

Force, displacement and strain nonlinear transfer function estimation.

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Abstract. The analysis of surface strain, displacement and acceleration data from the nonlinear response of a beam under base excitation is performed using frequency domain based reverse path identification. The uncorrelated nonlinear contributions of the dynamic response are used as inputs to produce repeatable transfer function estimates with the applied base acceleration as the output. The two stage process of displacements to strain to applied base excitation is proposed and also investigated. System identification of the reverse transfer functions is shown to be reasonably successful in quantifying the main frequency dependent contributions which occur at and around the fundamental natural frequency and harmonics due to the geometrical nonlinearity present.

1. Introduction

Acoustic fatigue and the subsequent nonlinear response of structures has been the subject of ongoing research within the aerospace community for over 60 years. As the loading has increased in magnitude, the use of simple linear relationships such as the single mode approximation by Rice [1] and Miles [2] have no longer been accepted as being applicable, often giving in some cases overly conservative fatigue life predictions. This paper discusses frequency domain analysis of nonlinear random vibration experimental data, from investigations performed at Wright Patterson Air Force Base (WPAFB) USA [3] [4], with the aim to quantify higher frequency content and also the nonlinear strain relationships as a function of the loading and displacement response. Subsequently such information can be used for fatigue damage estimation and prediction, especially with respect to the transformation of the nominal Gaussian random loading which produces a non-Gaussian and nonlinear response. The reverse path identification method [5] has been applied in three novel and different ways to determine linear and nonlinear frequency response functions between the measured input acceleration, displacement and strain spectra. The nonlinear frequency response functions, and especially their coherence functions, provide a keen insight into potential models that can best describe the response of the system in terms of the displacement and strain states. These FRF estimates have been subsequently used to establish some of the parameters for single degree of freedom nonlinear model representations and simulations. This is more preferable to linearized model approximations, as not only do they reproduce the frequency domain amplitude dependent behavior but also the statistical distribution for the response.



2. Frequency domain analysis of the wideband system response

The experiments have been discussed in several papers, see [3] [4], but it is worth briefly describing the experiments, the reasons for performing them and the available data. The experiments were designed to produce nonlinear responses for a simple physical vibrating system, with high quality random response data for comparison with predictions. The original intention was to consider validation of low-order nonlinear numerical modal models, which can be used for fatigue calculation. Here the data has been examined as a basis for quantifying and describing the changes as the system response progresses from mildly to strongly nonlinear. The structural nonlinearity is due to the large displacement response and geometric stiffening of the beam structure undergoing base excitation testing. The actual experiment comprised a clamped-clamped steel beam, with length = 229 mm, width = 12.7 mm and thickness = 0.79 mm. The measured data included the test base rms acceleration input ($g = 9.81 \text{ m/s}^2$) kept at a constant level over the 50 Hz to 500 Hz frequency range, plus displacement, and strain responses at the center of the beam. The frequency range covered the first (approximately 78 Hz) and third (approximately 414 Hz) bending modes of the beam, and with each loading case it is not expected that the second mode, which is antisymmetric about the beam center, is excited. Strain gauges were mounted to the top and bottom surfaces of the beam, which can allow both the bending strain and membrane strain to be evaluated. The total data record length was 88 s with a sample rate of 4096 Hz.

Frequency domain analysis was considered because the majority of the nonlinear aspects of the system response are more easily shown and determined in the frequency domain. The application of a standard reverse path identification technique [5] [6] has been applied, as the identification of several nonlinear frequency response functions help to explain the underlying nature of the nonlinear system. This can also be related and interpreted in terms of the physical behavior, i.e. geometrical nonlinear stiffening which can be modelled by an additional factor proportional to the cubic power of the displacement. These frequency response functions are subsequently used later to estimate the strain response from nonlinear displacement response data used as inputs.

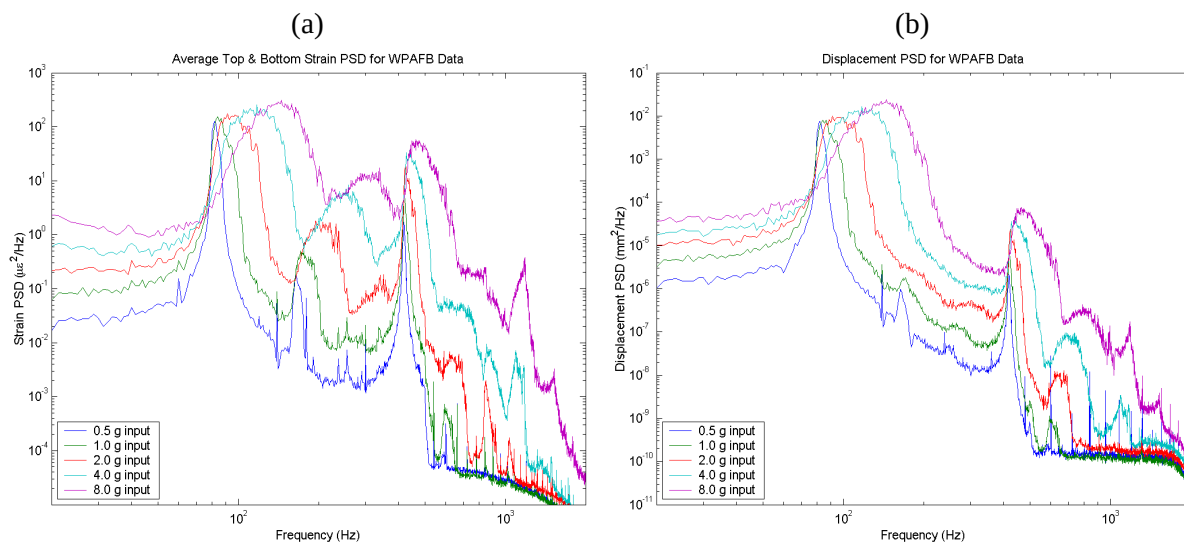


Figure 1. Beam center PSD responses determined from data at five base acceleration input levels: (a) averaged (top and bottom) strain and (b) displacement.

