical examples are road roughness[3], shallow water waves[4], wind pressure acting on low-rise buildings[5] and road vehicle vibrations[6]. The non-Gaussianity of such excitations is quite different from each other. The response of the system under non-Gaussian excitation is generally non-Gaussian and has the distinct characteristics from those in the case of Gaussian excitation. Thus, the response analysis in consideration of the non-Gaussianity of the excitation is important, and some analytical methods have been studied in recent years[3, 7–10]. However, Development of an analysis method for the system under non-Gaussian excitation is not being adequately performed. This is due to the reasons that the non-Gaussian excitation cannot be prescribed uniquely from the limited information on the correlation functions and the probability densities,



and that the mathematical treatment of the non-Gaussian excitation model is generally more involved than that of Gaussian excitation. Many methods which have been proposed so far are applicable only for a specific non-Gaussian excitation and are difficult to use when the non-Gaussianity of the excitation is different. It is therefore important to develop the analytical methods widely applicable to the systems subjected to excitations with various non-Gaussian distributions.

In this paper, equivalent non-Gaussian excitation method is proposed to obtain the statistical moments up to the 4th order of the response of non-Gaussian randomly excited systems. The non-Gaussian excitation is prescribed by the probability density function and the power spectrum, and is described by an Itô stochastic differential equation. Moment equations for the response can be derived by using the governing equations for the excitation and the system. However, the moment equations are generally not closed due to the nonlinearity of the diffusion coefficient in the governing equation for the excitation even though the system is linear. In the present method, the diffusion coefficient is replaced by the equivalent diffusion coefficient approximately to obtain a closed set of the moment equations. The square of the equivalent diffusion coefficient is expressed by a quadratic polynomial. The coefficients of the polynomial are determined according to the criterion of the least mean square error between the original diffusion coefficient and the equivalent diffusion coefficient.

In numerical examples, the proposed method is applied to a linear system under non-Gaussian excitations with bimodal and Rayleigh distributions. The analytical results are compared to the results obtained through Monte Carlo simulations. It is shown that the present method leads to the accurate results of the variance, skewness and kurtosis of the response for non-Gaussian excitation with the widely different probability densities and bandwidth. The probability distribution of the equivalent non-Gaussian excitation is also derived and the skewness and kurtosis of the distribution are investigated.

2. Dynamical model

2.1. Equation of motion

Consider a single-degree-of-freedom linear system described by

$$\ddot{X} + 2\zeta\dot{X} + X = U(t) \quad (1)$$

where ζ is the damping ratio and $U(t)$ is a stationary non-Gaussian random excitation.

2.2. Non-Gaussian random excitation

Non-Gaussian random excitation $U(t)$ is prescribed by the probability density function $p_U(u)$ and the power spectrum $S_U(\omega)$. In this paper, we consider $S_U(\omega)$ expressed by the following form:

$$S_U(\omega) = \frac{\alpha E[U^2]}{\pi(\omega^2 + \alpha^2)} \quad (2)$$

where α is the bandwidth parameter and $E[U^2]$ is the mean square of the excitation. The power spectra for $\alpha = 1, 0.1, 0.05, 0.02, 0.01$ and $E[U^2] = 1$ are shown in Fig.1. The excitation is wide-band when α is large, whereas the low-frequency component of the excitation dominates when α is small.

The excitation $U(t)$ with the specified probability distribution $p_U(u)$ and the power spectrum $S_U(\omega)$ given by Eq.(2) can be described by the following one-dimensional Itô stochastic differential equation[11]

$$dU = -\alpha U dt + D(U)dB(t) \quad (3)$$

where α is the bandwidth parameter in Eq.(2), $B(t)$ is a Wiener process and the diffusion coefficient $D(u)$ is expressed by

$$D^2(u) = -\frac{2\alpha}{p_U(u)} \int_{-\infty}^u sp_U(s)ds \quad (4)$$

It can be seen that the non-Gaussian distribution $p_U(u)$ is reflected in $D(u)$. On the other hand, the information on the power spectrum $S_U(\omega)$ is included in the drift coefficient in Eq.(3) (see Ref.[11]).

