

Complex Dynamics of Delay-Coupled Neural Networks

Xiaochen Mao^{1,2}

¹Department of Engineering Mechanics, College of Mechanics and Materials, Hohai University, Nanjing 210098, China

²State Key Laboratory of Mechanics and Control of Mechanical Structures, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

Email: maochen@hhu.edu.cn

Abstract. This paper reveals the complicated dynamics of a delay-coupled system that consists of a pair of sub-networks and multiple bidirectional couplings. Time delays are introduced into the internal connections and network-couplings, respectively. The stability and instability of the coupled network are discussed. The sufficient conditions for the existence of oscillations are given. Case studies of numerical simulations are given to validate the analytical results. Interesting and complicated neuronal activities are observed numerically, such as rest states, periodic oscillations, multiple switches of rest states and oscillations, and the coexistence of different types of oscillations.

1. Introduction

In recent years, coupled networks have attracted considerable research interest since they are quite common in nature and have extensive applications in science and engineering [1, 2]. For example, the biological nervous system is a complex information network composed of innumerable coupling neurons. Examples also include coupled semiconductor lasers, coupled oscillatory chemically reacting cells, neutrino oscillations, infectious diseases, josephson junction circuits, coupled logistic capacity models, and so on [3-6]. Coupled networks can exhibit rich behaviours, such as synchronization, phase trapping, phase locking, and oscillation death. For instance, synchronization occurs spontaneously in serious diseases, such as Parkinson's disease, essential tremor, and epilepsy [2].

Time delays occur naturally in many areas of physics, biology, chemistry, medicine, social science, and technology, such as nonlinear optics, electronic circuits, neuroscience, communication networks, traffic flows, etc [7, 8]. Time delays arise due to finite signal propagation and processing speeds, or may be introduced through closed-loop feedback control schemes. In fact, the information between the individual units may be transmitted by chemical agents in biological systems, by light in optical equipment, and by the motion of electrons in electronic devices. In all cases, it is never conveyed instantaneously, but only after some time delay, across space [9]. Undoubtedly, time delay is a fundamental reality for coupled systems. This leads to the inclusion of time-delay effects in the coupling terms. Time delay is often regarded as a source of instability and oscillations of the network and can lead to rich and complicated behaviours, such as periodic oscillations, multi-stability, quasi-periodic oscillations, and even chaotic phenomena [7, 10, 11]. In particular, the future states of the time-delay systems depend not only on the present state but also on the past state at discrete instant or in continuous time intervals. Hence, the solution space of such system is of infinite dimensions. This makes the theoretical analysis of delayed dynamic systems be a tough problem [8]. This issue still remains challenging.



Since the extensive applications of delay-coupled systems heavily depend on their dynamical behaviours, the analysis of dynamics is an important and necessary step for the practical design. The system performance is governed by the assumed dynamics of individual units and their reciprocal interactions. The dynamical properties arising from the interactions of individual units are often different from the behaviours in isolation. In fact, there has been an increasing interest and activity in the investigations on dynamics of delay-coupled systems, such as delay-coupled lasers [12], delay-coupled neurons [13], delay-coupled nonlinear oscillators [14], delay-coupled limit cycle systems [15], delay-coupled logistic capacity systems [6], and so on. For example, Burić et al. [16] investigated a pair of identical FitzHugh-Nagumo systems with internal and coupling time delays and obtained the bifurcational relations among coupling time-lag and coupling constant for different values of the internal time-lags. For related studies on the dynamics of delay-coupled systems with sub-networks, the reader is referred to the work in [17-20]. It should be noted that the research efforts mentioned in the literature have often concentrated on studying the behaviours of two coupled sub-networks because such models are fundamental and easily addressable. Although the models of two coupled sub-networks are relatively simple, it allows one to obtain vital information about the behaviour of the complex systems by studying the laws and rules that govern different dynamical regimes, including stability, cooperation, (de)synchronization, etc.

Over the past decades, the fresh concept and interesting behaviour of small-world networks have also called considerable attention to the network of large number of sites with a small number of short-cut connections added between randomly chosen pairs of sites since the pioneering work of Watts and Strogatz [21-23]. Small-world network has a short average path distance between nodes, as well as a large degree of clustering (in other words, many triangles in the network structure). These small-world properties can be found in many real-world structures. For instance, neural networks on the level of single neurons coupled through synapses or gap junctions, as well as on the level of cortex areas and their pathways exhibit small-world properties. In particular, the small-world architecture is optimal for processing and transmission of signals within and between brain areas because the brain has an architecture enabling both efficient global and local communication between neurons [24]. Usually, it is very difficult to gain an insight into the dynamics of small-world networks, especially when they involve time delays in signal transmissions. Nevertheless, it is necessary to study the effect of the small-world when any short-cut connection is added to a delayed neural network [17, 25].