2.1. Displacement and strain response power spectral densities

Typical response power spectral densities (PSDs) are shown in Figure 1, for the beam center strain and displacement respectively. It shows clearly the shift and broadening of the response, with increasing

contributions at harmonics of the fundamental mode frequency (78 Hz), and also response outside the bandwidth of the excitation. Note how the response in the frequency regions of the first and third modes becomes much broader; a characteristic that the system is becoming nonlinear as seen in equivalent Duffing type simulations [7]. Also observe that the second and third harmonics of the natural frequencies are evident even at the lowest input level.

The PSD of the beam center average surface strain (average of the top and bottom strain) response is shown in Figure 1(a). Note the similarities and differences to the displacement PSD; the second harmonic in the strain response is now much stronger, and the relative magnitude of the third mode strain response, at approximately 400 Hz, to the fundamental is much greater than in the corresponding displacement mode data, as shown in Figure 1(b). It is also apparent that the third harmonic of the fundamental becomes obscured by the response of the third mode at inputs above 2 g. These observations will be discussed further in the next sections, where results are shown indicating the possible sources of this frequency contribution.

3. Nonlinear system identification

The standard reverse path nonlinear system identification method [5] [6] was initially applied to determine the displacement to force relationships. The nonlinear displacement and the squared and cubed version of it were used as multiple “mathematical inputs” to a system with force as the “mathematical response”. The method estimates the parts of the response that are linearly related to the linear, squared and cubed displacement input, where the latter are processed to produce uncorrelated inputs before the transfer function estimates are produced. The force was presumed to be proportional to the measured base acceleration, using an equivalent mass of 17.96 grams. This is the equivalent inertial force on a beam of the same mass with the same acceleration.

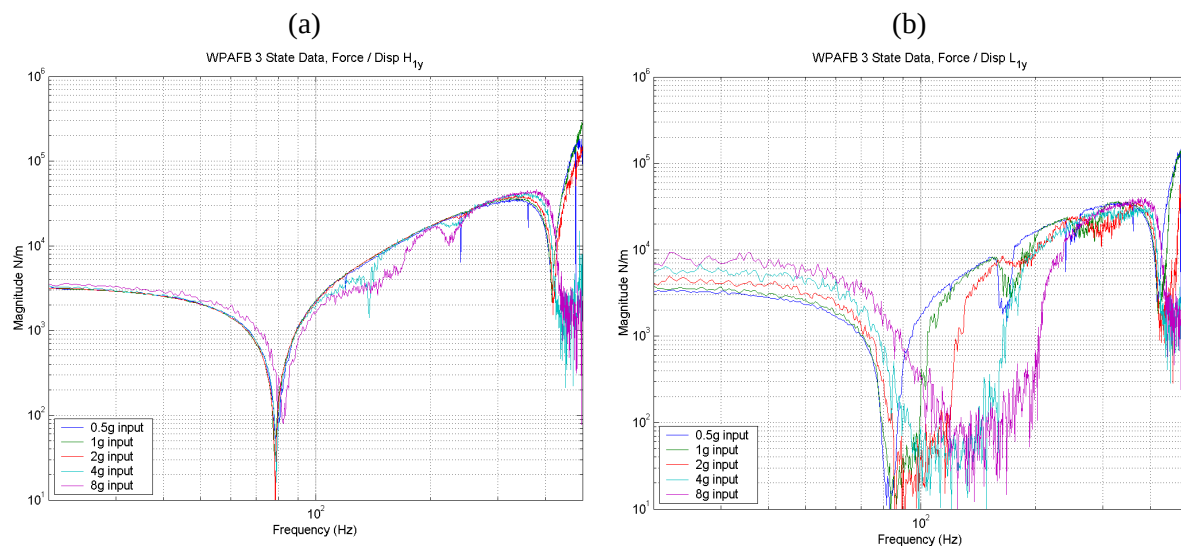


Figure 2. Displacement input to force output frequency response functions determined from: (a) uncorrelated displacement input to force output and (b) correlated displacement input to force output.

The estimated linear (see Figure 2) and cubic (see Figure 3(b)) displacement to force frequency response functions (FRFs) show consistent results for all test levels, while the squared displacement FRFs (see Figure 3(a)) exhibit markedly different, yet stable results in the low-frequency range at each level. When uncorrelated inputs are considered, Figure 2(a), the transfer functions are repeatable for different input acceleration levels. The dips in the transfer functions are in agreement with the fundamental and third natural frequency of the beam in bending. This clear correspondence is lost

when a straightforward correlated displacement to acceleration transfer function is estimated (Figure 3(b)). The coherence functions corresponding to the minimum and maximum spectral inputs, Figure 4, show that the nonlinear model is able to account for most of the response over all frequencies at the low 0.5 g input level (see Figure 4(a)), but, this nonlinear model did not fare as well at the higher input levels (see Figure 4(b)). At the lower level the response is mainly linear. It is interesting to see that at the higher level, the cubic displacement term accounted for the majority of the coherent response over the range between the first and second observed peaks in the system response. Some of the frequency content, between 100 Hz to 200 Hz, is not recovered by the polynomial terms and this corresponds to the broadened first peak in the displacement PSD (Figure 1(b)). The coherence plots show that the squared terms were noticeable at the low level, possibly caused by a “pre-buckled” condition, but they were insignificant at the highest level.

