

Mathematic study of the rotor motion with a pendulum self-balancing device

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Abstract. The rotary machines used in manufacturing may become unbalanced leading to vibration. In some cases, the problem may be solved by installing self-balancing devices (SBDs). Certain factors, however, exhibit a pronounced effect on the efficiency of these devices. The objective of the research comprised of establishing the most beneficial spatial position of pendulums to minimize the necessary time to repair the rotor unbalance. The mathematical research of the motion of a rotor with pendulum SBDs in the situation of their misalignment was undertaken. This objective was achieved by using the Lagrange equations of the second type. The analysis identified limiting cases of location of the rotor unbalance vector and the vector of housing's unbalance relative to each other, as well as the minimum capacity of the pendulum. When determining pendulums' parameters during the SBD design process, it is necessary to take into account the rotor unbalance and the unbalance of the machine body, which is caused by the misalignment of rotor axis and pendulum's axis of rotation.

1. Introduction

The rotary machines used in manufacturing may become unbalanced resulting in the machine's vibration. In some cases, the problem can be solved by installing self-balancing devices (SBDs) [1-6].

When determining pendulums' parameters during the SBD design process, it is necessary to take into account rotor unbalance and unbalance of the machine body, which is caused by the misalignment of rotor axis and pendulum's axis of rotation.

2. Equations of plane motion

2.1. Equations of plane motion of a rotor with self-balancing device (SBD).

Let us consider a mechanical system consisting of a rotor and a housing. The rotor is mounted in bearings of the housing, which, in its turn, is elastically connected to a stationary support. Pendulums are mounted movably on the rotor.

To comprise the equations of motion, the Lagrange equations of the second type were used. The notations for the Figure 1 include: the rotor's pivot point O_I , the center of rotor gravity C , and the pendulums' pivot point O_2 . The following coordinate systems are introduced: $\eta\xi$ is rigidly bound to the foundation; $x_I y_I$ performs translational pendulums, starting in the point O_I ; $x_R y_R$ is rigidly bound to the rotor; $x_{p1} y_{p1}$ and $x_{p2} y_{p2}$ as a partial cases of $x_p y_p$ in the Lagrange equations starting in the point O_2 ; the axis x_P is rigidly bound to the center of pendulum mass. The turning angle of the rotor is defined as angle φ , and that of the axis x_P as angle γ_k . Position of the point O_2 relative to the point O_I is the angle β and the eccentricity ε (Figure 1). Assume that the housing moves only progressively. Masses of the



housing, the rotor and the pendulums are denoted as M_H , M_R and m . Rigidities of the housing's connection to the stationary support are assumed to be C_η and C_ξ .

The inertia force is significantly larger, than the gravity force thus making the latter ignored.

In order to compose the Lagrange equations of the second type, the kinetic energy of the system T was determined:

$$T = T_R + T_H + \sum_{k=1}^n T_P^k$$

where T_R , T_H and T_P are kinetic energy of the rotor, the housing, and the pendulum, respectively; n is the quantity of pendulums.

The rotor performing plane motion has the kinetic energy as follows:

$$T_R = 0.5 \cdot I \dot{\varphi}^2 + 0.5 \cdot M_R V_{CR}^2$$

where I is the moment of the rotor's inertia relative to the axis passing through the mass center; V_{CR} is the velocity of the rotor's center of mass.

The kinetic energy of the progressively moving housing is:

$$T_H = 0.5 \cdot M_H V_1^2$$

where V_1 is the housing velocity.

The kinetic energy of the pendulum making plane motion is:

$$T_P = 0.5 \cdot m V_{CP}^2 + 0.5 \cdot I_P \dot{\gamma}_P^2$$

where V_{CP} is the velocity of the pendulum's center of mass, I_P is the pendulum's moment of inertia relative to the axis passing through the mass center.

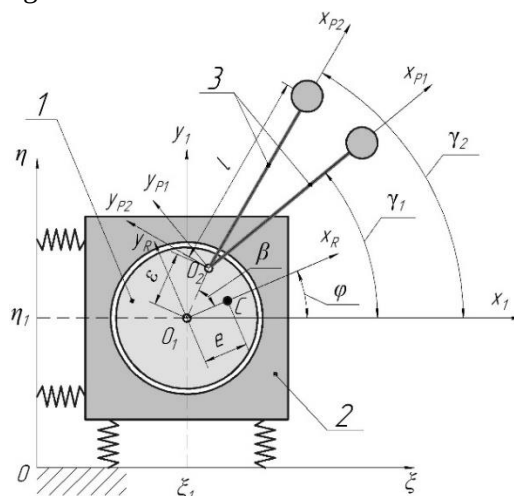


Figure 1. Outline of pendulum self-balancing device with axis misalignment::
1 – rotor; 2 – machine housing; 3 – pendulums.