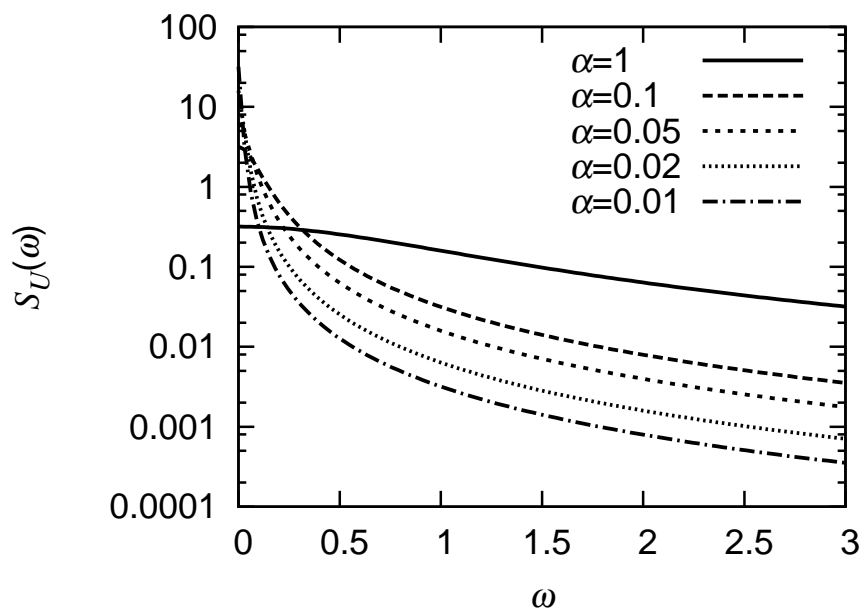


Figure 1. Power spectra of excitation ($E[U^2] = 1$).

3. Equivalent Non-Gaussian Excitation Method

3.1. Moment Equations

The augmented system including both the equation of motion (1) and the governing equation for the excitation (3) is described by a set of Itô stochastic differential equations as follows:

$$dX = \dot{X} dt \quad (5)$$

$$d\dot{X} = (-2\zeta\dot{X} - X + U)dt \quad (6)$$

$$dU = -\alpha U dt + D(U)dB(t) \quad (7)$$

Application of Itô's formula[12] to Eqs. (5)-(7) yields the $(i + j + k)$ th-order moment equations for the system response

$$\begin{aligned} \frac{d}{dt} E[X^i \dot{X}^j U^k] = & iE[X^{i-1} \dot{X}^{j+1} U^k] - jE[X^{i+1} \dot{X}^{j-1} U^k] - (2j\zeta + k\alpha) E[X^i \dot{X}^j U^k] \\ & + jE[X^i \dot{X}^{j-1} U^{k+1}] + \frac{1}{2}k(k-1)E[X^i \dot{X}^j U^{k-2} D^2(U)] \end{aligned} \quad (8)$$

$E[X^i \dot{X}^j U^{k-2} D^2(U)]$ on the right-hand side of Eq. (8) is generally complicated form because of the nonlinearity of the diffusion coefficient $D(u)$ given by Eq.(4), and thus the moment equations (8) are not closed. In this paper, equivalent non-Gaussian excitation method is proposed to obtain a closed set of the moment equations approximately.

3.2. Equivalent Non-Gaussian Excitation Method

The diffusion coefficient $D(U)$ in Eq.(7) is replaced approximately with the equivalent diffusion coefficient $D_{eq}(U)$. The square of $D_{eq}(U)$ is expressed by a quadratic polynomial

$$D_{eq}^2(U) = A_{eq}U^2 + B_{eq}U + C_{eq} \quad (9)$$

where the coefficients A_{eq} , B_{eq} and C_{eq} of the polynomial are determined according to the criterion of minimization of the mean square error $E[e^2]$ between the square of the original diffusion coefficient $D^2(U)$ and the square of the equivalent diffusion coefficient $D_{eq}^2(U)$

$$E[e^2] = E[(D^2(U) - D_{eq}^2(U))^2] = E[(D^2(U) - A_{eq}U^2 - B_{eq}U - C_{eq})^2] \quad (10)$$

Minimization of $E[e^2]$ is accomplished when the following three conditons hold

$$\frac{\partial E[e^2]}{\partial A_{eq}} = 0, \quad \frac{\partial E[e^2]}{\partial B_{eq}} = 0, \quad \frac{\partial E[e^2]}{\partial C_{eq}} = 0 \quad (11)$$

Eq.(11) yields

$$\begin{aligned} A_{eq} = & \frac{(E[U^2] - E[U]^2)(E[U^2 D^2(U)] - E[U^2]E[D^2(U)])}{(E[U^4] - E[U^2]^2)(E[U^2] - E[U]^2) - (E[U^3] - E[U]E[U^2])^2} \\ & - \frac{(E[U^3] - E[U]E[U^2])(E[U^2 D^2(U)] - E[U^2]E[D^2(U)])}{(E[U^4] - E[U^2]^2)(E[U^2] - E[U]^2) - (E[U^3] - E[U]E[U^2])^2} \end{aligned} \quad (12)$$

$$B_{eq} = \frac{(E[U^4] - E[U^2]^2)(E[U D^2(U)] - E[U]E[D^2(U)])}{(E[U^4] - E[U^2]^2)(E[U^2] - E[U]^2) - (E[U^3] - E[U]E[U^2])^2} \quad (13)$$