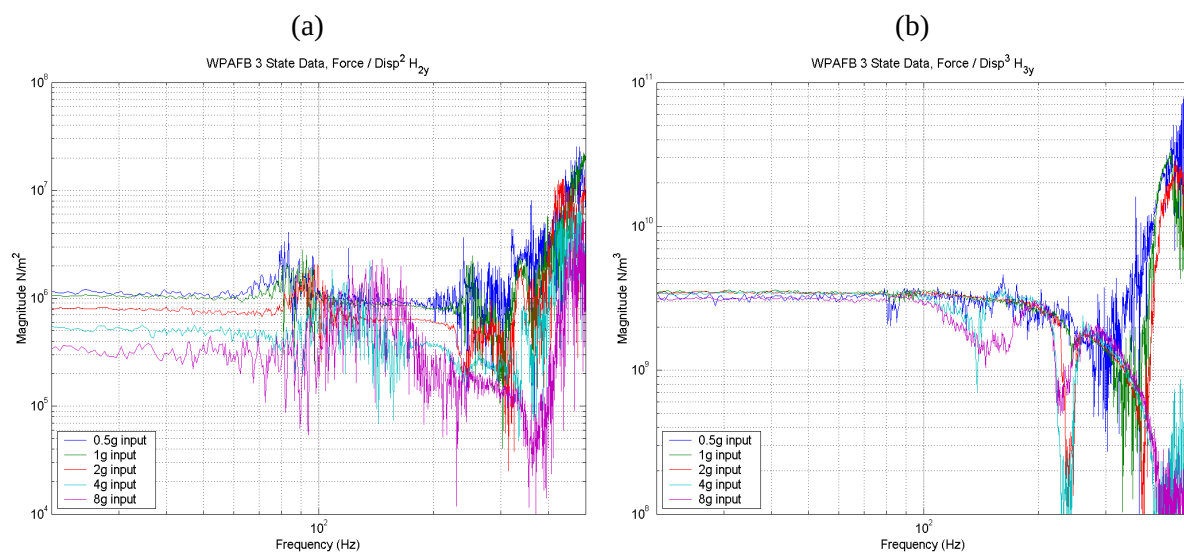


Figure 3. Nonlinear displacement input to force output frequency response functions determined from: (a) uncorrelated squared displacement input and (b) uncorrelated cubed displacement input.

The top surface strain response ε for a beam in bending is given by

$$\varepsilon = \frac{h}{2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad (1)$$

where h is the beam thickness, w the beam displacement and x the spatial coordinate along the length. The first term corresponds to the bending stiffness contribution, the second term to the axial or membrane positive strain due to the large deflection. For small displacements, or assuming linear behaviour, then the second term contribution is either small or negligible.

The reverse path identification method was subsequently applied to estimate the strain to force FRFs, but with limited success. The FRFs for linear (Figure 5), squared (Figure 6(a)) and cubed strains (Figure 6(b)) are not as clear when compared to the displacement to force transfer function estimates. The squared and cubic strain FRFs are especially noisy, although the linear relationship follows the same trend as the displacement to force results in Figure 2(a). An indication of the quality of these nonlinear FRF estimates can be seen in the coherence plots. At the lowest level strain input case, as shown in Figure 7(a), where respectable FRFs are produced, the corresponding coherence contributions are dominated by a linear relationship. At the higher levels (shown in Figure 7(b)) the

results indicate that the nonlinear combination of the strain to displacement, in conjunction with the nonlinear and unknown nonlinear displacement to force relationships leads to a very nonlinear and unidentifiable transformation. In this case, the inverse identification approach is perhaps less suitable and tractable for determination of a suitable relationship. A reason for this can be hypothesized from examining the displacement to strain relationship, which is nonlinear and given in equation (1), which has to be combined with the nonlinear but unknown base acceleration to displacement behavior. At low input levels both transformations might be reasonably assumed to be linear, which is why the coherence in Figure 7(a) shows that all of the response can be recovered.

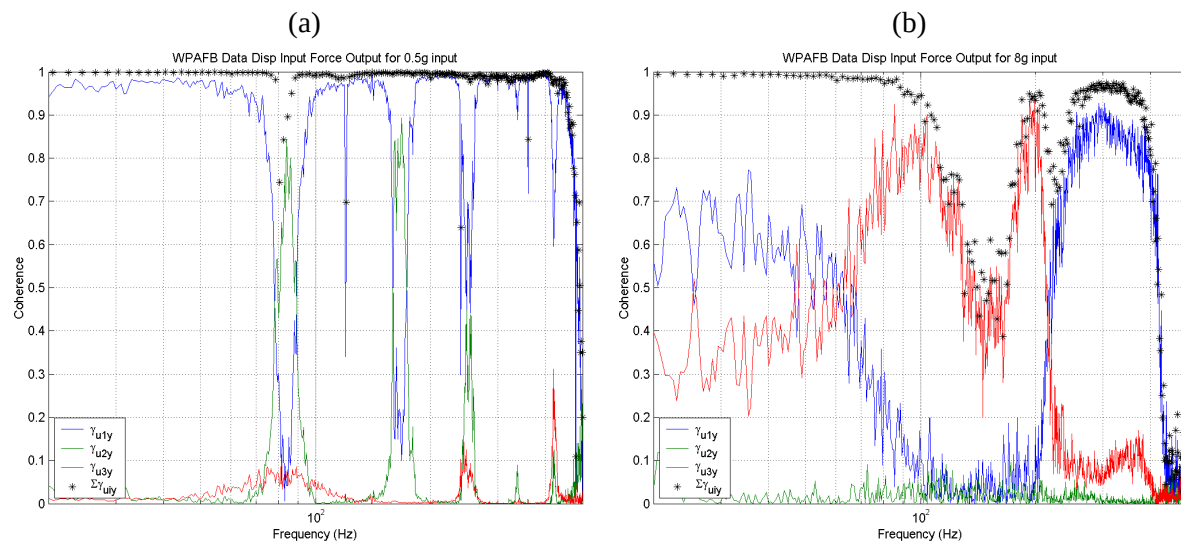


Figure 4. Coherence functions for displacement input to force output frequency response functions determined from: (a) 0.5 g data and (b) 8 g data.