The purpose of this paper is to investigate the dynamical behaviours of a network model that consists of a pair of sub-networks and two-way couplings of neurons between the individual sub-networks. Each sub-network has a ring of four identical elements with short-cut bidirectional connections. Ring networks of neurons are of wide interest in physiological and biological modeling since they are found in many neural structures and can be used as models of central pattern generators in the nervous system. For example, ring neural networks have been observed in the spinal cord, hippocampus, olfactory bulb, pyramidal cells and corticothalamic systems [26]. The coupled network system is constructed by adding bidirectional delayed couplings between the corresponding neurons in two sub-networks. In this network, the delays in the propagation of signals on the network-couplings are taken into account. Moreover, since the network-couplings between individual sub-networks may be faster than the internal dynamics, it is worthwhile and necessary to include internal short-cut time delays in internal connections within each sub-network. Thus, internal short-cut time delays and coupling time delays are introduced into the internal short-cut connections within the individual sub-networks and the couplings between sub-networks, respectively.

The rest of this paper is organized as follows. In Section 2, the stability of the network equilibrium is discussed and the conditions for the existence of oscillations are obtained. Numerical simulations are given to illustrate the analytical results and rich dynamical behaviours are observed in Section 3. Finally, conclusions and remarks are made in Section 4.

2. Stability and Bifurcation Analysis

The sub-network consists of four identical elements with nearest-neighbour and short-cut bidirectional delayed connections. The coupled network system is constructed by adding bidirectional delayed couplings between the corresponding neurons in every two sub-networks. The coupled network can be described by a set of delay differential equations as follows

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + af(x_{i-1}(t)) + af(x_{i+1}(t)) + bf(x_{i+2}(t-\sigma)) + cg(y_i(t-\tau)) \\ \dot{y}_i(t) = -y_i(t) + af(y_{i-1}(t)) + af(y_{i+1}(t)) + bf(y_{i+2}(t-\sigma)) + cg(x_i(t-\tau)) \end{cases} \quad (1)$$

where x_i and y_i denote the states of the i -th neuron in two sub-networks X and Y , respectively, a , b , and c are the weights of the nearest-neighbour links, short-cut connections, and network-couplings, respectively, σ and τ are the time delays of the internal short-cut connections and network-couplings, respectively, f and g are the internal and coupling activation functions, respectively, $i \bmod 4$.

The activation functions determine the input-output relation of neurons, which denote the intrinsic properties of neural systems. The nonlinear activation functions, f within the sub-networks and g between the sub-networks, are absolutely smooth. Without loss of generality, the nonlinear functions satisfy $f(0)=0$ and $g(0)=0$. A typical activation function is $\tanh$, which has been widely used for neural networks.

The linearization of Equation (1) at the trivial equilibrium of the network reads

$$\begin{cases} \dot{z}_i(t) = -z_i(t) + \alpha x_{i-1}(t) + \alpha x_{i+1}(t) + \beta x_{i+2}(t-\sigma) + \gamma y_i(t-\tau) \\ \dot{y}_i(t) = -y_i(t) + \alpha y_{i-1}(t) + \alpha y_{i+1}(t) + \beta y_{i+2}(t-\sigma) + \gamma x_i(t-\tau) \end{cases} \quad (2)$$

where $\alpha = af'(0)$, $\beta = bf'(0)$, and $\gamma = cg'(0)$. Equation (2) can be written as

$$\begin{aligned} \dot{z}(t) &= -z(t) + \alpha Pz(t) + \beta Qz(t-\sigma) + \gamma Rz(t-\tau) \\ &= -z(t) + \alpha \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{P} \end{bmatrix} z(t) + \beta \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{Q} \end{bmatrix} z(t-\sigma) + \gamma \begin{bmatrix} 0 & \tilde{R} \\ \tilde{R} & 0 \end{bmatrix} z(t-\tau) \end{aligned} \quad (3)$$

where $z(t) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)^T$, and

$$\tilde{P} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{4 \times 4}$$

where $I_{4 \times 4}$ is a 4×4 identity matrix.

The characteristic matrix of the coupled network is

$$\begin{aligned} M(\lambda, \sigma, \tau) &= (\lambda + 1)I - \alpha P - \beta e^{-\lambda\sigma} Q - \gamma e^{-\lambda\tau} R \\ &= \begin{bmatrix} (\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda\sigma} \tilde{Q} & -\gamma e^{-\lambda\tau} \tilde{R} \\ -\gamma e^{-\lambda\tau} \tilde{R} & (\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda\sigma} \tilde{Q} \end{bmatrix} \end{aligned} \quad (4)$$

Let $\chi = e^{(\pi/2)i}$ and $\varphi_j = [1, \chi^j, \chi^{2j}, \chi^{3j}]^T$, where $j=0,1,2,3$. Then, it is easy to check that $\tilde{P}\varphi_j = (\chi^j + \chi^{-j})\varphi_j$, $\tilde{Q}\varphi_j = \chi^{2j}\varphi_j$, and $\tilde{R}\varphi_j = \varphi_j$.