The velocities V_{CR} , V_1 and V_{CP} are determined using the method of coordinates:

$$V_1^2 = \dot{\xi}_1^2 + \dot{\eta}_1^2; \quad V_{CR}^2 = \dot{\xi}_{CR}^2 + \dot{\eta}_{CR}^2; \quad V_{CP}^2 = \dot{\xi}_{CP}^2 + \dot{\eta}_{CP}^2 \quad (1)$$

The mass centers' coordinates were found as follows:

$$\xi_{CR} = \xi_1 + e \cos \varphi \quad \eta_{CR} = \eta_1 + e \sin \varphi$$

$$\xi_{CP} = \xi_1 + \varepsilon \cos(\varphi + \beta) + l \cos \gamma \quad \eta_{CP} = \eta_1 + \varepsilon \sin(\varphi + \beta) + l \sin \gamma$$

where l is pendulum length, e is the displacement of the rotor's mass center relative to the axis of rotation.

After differentiating these expressions with respect to time with their subsequent substitution to the equations (1), the squares of velocities were found as follows:

$$V_{CR}^2 = (\dot{\xi}_1 - e \dot{\varphi} \sin \varphi)^2 + (\dot{\eta}_1 + e \dot{\varphi} \cos \varphi)^2$$

$$V_{CP}^2 = (\dot{\xi}_1 - \varepsilon \dot{\varphi} \sin(\varphi + \beta) - l \dot{\gamma} \sin \gamma)^2 + (\dot{\eta}_1 + \varepsilon \dot{\varphi} \cos(\varphi + \beta) + l \dot{\gamma} \cos \gamma)^2$$

Taking into account the obtained expressions, the system's kinetic energy equation becomes:

$$\begin{aligned} T = & 0.5 \cdot I_{CR} \dot{\varphi}^2 + 0.5 \cdot M_R \left[(\dot{\xi}_1 - e \dot{\varphi} \sin \varphi)^2 + (\dot{\eta}_1 + e \dot{\varphi} \cos \varphi)^2 \right] + 0.5 \cdot M_H (\dot{\xi}_1^2 + \dot{\eta}_1^2) + \\ & 0.5 \cdot \sum_{k=1}^n m \left[(\dot{\xi}_1 - \varepsilon \dot{\varphi} \sin(\varphi + \beta) - l \dot{\gamma}_k \sin \gamma_k)^2 + (\dot{\eta}_1 + \varepsilon \dot{\varphi} \cos(\varphi + \beta) + l \dot{\gamma}_k \cos \gamma_k)^2 \right] + \\ & + 0.5 \cdot \sum_{k=1}^n I_{P_k} \dot{\gamma}_k^2 \end{aligned}$$

After determining the partial derivatives and time derivatives, the system of Lagrange's equations of the second type was formulated:

$$\begin{cases}
\left[M_H + M_P + \sum_{k=1}^n m \right] \ddot{\xi}_1 - \left[M_R e \sin \varphi + \sum_{k=1}^n m \varepsilon \sin(\varphi + \beta) \right] \ddot{\phi} - \sum_{k=1}^n m l \sin \gamma_k \ddot{\gamma}_k - \\
- \left[M_R e \cos \varphi + \sum_{k=1}^n m \varepsilon \cos(\varphi + \beta) \right] \dot{\phi}^2 - \sum_{k=1}^n m l \cos \gamma_k \dot{\gamma}_k^2 = -C_\xi \ddot{\xi}_1 \\
\left[M_H + M_R + \sum_{k=1}^n m \right] \ddot{\eta}_1 + \left[M_R e \cos \varphi + \sum_{k=1}^n m \varepsilon \cos(\varphi + \beta) \right] \ddot{\phi} + \sum_{k=1}^n m l \cos \gamma_k \ddot{\gamma}_k - \\
- \left[M_R e \sin \varphi + \sum_{k=1}^n m \varepsilon \sin(\varphi + \beta) \right] \dot{\phi}^2 - \sum_{k=1}^n m l \sin \gamma_k \dot{\gamma}_k^2 = -C_\eta \ddot{\eta}_1 \\
\left[I_R + M_R e^2 + \sum_{k=1}^n m \varepsilon^2 \right] \ddot{\phi} - \left[M_R e \sin \varphi + \sum_{k=1}^n m \varepsilon \sin(\varphi + \beta) \right] \ddot{\xi}_1 + \left[M_R e \cos \varphi + \right. \\
+ \left. \sum_{k=1}^n m \varepsilon \cos(\varphi + \beta) \right] \ddot{\eta}_1 + \left[\sum_{k=1}^n m l \varepsilon \cos \gamma_k \cos(\varphi + \beta) + \sum_{k=1}^n m l \varepsilon \sin \gamma_k \sin(\varphi + \beta) \right] \ddot{\gamma}_k + \\
+ \left[\sum_{k=1}^n m l \varepsilon \cos \gamma_k \sin(\varphi + \beta) - \sum_{k=1}^n m l \varepsilon \sin \gamma_k \cos(\varphi + \beta) \right] \dot{\gamma}_k^2 + \sum_{k=1}^n m \varepsilon^2 \sin(\varphi + \beta) \cos(\varphi + \beta) \dot{\phi}^2 = \\
= M_{motor} - \chi \dot{\phi} - h_1 (\dot{\phi} - \dot{\gamma}_k) \\
\left[I_P + m l^2 \right] \ddot{\gamma}_k - m l \sin \gamma_k \ddot{\xi}_1 + m l \cos \gamma_k \ddot{\eta}_1 + m l \varepsilon \left[\sin \gamma_k \sin(\varphi + \beta) + \cos \gamma_k \cos(\varphi + \beta) \right] \ddot{\phi} \\
+ m l \varepsilon \left[\sin \gamma_k \cos(\varphi + \beta) - \cos \gamma_k \sin(\varphi + \beta) \right] \dot{\phi}^2 = -h_1 (\dot{\gamma}_k - \dot{\phi}), \quad (k=1, 2, \dots, n)
\end{cases} \quad (2)$$

where h_1 is the viscous drag force proportionality factor; M_{motor} is motor torque; χ – is the motor damping factor.

Further analysis was carried out for isotropic support stiffness ($C_\eta = C_\xi$).

2.2. Partial solution of a system of equations for plane motion of a rotor with self-balancing device

Let us address a case when the rotor acceleration has been completed, and the motion of the rotor–pendulums system becomes stationary. This way, we assume that:

$$\begin{aligned}
\varphi &= \Omega t \\
\dot{\phi} &= \dot{\gamma}_k = \text{const} = \Omega \\
\gamma_k &= (\Omega t + \theta_k) \\
\ddot{\phi} &= \ddot{\gamma}_k = 0
\end{aligned}$$

Under these conditions, the system of equations becomes the following form:

$$\left\{ \begin{aligned} & \left[M_H + M_R + \sum_{k=1}^n m \right] \ddot{\xi}_1 + C_\xi \xi_1 = \Omega^2 \left[M_R e \cos(\Omega t) + \sum_{k=1}^n m \varepsilon \cos(\Omega t + \beta) + \right. \\ & \left. + \sum_{k=1}^n m l \cos(\Omega t + \theta_k) \right] \\ & \left[M_H + M_R + \sum_{k=1}^n m \right] \ddot{\eta}_1 + C_\eta \eta_1 = \Omega^2 \left[M_R e \sin(\Omega t) + \sum_{k=1}^n m \varepsilon \sin(\Omega t + \beta) + \right. \\ & \left. + \sum_{k=1}^n m l \sin(\Omega t + \theta_k) \right] \\ & \sum_{k=1}^n m l \sin(\Omega t + \theta_k) \ddot{\xi}_1 - \sum_{k=1}^n m l \cos(\Omega t + \theta_k) \ddot{\eta}_1 - \Omega^2 \left[\sum_{k=1}^n m l \varepsilon \sin(\theta_k - \beta) \right] = 0 \quad (k=1, 2, \dots, n) \end{aligned} \right. \quad (3)$$