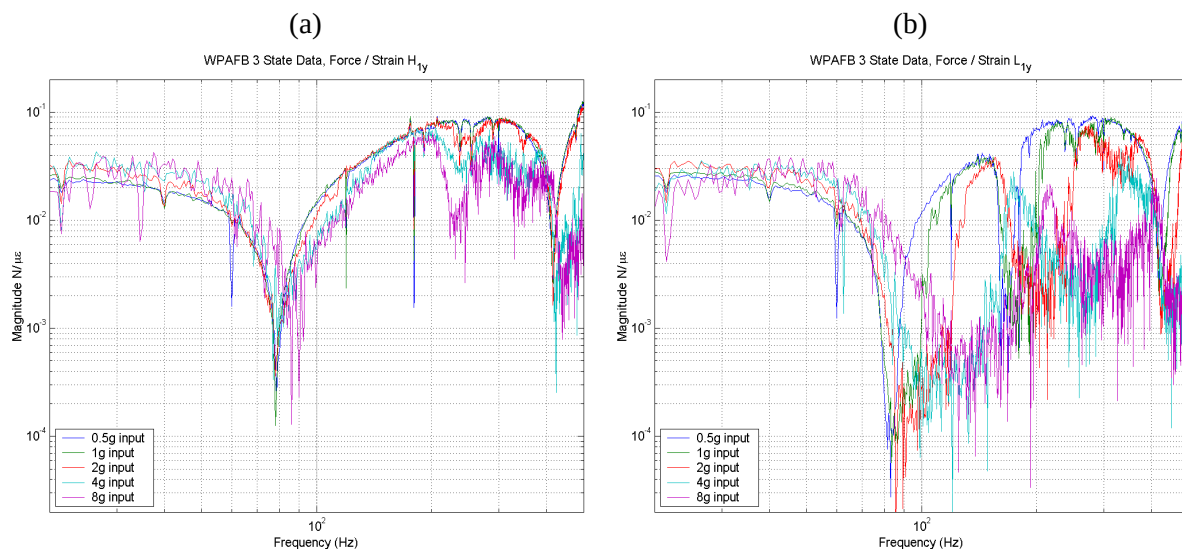


Figure 5. Top surface strain input to force output frequency response functions determined from: (a) uncorrelated top surface strain input to force output and (b) correlated top surface strain input to force output.

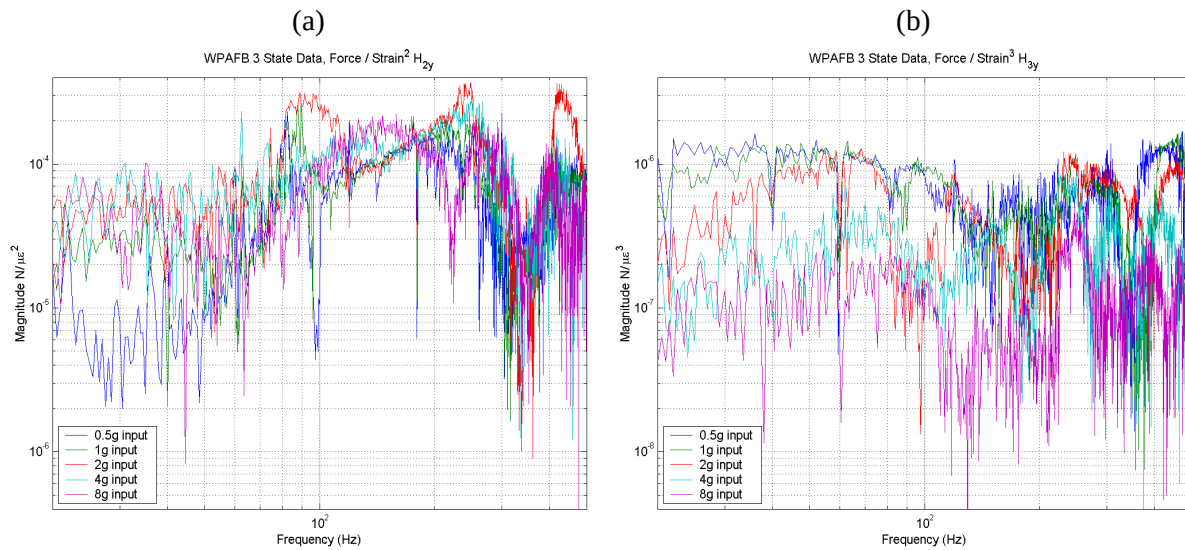


Figure 6. Nonlinear top surface strain input to force output frequency response functions determined from: (a) squared top surface strain input and (b) cubed top surface strain input.

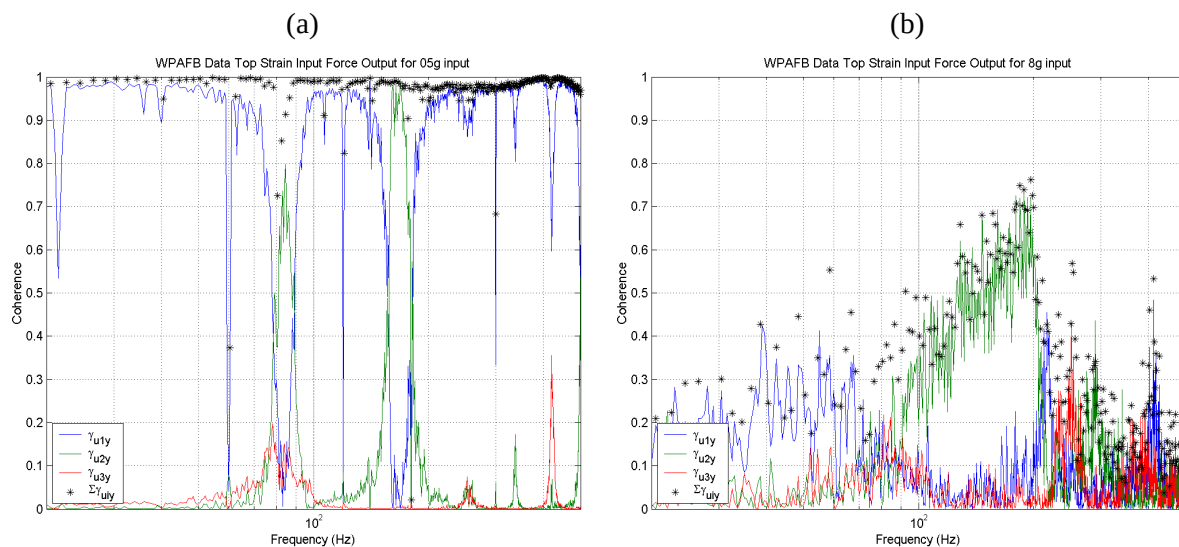


Figure 7. Coherence functions for top surface strain input to force output frequency response functions determined from: (a) 0.5 g data and (b) 8 g data.