$$C_{eq} = E[D^2(U)] - A_{eq}E[U^2] - B_{eq}E[U] \quad (14)$$

Then, we derive the 2nd, 3rd and 4th order moment equations for the stationary excitation $U(t)$ by using Eq.(8)

$$-2\alpha E[U^2] + E[D^2(U)] = 0 \quad (15)$$

$$-3\alpha E[U^3] + 3E[uD^2(U)] = 0 \quad (16)$$

$$-4\alpha E[U^4] + 6E[u^2D^2(U)] = 0 \quad (17)$$

From Eqs.(12)-(17), it is found that the coefficients A_{eq} , B_{eq} and C_{eq} are evaluated by the moments up to the 4th order of $U(t)$. The moments $E[U^n]$ ($n = 1, 2, 3, 4$) can be calculated analytically or numerically by using the non-Gaussian distribution $p_U(u)$, which is given in advance to prescribe the excitation $U(t)$.

By replacing $D(U)$ in Eq.(7) with $D_{eq}(U)$ obtained through the above procedure, the stochastic differential equation governing the equivalent non-Gaussian excitation can be obtained as

$$dU = -\alpha U dt + \sqrt{A_{eq}U^2 + B_{eq}U + C_{eq}} dB(t) \quad (18)$$

In this method, since the drift coefficient is not replaced, the power spectrum of the excitation given by Eq.(2) remains unchanged.

Eqs.(5), (6) and (18) lead to the $(i+j+k)$ th-order moment equations for the system response

$$\begin{aligned} \frac{d}{dt} E[X^i \dot{X}^j U^k] &= i E[X^{i-1} \dot{X}^{j+1} U^k] - j E[X^{i+1} \dot{X}^{j-1} U^k] \\ &+ \left(-2j\zeta - k\alpha + \frac{1}{2}k(k-1)A_{eq} \right) E[X^i \dot{X}^j U^k] \\ &+ j E[X^i \dot{X}^{j-1} U^{k+1}] + \frac{1}{2}k(k-1)B_{eq} E[X^i \dot{X}^j U^{k-1}] \\ &+ \frac{1}{2}k(k-1)C_{eq} E[X^i \dot{X}^j U^{k-2}] \end{aligned} \quad (19)$$

In Eq.(19), the moments of the order higher than $(i+j+k)$ do not appear. Therefore, Eq. (19) is closed form and can be solved analytically. When the stationary response is considered, the time derivative term on the left-hand side of Eq.(19) vanishes and Eq.(19) reduces to the algebraic equation.

4. Numerical example

Equivalent non-Gaussian excitation method will now be applied to a linear system subjected to non-Gaussian random excitations with two types of probability densities and the power spectrum given by Eq.(2). The variance, skewness and kurtosis of the system response obtained by the method are compared with the Monte Carlo simulation results to validate the effectiveness of the proposed method.

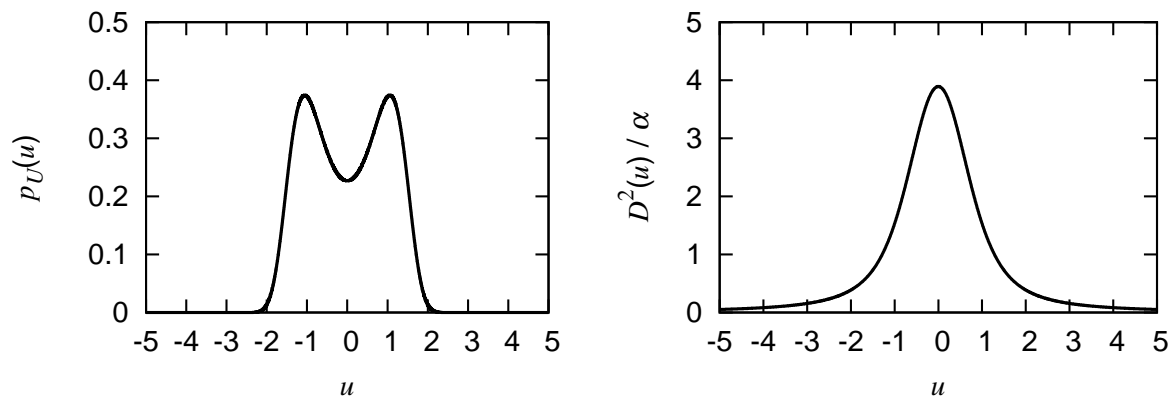


Figure 2. Bimodal distribution (left) and diffusion coefficient (right) ($a = 0.8935$, $b = -0.3991$, $C = 0.2269$).