There follows

$$\begin{aligned} |M_{\mp}(\lambda, \sigma, \tau)| &= |(\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda\sigma} \tilde{Q} \mp \gamma e^{-\lambda\tau} \tilde{R}| \\ &= \prod_{j=0}^3 [\lambda + 1 - \alpha(\chi^{-j} + \chi^j) - \beta e^{-\lambda\sigma} \chi^{2j} \mp \gamma e^{-\lambda\tau}] \end{aligned}$$

where $M_{\mp}(\lambda, \sigma, \tau) = (\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda\sigma} \tilde{Q} \mp \gamma e^{-\lambda\tau} \tilde{R}$. Thus, the characteristic equation of the coupled network reads

$$\begin{aligned}
\Delta(\lambda, \sigma, \tau) &= \det M(\lambda, \sigma, \tau) = \left| (\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda \sigma} \tilde{Q} - \gamma e^{-\lambda \tau} \tilde{R} \right| \cdot \left| (\lambda + 1)I_{4 \times 4} - \alpha \tilde{P} - \beta e^{-\lambda \sigma} \tilde{Q} + \gamma e^{-\lambda \tau} \tilde{R} \right| \\
&= \prod_{j=0}^3 [\lambda + 1 - \alpha(\chi^{-j} + \chi^j) - \beta e^{-\lambda \sigma} \chi^{2j} \mp \gamma e^{-\lambda \tau}] \\
&= \prod_{j=0}^3 [\lambda + 1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) e^{-\lambda \sigma} \mp \gamma e^{-\lambda \tau}] \\
&= \prod_{j=0}^3 \Delta_j^-(\lambda, \sigma, \tau) \Delta_j^+(\lambda, \sigma, \tau) = 0
\end{aligned} \tag{5}$$

where $\Delta_j^\mp(\lambda, \sigma, \tau) = \lambda + 1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) e^{-\lambda \sigma} \mp \gamma e^{-\lambda \tau}$, $j = 0, 1, 2, 3$. Clearly, the trivial equilibrium of the coupled system is locally asymptotically stable when all roots of $\Delta_j^\mp(\lambda, \sigma, \tau)$ have negative real parts, and unstable when at least one root of $\Delta_j^\mp(\lambda, \sigma, \tau)$ has a positive real part, where $j = 0, 1, 2, 3$.

Since internal time delays within the individual sub-networks are often simply neglected as a first approximation based on the assumption that coupling time delay is significantly longer [27, 28], the following results are useful for the coupled network free of internal short-cut time delay. Based on the local stability theory, one has the following conclusions.

Theorem 1 If $1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) \mp \gamma > 0$ hold for all $j = 0, 1, 2, 3$, then the trivial equilibrium of the coupled network free of internal short-cut time delay is locally asymptotically stable for all coupling time delays.

Proof. Let $\lambda = \pm i\nu$ ($\nu > 0$) be a pair of pure imaginary roots of $\Delta_j^\mp(\lambda, 0, \tau) = 0$, one obtains $\Delta_j^\mp(i\nu, 0, \tau) = i\nu + 1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) \mp \gamma e^{-i\nu\tau} = 0$. Separating the real and imaginary parts of $\Delta_j^\mp(i\nu, 0, \tau) = 0$ yields $1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) \mp \gamma \cos(\nu\tau) = 0$ and $\nu \pm \gamma \sin(\nu\tau) = 0$. Eliminating the harmonic terms gives $D_j^\mp(\nu) = \nu^2 + [1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j)]^2 - \gamma^2 = 0$. It is obvious that all $D_j^\mp(\nu) = 0$ has no positive roots. Moreover, one can check that $\Delta(\lambda, 0, 0) = 0$ have negative roots. Thus, all roots of $\Delta(\lambda, 0, \tau) = 0$ have negative real parts. This completes the proof.

The following study focuses on the case when the short-cut time delay σ is regarded as a parameter for fixed values of the coupling time delays. Let $\lambda = \pm i\omega$ ($\omega > 0$) be a pair of pure imaginary roots of $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$, one obtains

$$\begin{aligned}
\Delta_j^\mp(i\omega, \sigma, \tau) &= i\omega + 1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) [\cos(\omega\sigma) - i \sin(\omega\sigma)] \\
&\mp \gamma [\cos(\omega\tau) - i \sin(\omega\tau)] = 0
\end{aligned} \tag{6}$$

Separating the real and imaginary parts of $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ gives

$$\begin{cases} 1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) \cos(\omega\sigma) \mp \gamma \cos(\omega\tau) = 0 \\ \omega + \beta \cos(\pi j) \sin(\omega\sigma) \pm \gamma \sin(\omega\tau) = 0 \end{cases} \tag{7}$$

Eliminating σ in the above equation gives

$$\begin{aligned}
E_j^\mp(\omega) &= \omega^2 \pm 2\omega\gamma \sin(\omega\tau) \mp 2\gamma[1 - 2\alpha \cos(\pi j / 2)] \cos(\omega\tau) \\
&\quad + [1 - 2\alpha \cos(\pi j / 2)]^2 + \gamma^2 - \beta^2 \cos^2(\pi j) = 0
\end{aligned} \tag{8}$$

If $E_j^\mp(\omega)$ has no positive root, the system is delay-independent stable or unstable for any given short-cut time delay, depending on whether or not the system free of short-cut time delay is stable.