The reverse path method was then applied to the data to determine nonlinear top surface displacement to strain FRFs, where the displacement is the “mathematical input” and the strain is the response. The linear FRFs (see Figure 8) showed a very consistent set of results throughout the frequency range and at the different levels. The general rise with frequency in the strain response, as displacement increased, can be supported by considering the relationships between the mid-span strain and the response of the linear beam when excited only in the fundamental and third mode due to base excitation. It can be shown [8] that the modal response will be dominated, for light damping, by frequency content around the natural frequencies and proportional to the base acceleration. The corresponding mid-span surface strain is then, for the linear system, proportional to the natural frequency squared and a factor which incorporates the curvature in the bending of the mode. Taking

these factors into account, one has that the strain ratio of the third to first mode, when the amplitudes of the modes are set equal which is equivalent to the ratio of the transfer function estimation, which gives the strain ratio factor of approximately 6. As damping is incorporated into both the strain and displacement, then inspecting Figure 8(a) at around the fundamental and third natural frequency one can see that this is fair agreement. The squared (see Figure 9(a)) and cubed (see Figure 9(b)) displacement to strain FRFs are similar and reasonably consistent over the entire frequency range. The breakdown of the individual coherence functions and the summed coherent signal power (Figure 10) shows that the combined nonlinear model can account for almost all of the measured response. This nonlinear displacement to strain model is arguably better than the displacement to force model (compare the coherence results in Figure 4 and 10).

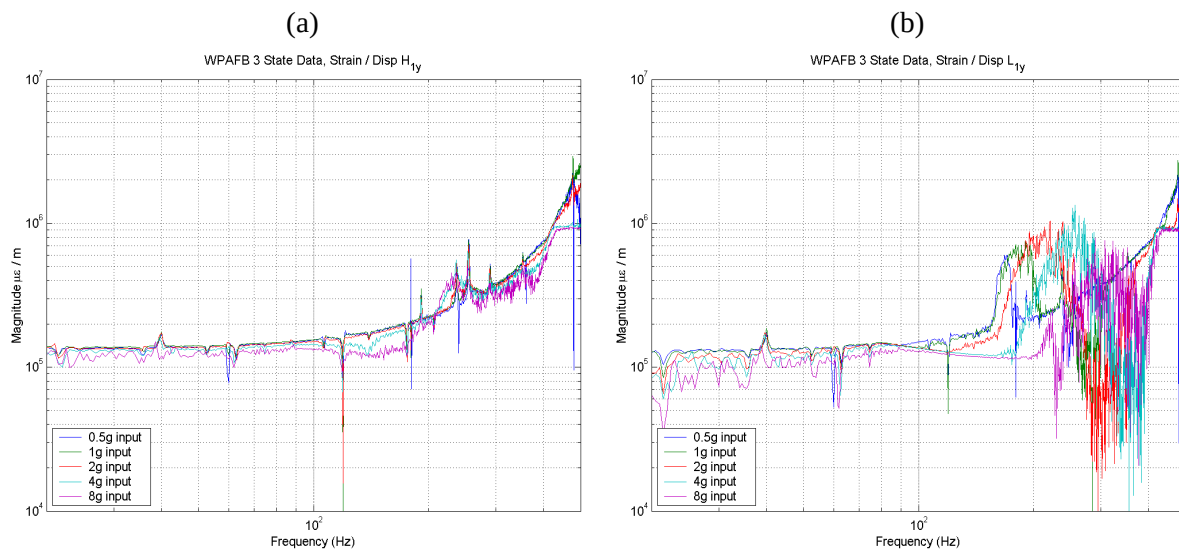


Figure 8. Displacement input to top surface strain output frequency response functions determined from: (a) uncorrelated displacement input and (b) correlated displacement input.

3.1. Linear frequency response function parameter estimation

One important outcome of this nonlinear FRF estimation process is a method to estimate the linear response of the system. The linear FRFs of displacement to force and strain to force were used to estimate the linear random response standard deviation of each input level. At first, an attempt was made to use the FRFs in their raw numerical format. This proved to be undesirable because small changes in the numerical values of the FRF estimates produced vastly different results. A parameter estimation of the multi degree of freedom (MDOF) transfer function proved to be more stable and informative.

The first step of this MDOF parameter estimation was to determine the force to displacement (and later force to strain) FRF. These were simply the reciprocal of the displacement to force FRF. The Laplace domain numerator and denominator polynomial representations of the transfer function were estimated and then converted into their equivalent roots (or poles) format. The modal roots are, assuming a viscous model for the modal damping,

$$\lambda_i = \omega_i \left(-\zeta_i \pm \sqrt{\zeta_i^2 - 1} \right) = -\omega_i \zeta_i \pm i \omega_i \sqrt{1 - \zeta_i^2} \quad (2)$$

This was useful for determining the modal parameters of the FRF

$$\omega_i = \sqrt{-[\text{real}(\lambda_i)^2 - \text{imag}(\lambda_i)^2]} \quad (3)$$

$$\zeta_i = -\text{real}(\lambda_i) / \omega_i$$

It was encouraging to find that the estimated parameters were comparatively consistent among the runs, as given in Table 1. There are two outliers: the 2 g (low ζ_1) and 8 g (large ζ_1 and higher f_{n1}) estimates. The average 1 g linear displacement response estimate σ_x , determined from the other three estimates, was 0.4293 mm.

Example FRFs and transfer function estimates are shown in Figure 11(a). The parameter estimates improved when the FFT frequency spacing was reduced to 0.25 Hz with a corresponding number of averages of 22. This was an interesting tradeoff; the smaller number of averages yielded a more variable point-to-point FRF estimate, but the larger number of points and the finer frequency spacing allowed for better estimates of the modal parameters.