4.1. Excitation distribution and diffusion coefficient

We consider bimodal and Rayleigh distributions for the excitation probability density. The bimodal distribution are expressed by

$$p_U(u) = C \exp(au^2 + bu^4) \quad (a > 0, b < 0) \quad (20)$$

where a and b are the shape parameters of the distribution, and C is the normalization constant. In this paper, in order to set the variance equals to 1, a , b and C are given as follows:

$$a = 0.8935, b = -0.3991, C = 0.2269 \quad (21)$$

Then, the kurtosis γ_{4U} of the bimodal distribution is

$$\gamma_{4U} \simeq -1.25446 \quad (22)$$

where γ_{4U} is defined so that the kurtosis of Gaussian distribution is 0. In Fig.2, the bimodal distribution with the parameters in Eq.(21) is shown. The diffusion coefficient $D(u)$ corresponding to the bimodal distribution is derived from Eqs.(4) and (20)

$$D^2(u) = \frac{\alpha}{2} \sqrt{\frac{\pi}{-b}} \exp \left\{ -b \left(u^2 + \frac{a}{2b} \right)^2 \right\} \operatorname{erfc} \left\{ \sqrt{-b} \left(u^2 + \frac{a}{2b} \right) \right\} \quad (23)$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad (24)$$

Fig.2 shows $D^2(u)/\alpha$ with the parameters in Eq.(21).

We consider the Rayleigh distribution which is shifted so that the probability density has zero mean.

$$p_U(u) = \frac{u + \theta\sqrt{\frac{\pi}{2}}}{\theta^2} \exp \left(-\frac{(u + \theta\sqrt{\frac{\pi}{2}})^2}{2\theta^2} \right), \quad u \geq -\theta\sqrt{\frac{\pi}{2}} \quad (25)$$

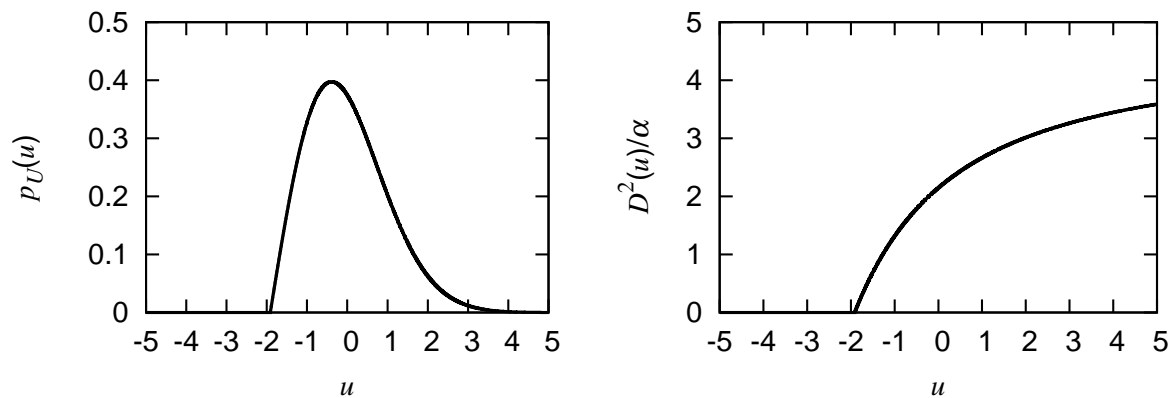


Figure 3. Shifted Rayleigh distribution (left) and diffusion coefficient (right) $\left(\theta = \sqrt{\frac{2}{4-\pi}} \right)$.

where θ is the scale parameter. In order to set the variance equals to 1, we use

$$\theta = \sqrt{\frac{2}{4-\pi}} \quad (26)$$

The Rayleigh distribution is shown in Fig.3. The skewness γ_{3U} and kurtosis γ_{4U} are given by

$$\gamma_{3U} = \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}} \simeq 0.631111, \quad \gamma_{4U} = -\frac{6\pi^2-24\pi+16}{(4-\pi)^2} \simeq 0.245089 \quad (27)$$

The diffusion coefficient $D(u)$ corresponding to the shifted Rayleigh distribution is obtained from Eqs.(4) and (25)

$$D^2(u) = 2\alpha\theta^2 \frac{\theta\sqrt{\frac{\pi}{2}} \left[1 - \operatorname{erf} \left(\frac{u}{\sqrt{2}\theta} + \frac{\sqrt{\pi}}{2} \right) \right] + u \exp \left[-\frac{1}{2\theta^2} \left(u + \theta\sqrt{\frac{\pi}{2}} \right)^2 \right]}{\left(u + \theta\sqrt{\frac{\pi}{2}} \right) \exp \left[-\frac{1}{2\theta^2} \left(u + \theta\sqrt{\frac{\pi}{2}} \right)^2 \right]}, \quad u \geq -\theta\sqrt{\frac{\pi}{2}} \quad (28)$$

where $\operatorname{erf}(\cdot)$ is the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (29)$$

$D^2(u)/\alpha$ with the parameters in Eq.(26) is shown in Fig.3.