If $E_j^\mp(\omega)$ has a number of positive and simple roots $\omega_{j,k}^\mp$, then $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ has following sets of critical short-cut time delays

$$\sigma_{j,k,n}^\mp = (\theta_{j,k}^\mp + 2n\pi) / \omega_{j,k}^\mp, \quad j = 0, 1, 2, 3, \quad k = 1, 2, \dots, \quad n = 0, 1, 2, \dots,$$

where $\theta_{j,k}^\mp \in [0, 2\pi)$ and $\theta_{j,k}^\mp$ satisfies

$$\begin{cases} \cos \theta_{j,k}^{\mp} = [1 - 2\alpha \cos(\pi j / 2) \mp \gamma \cos(\omega_{j,k}^{\mp} \tau)] / \beta \cos(\pi j), \\ \sin \theta_{j,k}^{\mp} = [-\omega_{j,k}^{\mp} \mp \gamma \sin(\omega_{j,k}^{\mp} \tau)] / \beta \cos(\pi j) \end{cases} \quad (9)$$

Once a pair of pure imaginary characteristic roots $\pm i\omega_{j,k}^{\mp}$ is found and the corresponding critical values of internal short-cut time delay in Equation (9) are determined, the variation direction of its real part with respect to the internal short-cut time delay σ can be studied through $\text{sgn Re}[\lambda_j^{\mp}(\sigma_{j,k,n}^{\mp})]'$. The following theorem enables one to avoid complicated computation.

Theorem 2 $\text{sgn Re}[\lambda_j^{\mp}(\sigma_{j,k,n}^{\mp})]' = \text{sgn}[E_j^{\mp}(\omega_{j,k}^{\mp})]'$

Proof. Differentiating λ with respect to σ in $\Delta_j^{\mp}(\lambda, \sigma, \tau) = 0$ gives

$$[\lambda_j^{\mp}(\sigma)]' - \beta \cos(\pi j) \{-\lambda_j^{\mp}(\sigma) - \sigma[\lambda_j^{\mp}(\sigma)]'\} e^{-\lambda \sigma} \pm \gamma \tau e^{-\lambda \tau} [\lambda_j^{\mp}(\sigma)]' = 0$$

Then, one arrives at

$$\begin{aligned} \text{Re}[\lambda_j^{\mp}(\sigma_{j,k,n}^{\mp})]' &= \text{Re} \left(\frac{-\beta \cos(\pi j) \omega_{j,k}^{\mp} i}{e^{i(\theta_{j,k}^{\mp} + 2n\pi)} + \beta \cos(\pi j)(\theta_{j,k}^{\mp} + 2n\pi) / \omega_{j,k}^{\mp} \pm \gamma \tau e^{i(\theta_{j,k}^{\mp} + 2n\pi)} e^{-i\omega \tau}} \right) \\ &= -\frac{\beta \cos(\pi j) \omega_{j,k}^{\mp} \text{Im } C_{j,k}^{\mp}}{[\text{Re } C_{j,k}^{\mp} + \beta \cos(\pi j)(2n\pi / \omega_{j,k}^{\mp})]^2 + [\text{Im } C_{j,k}^{\mp}]^2} \\ &= -\frac{\beta \cos(\pi j) \omega_{j,k}^{\mp} \{[1 \pm \gamma \tau \cos(\omega_{j,k}^{\mp} \tau)] \sin \theta_{j,k}^{\mp} \mp \gamma \tau \sin(\omega_{j,k}^{\mp} \tau) \cos \theta_{j,k}^{\mp}\}}{[\text{Re } C_{j,k}^{\mp} + \beta \cos(\pi j)(2n\pi / \omega_{j,k}^{\mp})]^2 + [\text{Im } C_{j,k}^{\mp}]^2} \end{aligned}$$

where $C_{j,k}^{\mp} = e^{i\theta_{j,k}^{\mp}} + \beta \cos(\pi j) \theta_{j,k}^{\mp} / \omega_{j,k}^{\mp} \pm \gamma \tau e^{i\theta_{j,k}^{\mp}} e^{-i\omega_{j,k}^{\mp} \tau}$, $e^{\lambda_j^{\mp} \sigma_{j,k,l}^{\mp}} = e^{i\omega_{j,k}^{\mp} \sigma_{j,k,l}^{\mp}} = e^{i(\theta_{j,k}^{\mp} + 2n\pi)} = e^{i\theta_{j,k}^{\mp}}$, $j = 0, 1, 2, 3$, $k = 1, 2, \dots$, $n = 0, 1, 2, \dots$.