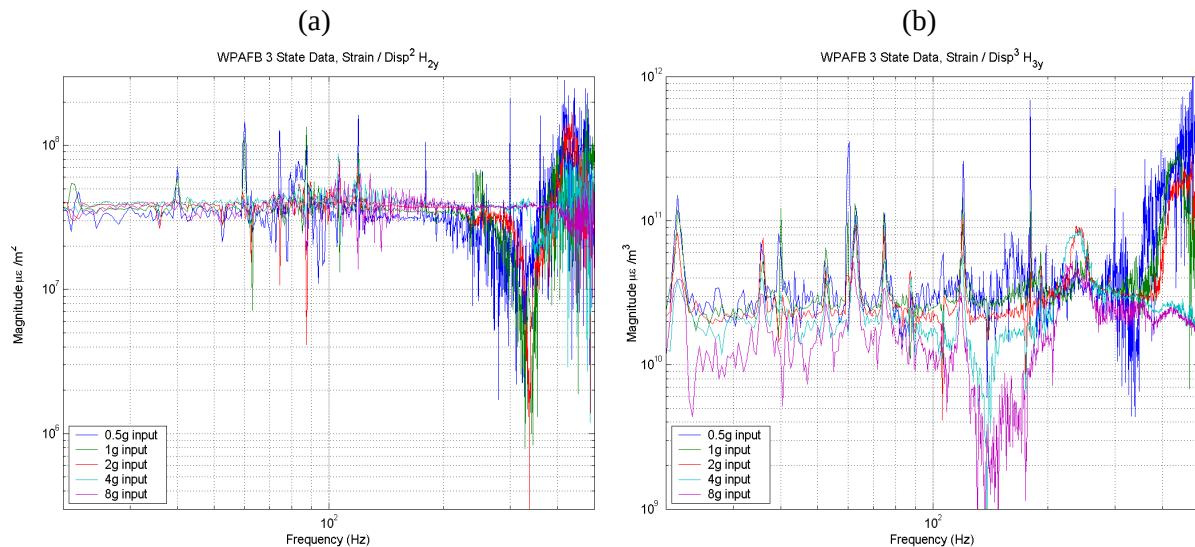


Figure 9. Nonlinear displacement input to force output frequency response functions determined from: (a) squared displacement input and (b) cubed displacement input.

Table 1. Multi-mode linear displacement response parameter estimation from WPAFB data.

Run	f_{n1} Hz	ζ_1	f_{n2} Hz	ζ_2	Est Linear σ_x mm	Est Linear 1g σ_x mm
0.5 g	78.9	0.23%	416.	0.52%	0.213	0.429
1 g	78.3	0.14%	416.	0.46%	0.447	0.452
2 g	78.4	0.064%	420.	0.38%	0.927	0.472
4 g	79.0	0.26%	438.	0.076%	1.582	0.404
8 g	83.9	0.51%	443.	0.031%	2.469	0.312

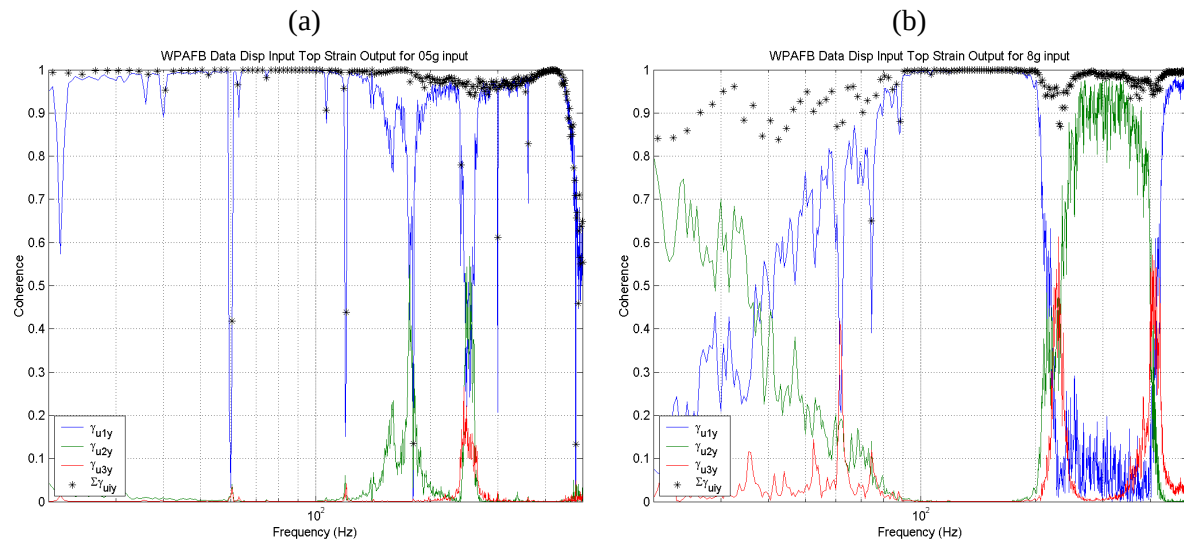


Figure 10. Coherence functions for displacement input to top surface strain output frequency response functions determined from: (a) 0.5 g data and (b) 8 g data.