4.2. Application of equivalent non-Gaussian excitation method

Since the diffusion coefficients $D(u)$ given by Eqs.(23) and (28) are the complicated expression, the moment equations (8) are not closed. Therefore, equivalent non-Gaussian excitation method presented in section 3.2 is applied. $D(U)$ is replaced approximately with the equivalent diffusion coefficient $D_{eq}(U)$ expressed by Eq.(9). The statistical moments $E[U^n]$ ($n = 1, 2, 3, 4$) in the coefficients A_{eq} , B_{eq} and C_{eq} of $D_{eq}(U)$ (Eqs.(12)-(17)) are calculated by using the excitation distribution $p_U(u)$.

In this example, for the bimodal-distributed excitation, the moments $E[U^n]$ ($n = 1, 2, 3, 4$) are evaluated numerically from Eq.(20). For the Rayleigh-distributed excitation, $E[U^n]$ ($n = 1, 2, 3, 4$) are obtained from the fact that the Rayleigh-distribution has the zero mean, unit variance, skewness and kurtosis given by Eq.(27).

Substituting A_{eq} , B_{eq} and C_{eq} into Eq.(18) yields the stochastic differential equation for the equivalent non-Gaussian excitation. Using this equation and Eqs.(5) and (6), the moment equations can be derived (Eq.(19)). In this paper, the moment equations up to the 4th order whose time derivative terms vanish are solved to obtain the variance, skewness and kurtosis of the stationary response. The moment equations are shown in appendix.

4.3. Results

(i) Bimodal-distributed excitation

The variance and kurtosis of the stationary displacement response to the bimodal-distributed excitation obtained by the equivalent non-Gaussian excitation method are shown in Fig.4. In Fig.4(b), the solid line and \odot denote the analytical results and the Monte Carlo simulation results, respectively. The horizontal axis indicates the non-dimensional parameter A , which represents the ratio of the excitation bandwidth α to the bandwidth ζ of the frequency response function of the system ($\zeta = 0.05$ in this numerical example).

$$A = \alpha/\zeta \quad (30)$$

$A < 1$ corresponds to the narrower excitation bandwidth than the system bandwidth, and $A > 1$ corresponds to the wider excitation bandwidth than the system bandwidth.

The stationary response variance σ_X^2 of a linear system is obtained exactly by using the excitation power spectrum $S_U(\omega)$ and the frequency response function $|H(\omega)|^2$ of the system

$$\sigma_X^2 = \int_{-\infty}^{\infty} |H(\omega)|^2 S_U(\omega) d\omega = \frac{E[U^2]}{2} \frac{A+2}{\zeta^2(1+A)^2 + 1 - \zeta^2} \quad (31)$$

The analytical result shown in Fig.4(a) perfectly agrees with this exact solution, because in the present method, the power spectrum of the excitation $S_U(\omega)$ given by Eq.(2) remains unchanged, as described in section 3.2. The analytical result of the response kurtosis in Fig.4(b) is in good agreement with the Monte Carlo simulation result for a wide range of the bandwidth ratio A .

(ii) Shifted Rayleigh-distributed excitation

The skewness and kurtosis of the stationary displacement response to the shifted-Rayleigh-distributed excitation are shown in Fig.5. The result of the response variance is the same as that shown in Fig.4(a). The skewness and the kurtosis obtained with the present method are in good agreement with the simulated results.

The two numerical examples shown above illustrate that the equivalent non-Gaussian excitation method is applicable to non-Gaussian excitation with the widely different probability densities and bandwidth.

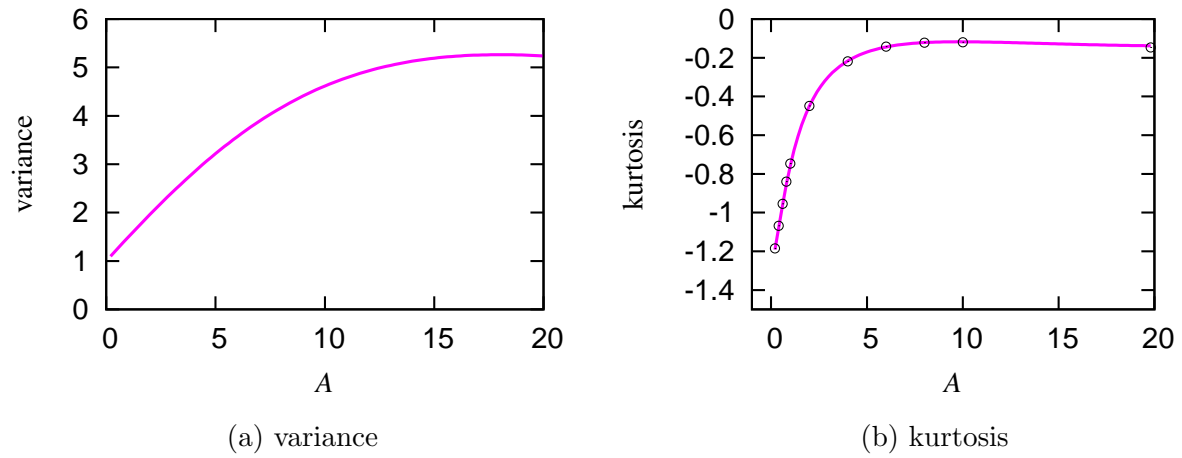


Figure 4. Variance and kurtosis of stationary displacement response to bimodal-distributed excitation. — : present method, \odot : Monte Carlo simulation.