From Equation (9), one arrives at

$$\begin{aligned} \text{Re}[\lambda_j^{\mp}(\sigma_{j,k,l}^{\mp})]' &= \frac{\omega_{j,k}^{\mp} \{[1 \pm \gamma \tau \cos(\omega_{j,k}^{\mp} \tau)] [\omega_{j,k}^{\mp} \pm \gamma \sin(\omega_{j,k}^{\mp} \tau)]\}}{[\text{Re } C_{j,k}^{\mp} + \beta \cos(\pi j)(2l\pi / \omega_{j,k}^{\mp})]^2 + [\text{Im } C_{j,k}^{\mp}]^2} \\ &\quad \pm \frac{\omega_{j,k}^{\mp} \{\gamma \tau \sin(\omega_{j,k}^{\mp} \tau) [1 - 2\alpha \cos(\pi j / 2) \mp \gamma \cos(\omega_{j,k}^{\mp} \tau)]\}}{[\text{Re } C_{j,k}^{\mp} + \beta \cos(\pi j)(2l\pi / \omega_{j,k}^{\mp})]^2 + [\text{Im } C_{j,k}^{\mp}]^2} \\ &= \frac{\omega_{j,k}^{\mp} [E_j^{\mp}(\omega_{j,k}^{\mp})]'}{2\{[\text{Re } C_{j,k}^{\mp} + \beta \cos(\pi j)(2n\pi / \omega_{j,k}^{\mp})]^2 + [\text{Im } C_{j,k}^{\mp}]^2\}} \end{aligned}$$

The proof is completed.

Based on the above results, the conditions for local stability of the trivial equilibrium of the coupled network are obtained as follows.

Theorem 3 The following assertions are true if $1 - 2\alpha \cos(\pi j / 2) - \beta \cos(\pi j) \mp \gamma > 0$ hold for all $j = 0, 1, 2, 3$

- If each $E_j^{\mp}(\omega)$ has no positive roots, then the network is locally asymptotically stable for all coupling time delays.
- If $E_j^{\mp}(\omega)$ has only one positive and simple root, there exists a σ_c such that the network is locally asymptotically stable for $\sigma \in [0, \sigma_c)$, and at least the network undergoes a Hopf bifurcation when $\sigma = \sigma_c$, where $\sigma_c = \min\{\sigma_{j,0}^{\mp}\}$.
- If $E_j^{\mp}(\omega)$ has at least two positive and simple roots, a finite number of stability switches of the network occur as the internal short-cut time delay increases from zero to the infinity, and at least the system becomes unstable at last.

Proof. (a) From Theorem 1, It is easy to check that all roots of $\Delta(\lambda, \sigma, \tau) = 0$ have negative real parts.

(b) Since the leading coefficient of $E_j^\mp(\omega)$ is positive, $E_j^\mp(\omega) > E_j^\mp(\omega_{j,0}^\mp) = 0$ for all $\omega > \omega_{j,0}^\mp$ and $E_j^\mp(\omega) < E_j^\mp(\omega_{j,0}^\mp) = 0$ for $\omega \in [0, \omega_{j,0}^\mp)$. There follows $[E_j^\mp(\omega_{j,0}^\mp)]' > 0$. From Theorem 2, $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ has a new pair of conjugate roots with positive real parts when the short-cut time delay is crossing a critical value $\sigma_{j,k}^\mp$, and the number of roots with positive real part of $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ cannot decrease as the short-cut time delay increases.

(c) $E_j^\mp(\omega)$ has a number of positive and simple roots denoted by $\omega_{j,1}^\mp > \dots > \omega_{j,k}^\mp > \omega_{j,k+1}^\mp > \dots > \omega_{j,h}^\mp > 0$, $h \geq 2$. The crossing of real parts of the roots of $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ at two adjacent simple roots $\omega_{j,k}^\mp$ and $\omega_{j,k+1}^\mp$ must be in opposite directions since $[E_j^\mp(\omega_{j,k}^\mp)]'$ and $[E_j^\mp(\omega_{j,k+1}^\mp)]'$ have opposite signs. More specifically, $E_j^\mp(\omega) > E_j^\mp(\omega_{j,1}^\mp) = 0$ holds for all $\omega \in (\omega_{j,1}^\mp, +\infty)$ and for all possible $\omega \in (\omega_{j,2k+1}^\mp, \omega_{j,2k}^\mp)$, and $E_j^\mp(\omega) < E_j^\mp(\omega_{j,1}^\mp) = 0$ holds for all possible $\omega \in (\omega_{j,2k}^\mp, \omega_{j,2k+1}^\mp)$. Then, both $[E_j^\mp(\omega_{j,2k-1}^\mp)]' > 0$ and $[E_j^\mp(\omega_{j,2k}^\mp)]' < 0$ are true for $k \geq 1$. According to Theorem 2, as the short-cut time delay varies from zero to the infinity, $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ always adds a new pair of conjugate roots with positive real parts for each crossing at $\sigma_{j,2k-1,n}^\mp$, but reduces such a pair for each crossing at $\sigma_{j,2k,n}^\mp$.