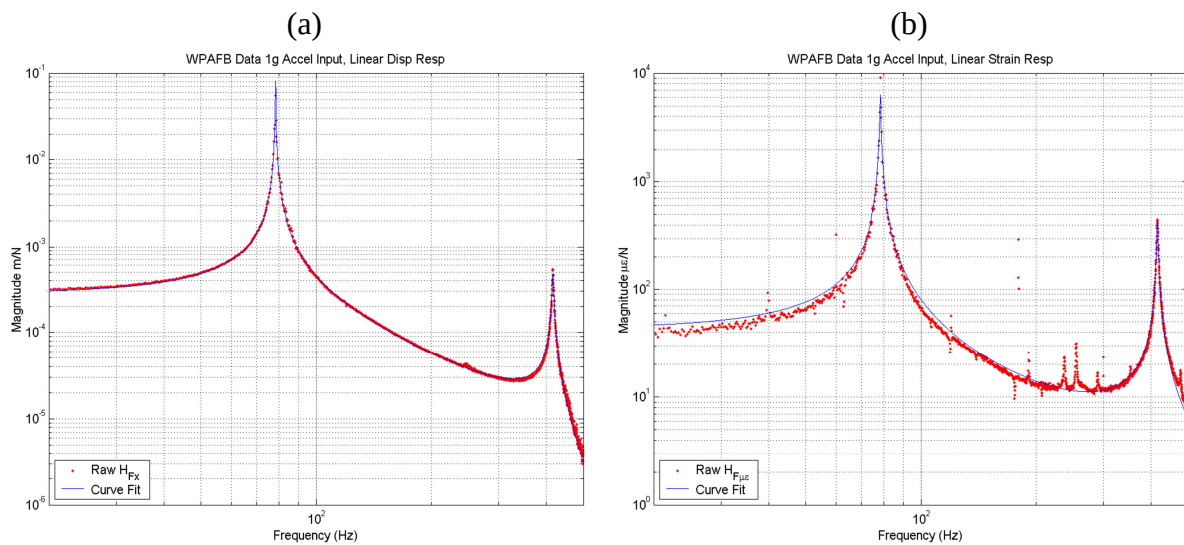


Figure 11. Estimated force input transfer functions from WPAFB 1 g input data: (a) linear displacement response and (b) linear top surface strain response.

A typical numerical example of the force input to displacement output transfer function for the 1 g input is

$$H_{Fx} = \frac{x}{F} = \frac{41s^2 - 2.0 \times 10^4 s + 4.7 \times 10^8}{s^4 + 25s^3 + 7.1 \times 10^6 s^2 + 1.5 \times 10^7 s + 1.7 \times 10^{12}} \frac{\text{m}}{\text{N}} \quad (4)$$

Table 2. Multi-mode linear top surface strain response for force input parameter estimation from WPAFB data.

Run	f_{n1} Hz	ζ_1	f_{n2} Hz	ζ_2	Est Linear $\sigma_\varepsilon (\mu\varepsilon)$	Est Linear 1g $\sigma_\varepsilon (\mu\varepsilon)$
0.5 g	79.2	0.42%	416.	0.49%	23.8	48.1
1 g	78.3	0.37%	416.	0.41%	48.9	49.5
2 g	78.0	0.16%	417.	0.28%	150	76.4
4 g	100	0.91%	430.	0.027%	241	61.8
8 g	108	2.8%	442.	0.078%	420	53.6

The force input to strain output modal parameters were estimated in a similar fashion. The eigenvalue (modal frequency and damping) estimates (see Table 2) are similar to those estimated from the force input to displacement output FRFs. The force to strain graphs show strong similarities to the force to displacement graphs (compare Figure 11(a) and Figure 11(b)). Note that the second-mode strain response is proportionately higher than the displacement response. It is also interesting to note that the first-mode damping estimates based on strain tend to be about twice as large as that of the displacement estimates, while the second-mode estimates are much closer in value (see Table 2). A typical numerical example of the force input to strain output transfer function for the 1 g input is

$$H_{F\varepsilon} = \frac{\varepsilon}{F} = \frac{4.4 \times 10^6 s^2 + 4.7 \times 10^{10} s - 7.2 \times 10^{13}}{s^4 + 25s^3 + 7.1 \times 10^6 s^2 + 3.0 \times 10^7 s + 1.7 \times 10^{12}} \frac{\mu\varepsilon}{N} \quad (5)$$

Note the similarity in the denominators of this function and the displacement transfer function given in equation (4).

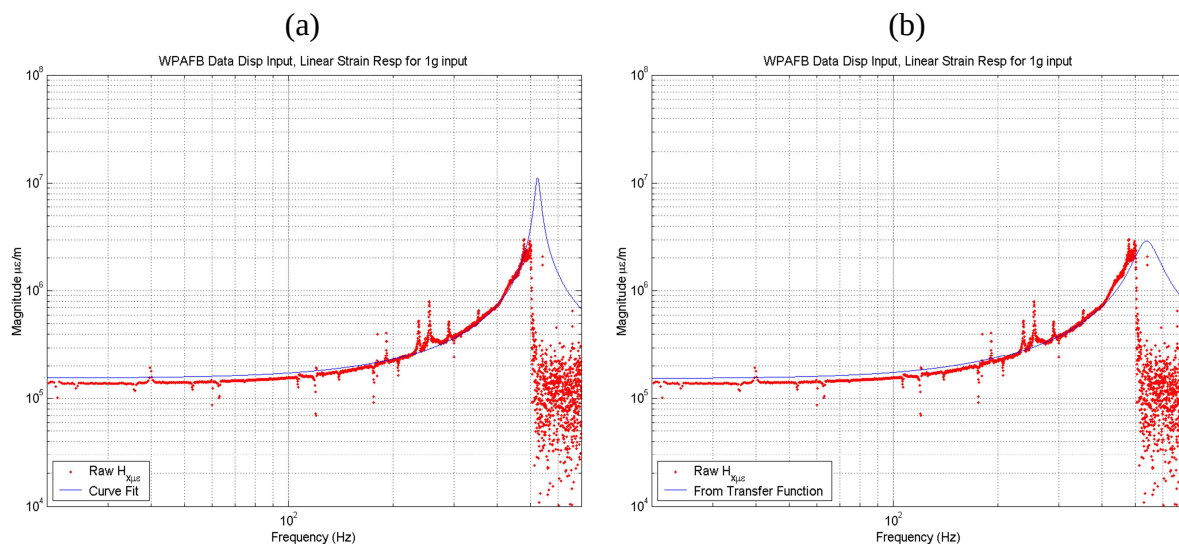


Figure 12. Estimated displacement input, linear top strain output transfer functions for WPAFB 1 g input data: (a) MDOF parameter estimate from raw strain to displacement data and (b) estimate from product of strain to force and force to displacement transfer functions.