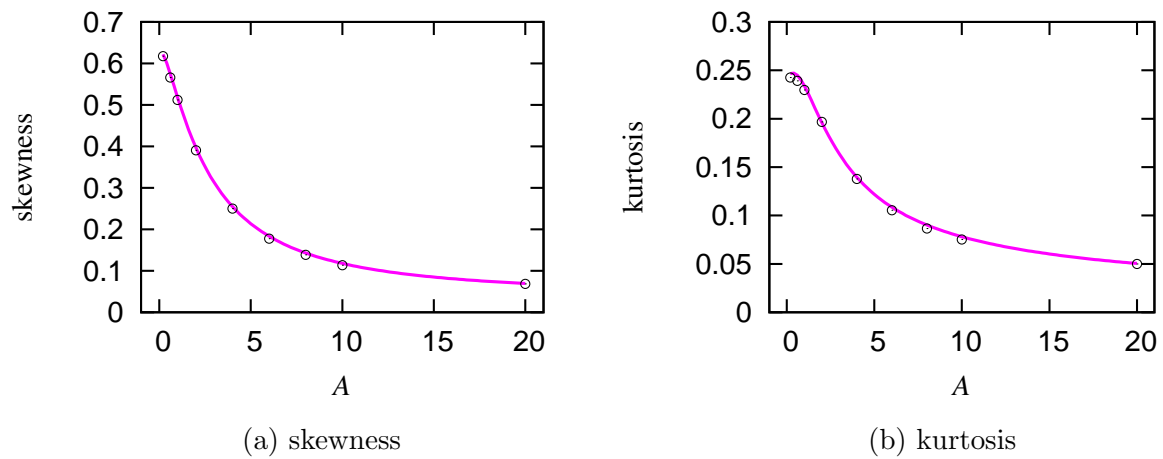


Figure 5. Skewness and kurtosis of stationary displacement response to shifted-Rayleigh-distributed excitation. — : present method, \odot : Monte Carlo simulation.

5. Comparison of skewness and kurtosis between equivalent excitation distribution and original excitation distribution

The reduced Fokker-Planck equation corresponding to Eq.(18) is derived as

$$-\frac{d}{du} \left(\alpha u p_{Ueq}(u) + \frac{1}{2} \frac{d}{du} [(A_{eq}u^2 + B_{eq}u + C_{eq})p_{Ueq}(u)] \right) = 0 \quad (32)$$

where $p_{Ueq}(u)$ is the probability density of the equivalent non-Gaussian excitation described by Eq.(18). Solving Eq. (32) subject to the boundary conditions given by

$$p_{Ueq}(\pm\infty) = 0, \quad \frac{d}{du} p_{Ueq}(\pm\infty) = 0 \quad (33)$$

yields the following probability distribution

$$4A_{eq}C_{eq} - B_{eq}^2 < 0:$$

$$p_{Ueq}(u) = C \exp \left[-\frac{A_{eq} + \alpha}{A_{eq}} \ln(A_{eq}u^2 + B_{eq}u + C_{eq}) + \frac{2B_{eq}}{\sqrt{B_{eq}^2 - 4A_{eq}C_{eq}}} \left(1 - \frac{A_{eq} + \alpha}{A_{eq}} \right) \operatorname{arctanh} \left(\frac{2A_{eq}u + B_{eq}}{\sqrt{B_{eq}^2 - 4A_{eq}C_{eq}}} \right) \right] \quad (34)$$

$$4A_{eq}C_{eq} - B_{eq}^2 = 0:$$

$$p_{Ueq}(u) = C \exp \left[-\frac{A_{eq} + \alpha}{A_{eq}} \ln(A_{eq}u^2 + B_{eq}u + C_{eq}) + \frac{2B_{eq}}{2A_{eq}u + B_{eq}} \left(1 - \frac{A_{eq} + \alpha}{A_{eq}} \right) \right] \quad (35)$$

$$4A_{eq}C_{eq} - B_{eq}^2 > 0:$$

$$p_{Ueq}(u) = C \exp \left[-\frac{A_{eq} + \alpha}{A_{eq}} \ln(A_{eq}u^2 + B_{eq}u + C_{eq}) - \frac{2B_{eq}}{\sqrt{4A_{eq}C_{eq} - B_{eq}^2}} \left(1 - \frac{A_{eq} + \alpha}{A_{eq}} \right) \arctan \left(\frac{2A_{eq}u + B_{eq}}{\sqrt{4A_{eq}C_{eq} - B_{eq}^2}} \right) \right] \quad (36)$$