The difference between two critical values of short-cut time delay corresponding to a given pair of roots $\pm i\omega_{j,k}^\mp$ satisfies

$$\sigma_{j,k,l+1}^\mp - \sigma_{j,k,l}^\mp = \frac{2\pi}{\omega_{j,k}^\mp} < \frac{2\pi}{\omega_{j,k+1}^\mp} = \sigma_{j,k+1,l+1}^\mp - \sigma_{j,k+1,l}^\mp, \quad k = 1, 2, \dots, h-1, \quad n = 0, 1, 2, \dots$$

Given a long short-cut time delay $\tilde{\sigma}$, the interval $[0, \tilde{\sigma}]$ includes more $\sigma_{j,1,m}^\mp$ corresponding to $\pm i\omega_{j,1}^\mp$ than $\sigma_{j,2,l}^\mp$ to $\pm i\omega_{j,2}^\mp$. Hence, more characteristic roots of $\Delta_j^\mp(\lambda, \sigma, \tau) = 0$ change their sign of real parts from the negative to the positive at $\sigma_{j,1,m}^\mp$ than those changing the sign of real parts from the positive to the negative at $\sigma_{j,2,l}^\mp$ with an increase of short-cut time delay in the interval $[0, \tilde{\sigma}]$. A similar assertion holds also true for the short-cut time delay crossing at $\sigma_{j,3,r}^\mp$ and $\sigma_{j,4,s}^\mp$ corresponding to $\pm i\omega_{j,3}^\mp$ and $\pm i\omega_{j,4}^\mp$, and so forth. As a result, the system must become unstable with an increase of the short-cut time delay and the number of stability switches is finite. This completes the proof.

3. Numerical Simulations

This section presents numerical examples for the coupled network to validate the theoretical analysis and explore interesting network behaviours. The coupled network parameters and nonlinear activation functions are chosen as $a = -0.2$, $b = -0.4$, $c = 0.5$, $f = g = \tanh$. From Theorem 1, in this case, the coupled network free of the short-cut time delays is stable for all coupling time delays.

(1) $\tau = 2$. Solving $E_2^-(\omega) = 0$ gives only one positive root $\omega_{2,1}^- = 0.19$. Then, one obtains a set of critical short-cut time delays $\sigma_{2,1,n}^- = 10.07, 43.07, \dots$. In addition, it is easy to check that $E_0^\mp(\omega) = 0$, $E_1^\mp(\omega) = 0$, and $E_2^+(\omega) = 0$ have no positive roots. According to Theorem 3, the coupled network is stable when $\sigma \in [0, \sigma_c)$ and loses its stability when $\sigma \geq \sigma_c$, where $\sigma_c = \sigma_{2,1,0}^-$. Figure 1(a) shows the stable rest states of the coupled network when $\sigma = 9$. Figure 1(b) illustrates the asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$ when $\sigma = 11$. In this case, the trivial equilibrium of the coupled network loses its stability and a branch of periodic oscillations comes into being. The responses of the coupled network, as shown in Figure 1, fully support the statements in Theorem 3.

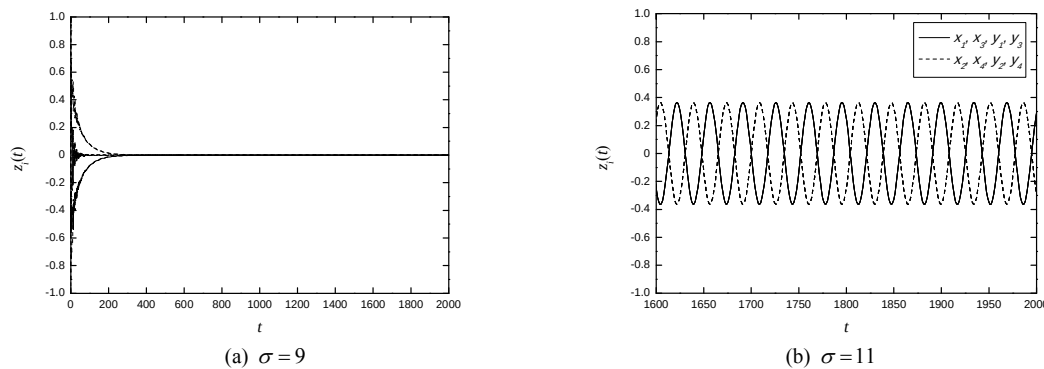


Figure 1 Responses of the coupled network when $a = -0.2$, $b = -0.4$, $c = 0.5$, $\tau = 2$. (a) rest states. (b) asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$.