At that, the last two equations of the system (3) are independent. Let us consider the solution of the first equation of the system and reduce it to the following form:

$$\ddot{\xi}_1 + k^2 \xi_1 = H_R \cos(\Omega t) + H_O \cos(\Omega t + \beta) + H_P \sum_{k=1}^n \cos(\Omega t + \theta_k)$$

where

$$\begin{aligned} k^2 &= \frac{C_\xi}{M_H + M_R + \sum_{k=1}^n m} & H_R &= \frac{\Omega^2 M_R e}{M_H + M_R + \sum_{k=1}^n m} \\ H_P &= \frac{\Omega^2 \sum_{k=1}^n m l}{M_H + M_R + \sum_{k=1}^n m} & H_O &= \frac{\Omega^2 m \varepsilon}{M_H + M_R + \sum_{k=1}^n m} \end{aligned}$$

Let us find the partial solution of the complete equation in the form:

$$\xi = A \sin(\Omega t) + B \cos(\Omega t)$$

After the transformations, we obtain:

$$\xi = \left[\left(H_R + H_O \cos \beta + H_P \sum_{k=1}^n \cos \theta_k \right) / (k^2 - \Omega^2) \right] \cos(\Omega t) - \left[\left(H_O \sin \beta + H_P \sum_{k=1}^n \sin \theta_k \right) / (k^2 - \Omega^2) \right] \sin(\Omega t)$$

Similarly, by solving the second equation of the system we find:

$$\eta = \left[\left(H_R + H_O \cos \beta + H_P \sum_{k=1}^n \cos \theta_k \right) / (k^2 - \Omega^2) \right] \sin(\Omega t) + \left[\left(H_O \sin \beta + H_P \sum_{k=1}^n \sin \theta_k \right) / (k^2 - \Omega^2) \right] \cos(\Omega t) \quad (4)$$

Let us address the last equations of the system (3). For the case when the number of pendulums

$n = 2$, we obtain a system of two equations:

$$\begin{cases} ml \sin(\Omega t + \theta_1) \ddot{\xi}_1 - ml \cos(\Omega t + \theta_1) \ddot{\eta}_1 - ml \varepsilon \sin(\theta_1 - \beta) \Omega^2 = 0 \\ ml \sin(\Omega t + \theta_2) \ddot{\xi}_1 - ml \cos(\Omega t + \theta_2) \ddot{\eta}_1 - ml \varepsilon \sin(\theta_2 - \beta) \Omega^2 = 0 \end{cases} \quad (5)$$

We substitute the obtained solutions (4) to the system (5) in accordance with the coordinates ξ and η , average the equation for the period $2\pi/\Omega$ and, taking $\Omega^2 \gg k^2$ for the area located far from the resonance, find:

$$\begin{cases} M_R \varepsilon \sin \theta_1 + ml \sin(\theta_1 - \theta_2) - (M_R + M_H) \varepsilon \sin(\theta_1 - \beta) = 0 \\ M_R \varepsilon \sin \theta_2 - ml \sin(\theta_1 - \theta_2) - (M_R + M_H) \varepsilon \sin(\theta_2 - \beta) = 0 \end{cases}$$

The obtained system allows four solutions, having one of them presenting practical interest in automatic rotor balancing. Quasi-steady rotation of the rotor–pendulums system is viable if the system's rotation center coincides with the pendulums' axis of rotation, and the pendulums are located as shown in Figure 2. In this case, the housing circumscribes the radius ε . Therefore, there are two vectors of unbalance: the unbalance of the rotor U_R , and the unbalance of the housing U_H . Pendulums are positioned in a way that removes their total unbalance U_Σ .

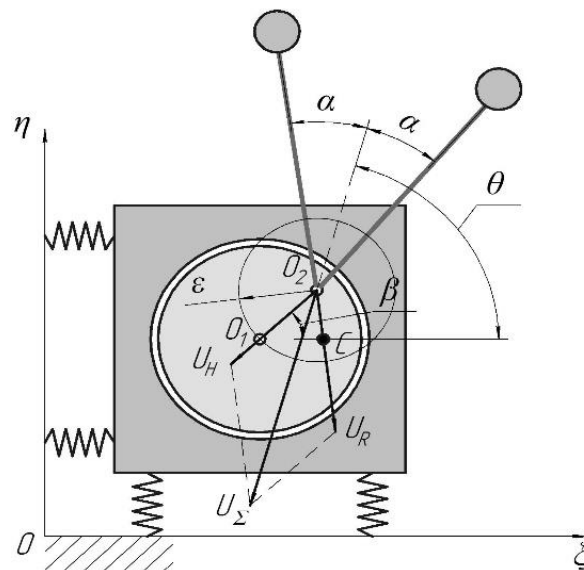


Figure 2. Quasi-steady rotation of the rotor–pendulums system

Let us define $\theta_1 = \theta + \alpha$; $\theta_2 = \theta - \alpha$. Then we receive the following solution of this system of equations:

$$\theta = \beta + \arccos \left(\frac{U_H^2 + U_\Sigma^2 - U_R^2}{2U_H U_\Sigma} \right) \quad (6)$$

$$\alpha = \arccos\left(\frac{\mathbf{U}_\Sigma}{2ml}\right)$$

where

$$\mathbf{U}_H = M_H \varepsilon$$

$$\mathbf{U}_R = M_R (\varepsilon^2 + e^2 - 2\varepsilon e \cos \beta)^{1/2}$$

$$\mathbf{U}_\Sigma = (\mathbf{U}_R^2 + \mathbf{U}_H^2 + 2\mathbf{U}_R \mathbf{U}_H \cos(\angle O_1 O_2 C))^{1/2}$$

The angle $\angle O_1 O_2 C$ may be found from the equation:

$$\sin(\angle O_1 O_2 C) = \frac{\sin \beta e}{(\varepsilon^2 + e^2 - 2\varepsilon e \cos \beta)^{1/2}}$$

Let us consider the limiting cases of the partial solution (6).

1. The angle $\beta = 180^\circ$. It is the worst case since the directions of the vectors $\mathbf{U}_R$ and $\mathbf{U}_H$ coincide. Therefore,

$$\mathbf{U}_\Sigma = \mathbf{U}_R + \mathbf{U}_H = M_R (\varepsilon + e) + M_H \varepsilon$$

$$\alpha = \arccos\left(\frac{M_R (\varepsilon + e) + M_H \varepsilon}{2ml}\right)$$

From the last equation, we may find the minimum required capacity of the pendulum

$$U_{\min} = \frac{M_R (\varepsilon + e) + M_H \varepsilon}{2}.$$

2. The angle $\beta = 0^\circ$, $\mathbf{U}_R > \mathbf{U}_H$. The vectors $\mathbf{U}_R$ and $\mathbf{U}_H$ are directed oppositely and partially cancel each other out:

$$\mathbf{U}_\Sigma = \mathbf{U}_R - \mathbf{U}_H = M_R e - M_H \varepsilon,$$

$$\alpha = \arccos\left(\frac{M_R e - M_H \varepsilon}{2ml}\right).$$

3. The angle $\beta = 0^\circ$, $U_R < U_H$. The vectors U_R and U_H are directed oppositely and do not cancel each other out:

$$U_\Sigma = U_H - U_R = M_H \varepsilon - M_R e ,$$

$$\alpha = \arccos\left(\frac{M_H \varepsilon - M_R e}{2ml}\right).$$

3. Conclusions

1. If there is misalignment of the axis of pendulums' rotation with the rotor axis, pendulums tend to bring the system's central axis of inertia to their own axis of rotation.

2. The housing circumscribes with the radius of eccentricity ε with a frequency equal to the rotor's rotation rate.

3. Designing the SBD, i.e. determining the minimum required capacity of pendulums, one should consider not only the possible rotor unbalance, but also the unbalance of the machine housing caused by the presence of misalignment of the pendulums' axis of rotation with the rotor axis.

References

- [1] Bai Cai-Bo, Teng Peng-fei, Ma Rui, Sun Chang-Jing 2011 Summary on On-line Balancing Techniques and Devices of Spindle *Advanced Materials Research Vols. 287-290 pp 2906-2910*
- [2] Zhang Shihai, Wu Liangsheng, Teng Xianbin 2010 Research on Double-face Online Dynamic Balance Technology of Machine Tool Spindle *Advanced Materials Research Vols. 44-47 pp 112-116*
- [3] Lee Chong-Won 2006 Mechatronics in Rotating Machinery *7th IFToMM-Cinference on Rotor Dynamics*
- [4] Blanco-Ortega A, Silva-Navarro G, Colin-Ocampo J, Oliver-Salazar M, Vela-Valdes G 2012 Automatic Balancing of Rotor-Bearing Systems *INTECH DOI 5772/48621 (2012)*
- [5] Zhongkun Hong, Yunfei Chen Dynamic Modeling and Optimization on the Automatic Balancing of a Washing Machine *Applied Mechanics and Materials Vol. 620 (2014) pp 304-309*
- [6] Marijonas Bogdevicius, Jolanta Januteniene Influence of Dynamic Viscosity on Automatic Dynamic Balance *Solid State Phenomena Vol. 164 (2010) pp 127-132*