where C represents the normalization constant and the domain of $p_{Ueq}(u)$ is $\{u \mid A_{eq}u^2 + B_{eq}u + C_{eq} > 0\}$. Although the excitation bandwidth parameter α appears in these equations, it can be confirmed that the shape of $p_{Ueq}(u)$ does not depend on α by substituting the numerical values into A_{eq} , B_{eq} and C_{eq} and rearranging the expression. For each excitation distribution $p_U(u)$, $p_{Ueq}(u)$ takes one of the forms given by Eqs.(34)-(36) depending on the value of $4A_{eq}C_{eq} - B_{eq}^2$.

The skewness and kurtosis of the equivalent excitation distribution $p_{Ueq}(u)$ and the original excitation distribution $p_U(u)$ is compared. The skewness and kurtosis of $p_{Ueq}(u)$ can be obtained numerically by using Eqs.(34)-(36) or analytically by using the moment equations derived from Eq.(18). For the bimodal-distributed excitation, $p_{Ueq}(u)$ is expressed by Eq.(34). The kurtosis of $p_{Ueq}(u)$ is -1.25394 , and is almost the same as the kurtosis of the original bimodal distribution (Eq.(22)). For the Rayleigh-distributed excitation, $p_{Ueq}(u)$ is given by Eq.(34) and has the skewness equals to 0.631112 and the kurtosis equals to 0.245088 . These values also agree very well with the skewness and kurtosis of the original Rayleigh distribution (Eq.(27)). The results can be explained by the fact that as shown in Eqs.(12)-(17), the coefficients A_{eq} , B_{eq} and C_{eq} , which determine the shape of $p_{Ueq}(u)$, are evaluated by the moments up to the 4th order of $p_U(u)$. Due to this, $p_{Ueq}(u)$ can retain these four moments accurately. This feature allows us to obtain the variance, skewness and kurtosis of the response with high accuracy by the proposed method.

6. Conclusions

Equivalent non-Gaussian excitation method has been proposed to obtain the statistical moments up to the 4th order of the response of non-Gaussian randomly excited systems. The non-Gaussian excitation is prescribed by the probability density function and the power spectrum, and is described by an Itô stochastic differential equation. Moment equations for the response can be derived from the stochastic differential equations for the excitation and the system. However, the moment equations are not closed due to the complex nonlinearity of the diffusion coefficient in the governing equation for the excitation.

In equivalent non-Gaussian excitation method, the diffusion coefficient is replaced with the equivalent diffusion coefficient approximately to obtain a closed set of the moment equations. The square of the equivalent diffusion coefficient is expressed by a quadratic polynomial. The coefficients of the polynomial are determined according to the criterion of minimization of the mean square error between the original diffusion coefficient and the equivalent diffusion coefficient.

In order to demonstrate the validity of the present method, the method has been applied to a linear system subjected to non-Gaussian random excitations with bimodal and Rayleigh distributions. The results obtained by the present method have been compared with the results of the Monte Carlo simulations. The comparison shows that the proposed method yields the variance, skewness and kurtosis of the response accurately for non-Gaussian excitations with the widely different probability densities and bandwidth.

The probability density function of the equivalent non-Gaussian excitation has also been derived from the Fokker-Planck equation. It has been shown that the skewness and kurtosis of the equivalent excitation distribution agree very well with those of the original excitation distribution.

Appendix

The moment equations up to the 4th order for stationary response are shown below. These equations can be derived from Eq.(19).

1st-order moment equations

$$E[\dot{X}] = 0 \quad (\text{A.1})$$

$$-E[X] - 2\zeta E[\dot{X}] + E[U] = 0 \quad (\text{A.2})$$

$$-\alpha E[U] = 0 \quad (\text{A.3})$$

2nd-order moment equations

$$2E[X\dot{X}] = 0 \quad (\text{A.4})$$

$$-E[X^2] + E[\dot{X}^2] - 2\zeta E[X\dot{X}] + E[XU] = 0 \quad (\text{A.5})$$

$$-\alpha E[XU] + E[\dot{X}U] = 0 \quad (\text{A.6})$$

$$-2E[X\dot{X}] - 4\zeta E[\dot{X}^2] + 2E[\dot{X}U] = 0 \quad (\text{A.7})$$

$$-E[XU] - (2\zeta + \alpha)E[\dot{X}U] + 2E[U^2] = 0 \quad (\text{A.8})$$

$$(-2\alpha + A_{eq})E[U^2] + B_{eq}E[U] + C_{eq} = 0 \quad (\text{A.9})$$

3rd-order moment equations

$$3E[X^2\dot{X}] = 0 \quad (\text{A.10})$$

$$-E[X^3] + 2E[X\dot{X}^2] - 2\zeta E[X^2\dot{X}] + E[X^2U] = 0 \quad (\text{A.11})$$