(2) $\tau = 12$. By solving $E_2^-(\omega) = 0$ gives three positive and simple roots $\omega_{2,1}^- = 0.503$, $\omega_{2,2}^- = 0.427$, and $\omega_{2,3}^- = 0.053$. Then, one arrives at three sets of critical short-cut time delays $\sigma_{2,1,n}^- = 3.698, 16.180, 28.662, \dots$, $\sigma_{2,2,n}^- = 7.533, 22.240, 36.948, \dots$, $\sigma_{2,3,n}^- = 39.6, 158.8, 278.0, \dots$. Based on Theorem 2, as the short-cut time delay σ increases from zero to infinity, $\Delta_2^-(\lambda, \sigma, \tau) = 0$ always adds a new pair of conjugate roots with positive real parts for each crossing at $\sigma = \sigma_{2,1,n}^-$ and $\sigma = \sigma_{2,3,n}^-$, but reduces such a pair for each crossing at $\sigma = \sigma_{2,2,n}^-$. Similarly, solving $E_2^+(\omega) = 0$ gives two positive and simple roots $\omega_{2,1}^+ = 0.279$ and $\omega_{2,2}^+ = 0.180$. Then, one has two sets of critical short-cut time delays $\sigma_{2,1,n}^+ = 6.63, 29.12, 51.61, \dots$, $\sigma_{2,2,n}^+ = 20.98, 55.91, 90.84, \dots$. It follows that $\Delta_2^+(\lambda, \sigma, \tau) = 0$ always adds a new pair of conjugate roots with positive real parts for each crossing at $\sigma = \sigma_{2,1,n}^+$, but reduces such a pair for each crossing at $\sigma = \sigma_{2,2,n}^+$. In addition, all roots of $\Delta_j^+(\lambda, \sigma, \tau) = 0$ have negative real parts since $E_j^+(\omega) = 0$ have no positive roots, where $j = 0, 1$.

From simple comparisons, the above critical values of internal short-cut time delays $\sigma_{2,1,n}^-$, $\sigma_{2,2,n}^-$, $\sigma_{2,3,n}^-$, $\sigma_{2,1,n}^+$, and $\sigma_{2,2,n}^+$ can be ranked as

$$0 < \sigma_{2,1,0}^- < \sigma_{2,1,0}^+ < \sigma_{2,2,0}^- < \sigma_{2,1,1}^- < \sigma_{2,2,0}^+ < \sigma_{2,2,1}^- \\ < \sigma_{2,1,2}^- < \sigma_{2,1,1}^+ < \sigma_{2,2,2}^- < \sigma_{2,3,0}^- < \dots$$

Therefore, the coupled network is locally stable when $\sigma \in [0, \sigma_{2,1,0}^-) \cup (\sigma_{2,2,1}^-, \sigma_{2,1,2}^-)$ and unstable when $\sigma \in (\sigma_{2,1,0}^-, \sigma_{2,2,1}^-) \cup (\sigma_{2,1,2}^-, +\infty)$. That is, as the short-cut time delays increases from zero to infinity, the trivial equilibrium undergoes a finite number of stability switches and becomes unstable at last.

Figures 2(a) and (b) illustrate the stable rest states and asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$ when $\sigma = 3 \in (0, \sigma_{2,1,0}^-)$ and $\sigma = 5 \in (\sigma_{2,1,0}^-, \sigma_{2,1,1}^-)$, respectively. Figures 2(c) and (d) show the coexistence of the asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$ and asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = -y_i = y_{i+1} = -y_{i+2}$ under different initial conditions when $\sigma = 7 \in (\sigma_{2,1,0}^+, \sigma_{2,2,0}^-)$. Figure 2(e) gives the asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = -y_i = y_{i+1} = -y_{i+2}$ for $\sigma = 10 \in (\sigma_{2,2,0}^-, \sigma_{2,1,1}^-)$. Figure 2(f) depicts the unstable rest states become stable again when $\sigma = 26 \in (\sigma_{2,2,1}^-, \sigma_{2,1,2}^-)$. The responses of the coupled network, as shown in Figure 2, well coincide with the assertions in Theorem 3 again.

