

# Insight into the dynamic behaviour of the Van der Pol/Raleigh oscillator using the internal stiffness and damping forces

M J Brennan<sup>1</sup>, B Tang<sup>2</sup> and J C Carranza<sup>1</sup>

<sup>1</sup> Department of Mechanical Engineering, UNESP, Ilha Solteira, 15385-000, Sao Paulo, Brazil.

<sup>2</sup> Institute of Internal Combustion Engine, Dalian University of Technology, Dalian, 116023, China.

mjbrennan0@btinternet.com

**Abstract.** The van der Pol oscillator is an archetypal nonlinear oscillator that has been studied for many years. It is a self-sustaining oscillator that vibrates in a limit cycle, and has the characteristic that it generates energy in the part of the cycle when the displacement is small and dissipates energy in the part of the cycle when the displacement is large. Almost all analyses for this type of oscillator have been conducted in a strict mathematical framework using the displacement and velocity of the mass to describe the motion in the phase plane. Physical insight into the behaviour is then generally only possible for very small or for very large damping nonlinearity. In this paper a fresh approach is taken. The internal forces of the Rayleigh oscillator are studied rather than van der Pol's equation as the key damping force is a function of only velocity. Simulations are presented which show how the stiffness and damping forces vary when the system is vibrating in a steady-state limit cycle.

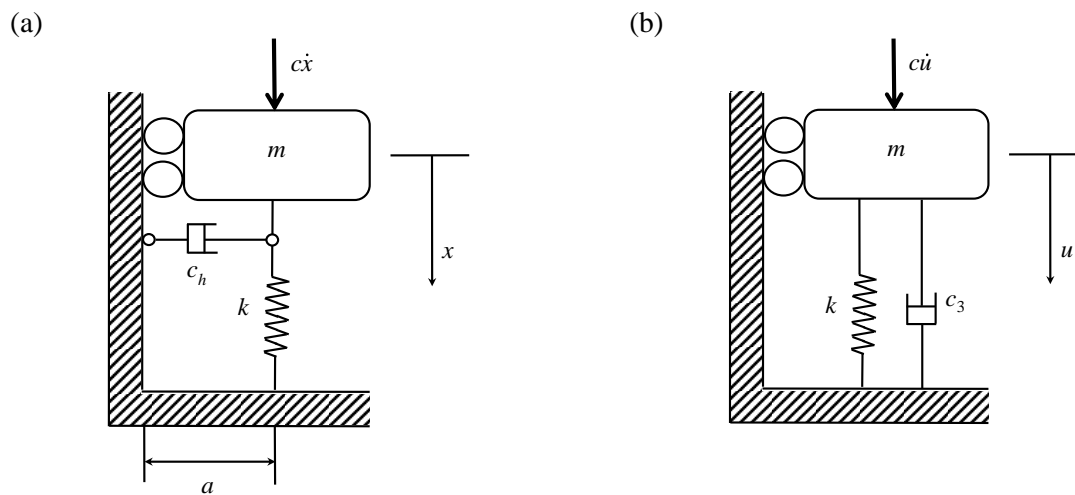
## 1. Introduction

This paper is concerned with two archetypal oscillators that were first described many years ago, namely the Rayleigh oscillator [1] and the van der Pol oscillator [2]. Both of these are self-excited oscillators in which there is some feedback of the output which results in an unstable system. If the systems are linear then their responses are unbounded. To achieve a bounded response, a nonlinear damping force is assumed. Although the oscillators are often discussed in a similar way by many researchers, their origins are very different. Rayleigh considered the self-sustained vibration of a clarinet reed. He assumed that the air flow over the reed had the same effect as the reed having negative linear viscous damping. Further, he introduced positive viscous damping that was proportional to the cube of the reed velocity. Appleton and van der Pol, on the other hand, derived a similar equation for an electrical circuit involving a triode vacuum tube [3]. Van der Pol preferred to work with an alternative version of the equation, which he described in his now famous paper [2]. In equivalent mechanical terms, the negative damping was also of the linear viscous type, but the positive viscous damping was proportional to the product of the square of the displacement and the velocity. van der Pol's paper has attracted the attention of many scientists and applied mathematicians who have made innumerable attempts to solve his equation analytically, see for example [4-6] and the references therein. More recently, modifications have been made to van der Pol's original equation in terms of the relationship between the damping and stiffness forces to velocity and displacement respectively, and then these equations have been studied, for example [7,8].



Alongside the more theoretical work described above, some researchers have tried to model physical systems using van der Pol and Rayleigh oscillators. A very good review of the work on electrical systems which behave as self-sustaining oscillators, that occurred before and just after the van der Pol's paper, has been presented by Ginoux [9]. Work on vibrations caused by fluid flow past structures has kept work on the Rayleigh oscillator alive [10, 11] as it has been shown to be a good approximate model to capture the physics of this phenomenon. Some other examples involving the Rayleigh oscillator are given by Cveticanin [12].

In this paper an alternative approach is taken in the study of van der Pol and Rayleigh oscillators. Rather than attempting to solve the equations directly, simulations are performed and the internal forces are studied. The damping term is varied from being very large when the system behaves as a relaxation oscillator, to being very small when the system behaves similar to a harmonic oscillator. Physical interpretation of their behaviour is facilitated by studying the Rayleigh oscillator rather than the van der Pol oscillator as this has the desirable quality that the damping force is only a function of velocity, which is not the case for the van der Pol oscillator.