$$E[\dot{X}^3] - 2E[X^2\dot{X}] - 4\zeta E[X\dot{X}^2] + 2E[X\dot{X}U] = 0 \quad (\text{A.12})$$

$$2E[X\dot{X}U] - \alpha E[X^2U] = 0 \quad (\text{A.13})$$

$$E[\dot{X}U^2] + (-2\alpha + A_{eq})E[XU^2] + B_{eq}E[XU] + C_{eq}E[X] = 0 \quad (\text{A.14})$$

$$E[\dot{X}^2U] - E[X^2U] - (2\zeta + \alpha)E[X\dot{X}U] + E[XU^2] = 0 \quad (\text{A.15})$$

$$-3E[X\dot{X}^2] - 6\zeta E[\dot{X}^3] + 3E[\dot{X}^2U] = 0 \quad (\text{A.16})$$

$$-2E[X\dot{X}U] - (4\zeta + \alpha)E[\dot{X}^2U] + 2E[\dot{X}U^2] = 0 \quad (\text{A.17})$$

$$-E[XU^2] - (2\zeta + 2\alpha - A_{eq})E[\dot{X}U^2] + E[U^2] + B_{eq}E[\dot{X}U] + C_{eq}E[\dot{X}] = 0 \quad (\text{A.18})$$

$$(-3\alpha + 3A_{eq})E[U^3] + 3B_{eq}E[U^2] + 3C_{eq}E[U] = 0 \quad (\text{A.19})$$

4th-order moment equations

$$4E[X^3\dot{X}] = 0 \quad (\text{A.20})$$

$$-E[X^4] - 2\zeta E[X^3\dot{X}] + 3E[X^2\dot{X}^2] + E[X^3U] = 0 \quad (\text{A.21})$$

$$-2E[X^3\dot{X}] - 4\zeta E[X^2\dot{X}^2] + 2E[X\dot{X}^3] + E[X^2\dot{X}U] = 0 \quad (\text{A.22})$$

$$-3E[X^2\dot{X}^2] - 6\zeta E[X\dot{X}^3] + 3E[X\dot{X}^2U] + E[\dot{X}^4] = 0 \quad (\text{A.23})$$

$$-\alpha E[X^3U] + 3E[X^2\dot{X}U] = 0 \quad (\text{A.24})$$

$$(-2\alpha + A_{eq})E[X^2U^2] + 2E[X\dot{X}U^2] + B_{eq}E[X^2U] + C_{eq}E[X^2] = 0 \quad (\text{A.25})$$

$$(-3\alpha + 3A_{eq})E[XU^3] + E[\dot{X}U^3] + 3B_{eq}E[XU^2] + 3C_{eq}E[XU] = 0 \quad (\text{A.26})$$

$$-E[X^3U] - (2\zeta + \alpha)E[X^2\dot{X}U] + E[X^2U^2] + 2E[X\dot{X}^2U] = 0 \quad (\text{A.27})$$

$$-2E[X^2\dot{X}U] - (4\zeta + \alpha)E[X\dot{X}^2U] + 2E[X\dot{X}U^2] + E[\dot{X}^3U] = 0 \quad (\text{A.28})$$

$$-E[X^2U^2] - (2\zeta + 2\alpha - A_{eq})E[X\dot{X}U^2] + E[XU^3] + E[\dot{X}^2U^2] + B_{eq}E[X\dot{X}U] + C_{eq}E[X\dot{X}] = 0 \quad (\text{A.29})$$

$$-4E[X\dot{X}^3] - 8\zeta E[\dot{X}^4] + 4E[\dot{X}^3U] = 0 \quad (\text{A.30})$$

$$-3E[X\dot{X}^2U] - (6\zeta + \alpha)E[\dot{X}^3U] + 3E[\dot{X}^2U^2] = 0 \quad (\text{A.31})$$

$$-2E[X\dot{X}U^2] - (4\zeta + 2\alpha - A_{eq})E[\dot{X}^2U^2] + 2E[\dot{X}U^3] + B_{eq}E[\dot{X}^2U] + C_{eq}E[\dot{X}^2] = 0 \quad (\text{A.32})$$

$$-E[XU^3] - (2\zeta + 3\alpha - 3A_{eq})E[\dot{X}U^3] + E[U^4] + 3B_{eq}E[\dot{X}U^2] + 3C_{eq}E[\dot{X}U] = 0 \quad (\text{A.33})$$

$$(-4\alpha + 6A_{eq})E[U^4] + 6B_{eq}E[U^3] + 6C_{eq}E[U^2] = 0 \quad (\text{A.34})$$

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