MDOF parameter estimates were then made with the displacement to strain FRFs (see for example, Figure 12(a)). A typical numerical example of this displacement input to strain output transfer function for the data from the 1 g input case is

$$H_{\varepsilon x} = \frac{\varepsilon}{x} = \frac{1.6 \times 10^5 s^2 + 7.5 \times 10^8 s - 1.7 \times 10^{12}}{s^2 + 115s + 1.1 \times 10^7} \frac{\mu\varepsilon}{m} \quad (6)$$

At first this is not informative, but considering that the ratio of transfer function equations (5) and (4) (assuming that the denominators are equal) is

$$\begin{aligned} \frac{\varepsilon}{x} &= \frac{\varepsilon}{F} \frac{F}{x} = \frac{4.4 \times 10^6 s^2 + 4.7 \times 10^{10} s - 7.2 \times 10^{13}}{41s^2 - 2.0 \times 10^4 s + 4.7 \times 10^8} \frac{\mu\varepsilon}{m} \\ \frac{\varepsilon}{x} &= \frac{1.1 \times 10^5 s^2 + 1.1 \times 10^9 s - 1.7 \times 10^{12}}{s^2 - 493s + 1.1 \times 10^7} \frac{\mu\varepsilon}{m} \end{aligned} \quad (7)$$

one can recognize the similarity to equation (6). A plot of this transfer function estimate and the raw FRF from the reverse identification method are shown in Figure 12(b).

3.2. Flow diagrams of nonlinear strain and displacement models

The nonlinear transfer function estimation results suggest two methods for representing the nonlinear force to strain model; one based on a series sequential connection of nonlinear displacement and strain FRFs, and a second based on a parallel connection of modal models.

The first model, shown in Figure 13, is based on the two most successful nonlinear system identification models from the previous section, namely, the displacement to force and displacement to strain nonlinear FRFs. The second model is an attempt to show how the modal models (labelled osc1 and osc2 in Figure 14) are linked to both the displacement and strain responses. It may be possible to refine this latter modal model further to allow more of the nonlinear combination of terms to be described in the “modal domain.”

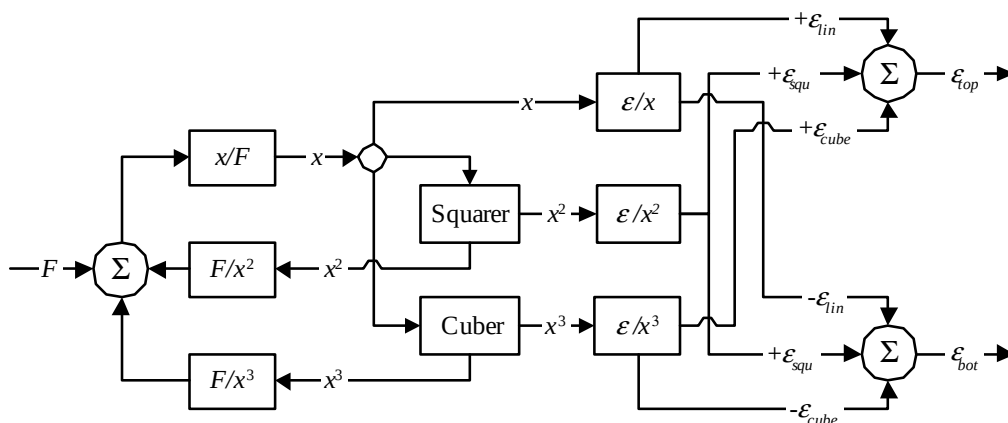


Figure 13. Flow diagram of nonlinear clamped-clamped beam model with series connection of displacement and strain functions.

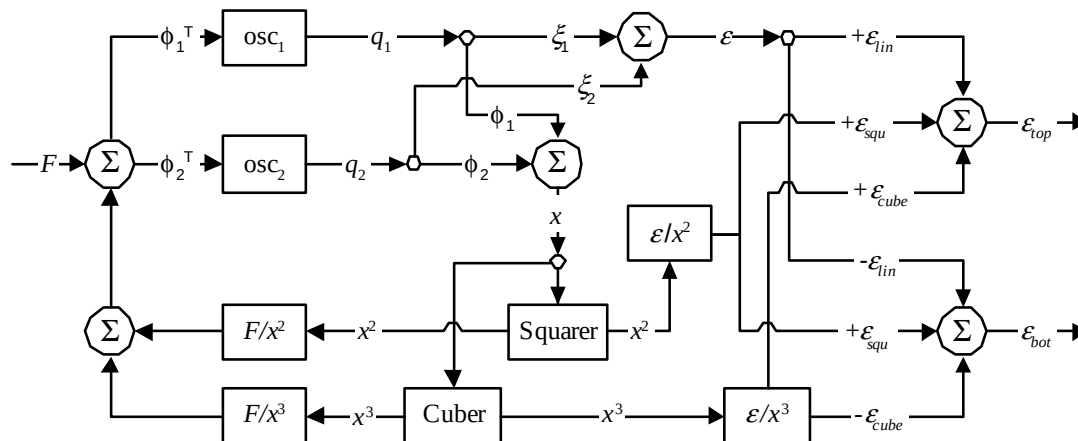


Figure 14. Flow diagram of nonlinear clamped-clamped beam model with modal superposition for displacement and strain functions. The transformations are, say, into the modal domain for the linear modes.

4. Summary and conclusions

The reverse path nonlinear technique has been shown to be very effective in identifying nonlinear models from measured displacement and strain data. The identified frequency response and coherence functions gave keen insight into models for both the force to nonlinear displacement response and the displacement to nonlinear strain response. The relationships between force and surface strain appear not to be related so well by a polynomial type expansion. The separation of the surface strain into the bending strain and membrane strain, with subsequent identification taking place, will be the subject of future investigation.

Modal models of the nonlinear system response have also been proposed.

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