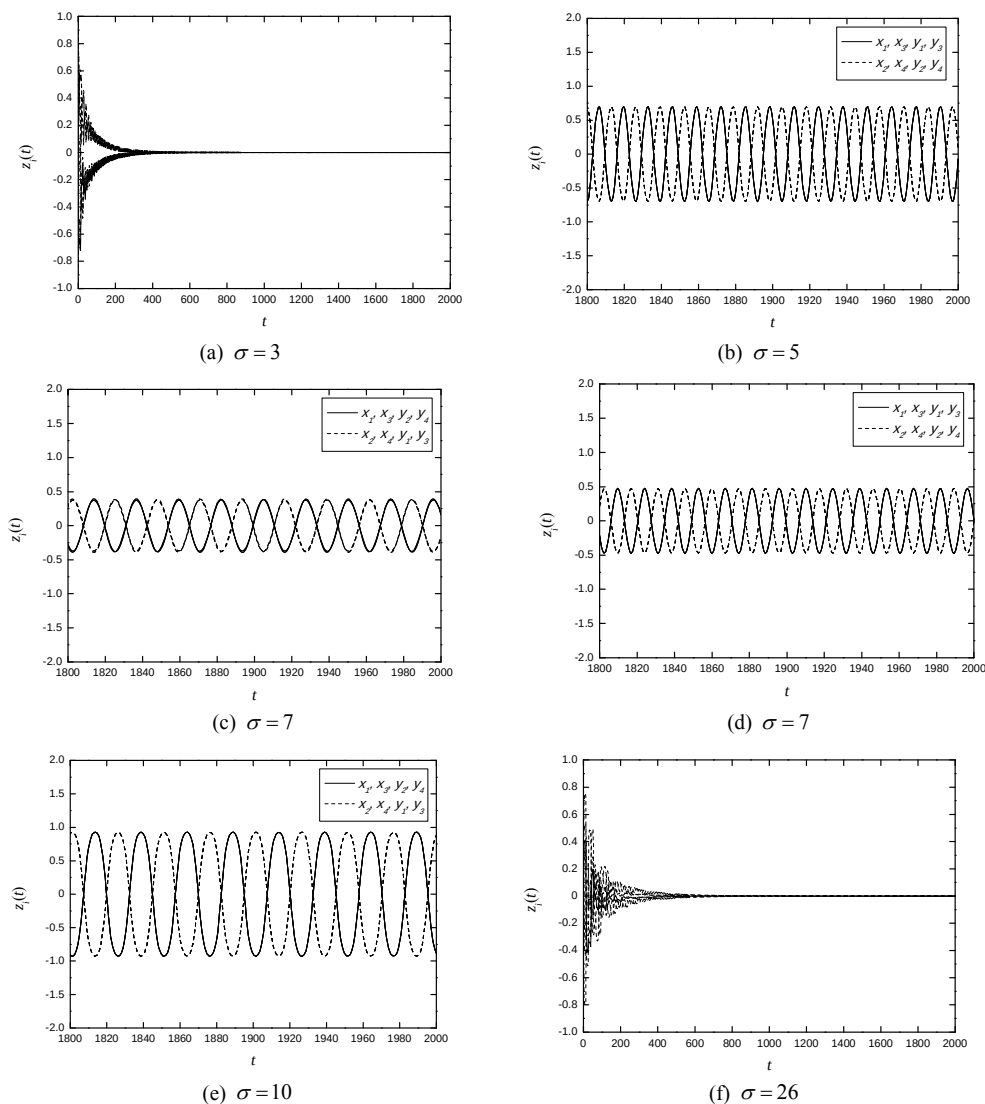


Figure 2 Responses of the coupled network when $a = -0.2$, $b = -0.4$, $c = 0.5$, $\tau = 12$. (a) rest states; (b) asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$; (c) and (d) coexistence of asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = y_i = -y_{i+1} = y_{i+2}$ and asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = -y_i = y_{i+1} = -y_{i+2}$ under different initial conditions; (e) asynchronous periodic oscillations with $x_i = -x_{i+1} = x_{i+2} = -y_i = y_{i+1} = -y_{i+2}$; (f) rest states.

4. Conclusions and Remarks

This paper presents a coupled network model that consists of two coupled neural networks and multiple couplings. The conditions for the stability switches of the network equilibrium and the existence of bifurcated oscillations are discussed. Interesting neuro-computational activities have been shown, such as rest state, different types of asynchronous periodic oscillations, multiple stability switches between the rest states and periodic activities, and multi-stability. It is possible to design the network activity from a rest state to a synchronous/asynchronous state, and/or from a synchronous/asynchronous state to a rest state, which may be helpful to the treatment of epilepsy and Parkinson's disease [29]. The system can exhibit multi-stability phenomena, such as the coexistence of different oscillations. Multi-stability is one of the most important dynamical features and has been observed in nervous systems composed of the delay-coupling neurons [27, 30, 31]. For example, the

delayed coupling neural network composed of a pair of Hindmarsh–Rose neurons exhibited the bistable dynamics, which is stable rest state and periodic activity [31]. In fact, the phenomena of multi-stability have found applications in many fields, such as content-addressable memory, pattern recognition, automatic control, and so on.

Although the system studied here is too simple to draw definite conclusions about real coupled neural networks, results are potentially useful because the dynamical phenomena revealed in this model can provide promising information for understanding the evolution patterns and rhythms in biological and physiological systems.

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