**Figure 1.** Mechanical systems modelled by (a) van der Pol's equation (b) Rayleigh's equation.

## 2. Formulation of the equations of van der Pol and Rayleigh systems

An approximate mechanical representation of the van-der Pol equation is shown in figure 1(a). It consists of a conventional mass-spring-damper system, but with the damper configured so that it is orthogonal to the spring. The free vibration of this system (without the forcing term  $c\dot{x}$ ) has been considered in [13] and the harmonically forced vibration of the system in [14]. As mentioned in the Introduction, a particular feature of the van der Pol system is that it is self-excited, with the forcing function being proportional to the velocity of the mass as shown in figure 1(a). The equation of motion for this system is given by,

$$m\ddot{x} + c_h \frac{x^2}{a^2 + x^2} \dot{x} + kx = c\dot{x} \quad (1a)$$

where  $m$  and  $k$  are the mass and stiffness respectively,  $c_h$  is the linear viscous damping coefficient and  $a$  is the length of the damper; the overdots denote differentiation with respect to time. Provided that the displacement of the system is such that  $|x/a| < 0.2$ , then equation (1a) can be written as van der Pol's equation,

$$m\ddot{x} - (c - c_h x^2) \dot{x} + kx = 0 \quad (1b)$$

Note that when the equation is written this way, the forcing term and the dissipative term are combined into a nonlinear damping term  $(-c + c_h x^2)$  which is negative in the part of the cycle when  $|x| < \sqrt{c/c_h}$  and positive when  $|x| > \sqrt{c/c_h}$ . Letting the undamped natural frequency  $\omega_n = \sqrt{k/m}$ , the viscous damping ratio  $\zeta = c_1/(2\sqrt{mk})$ , the non-dimensional displacement  $y = x/\sqrt{c/c_h}$ , and the non-dimensional time  $\tau = \omega_n t$ , equation (1b) can be written in non-dimensional form as,

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0 \quad (2)$$

where  $\mu = 2\zeta$  and the overdots now denote differentiation with respect to non-dimensional time.

The equation of motion for the Rayleigh system shown in Fig 1(b) is given by,

$$m\ddot{u} - (c - c_3\dot{u}^2)\dot{u} + ku = 0 \quad (3)$$

where the damping terms have been collected on the left-hand side of the equation as in equation (1b). Letting,  $w = u/\sqrt{c/3c_3}$  and using the previously defined non dimensional variables results in,

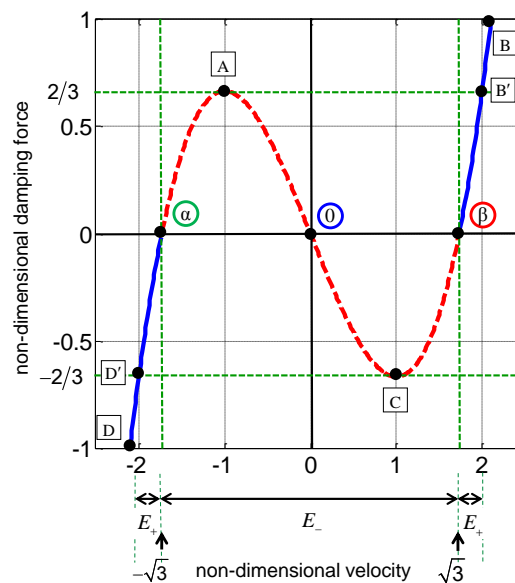
$$\ddot{w} - \mu\left(1 - \frac{\dot{w}^2}{3}\right)\dot{w} + w = 0 \quad (4)$$

If equation (4) is differentiated with respect to non-dimensional time and  $\dot{w}$  is set to be equal to  $y$ , then the result is the van der Pol equation given in equation (2). Thus, it can be seen that if  $c_h = 3c_3$  then the displacement and velocity of the system in figure 1(a) are the same as the velocity and acceleration of the system in figure 1(b). Thus, provided these relationships are noted, either the van der Pol system or the Rayleigh system can be studied to determine the dynamic responses of the systems shown in figures 1(a) and (b). Although this is the case, the research community has almost exclusively studied van der Pol's equation rather than Rayleigh's equation. This is probably because the dynamics of the system have been portrayed in the time domain and in the phase plane  $(\dot{y}, y)$ , and the point in the cycle when the damping in van der Pol's system changes from being negative to positive conveniently occurs when  $y = 1$ . However, in Rayleigh's equation the damping force is only a function of velocity which makes it easier to study (in van der Pol's equation it is a function of both displacement and velocity). The damping force is the key force in this system, as it predominantly determines the dynamic behaviour, so there are some advantages in studying Rayleigh's equation rather than van der Pol's equation. Thus the remainder of this paper is devoted to the study of equation (4).

### 3. Damping force

The difference in the systems discussed in figure 1 compared to simple linear oscillators is the damping force, which is given by  $F_d = \mu(1 - y^2)\dot{y}$  in Van der Pol's system and by  $F_d = \mu(1 - \dot{w}^2/3)\dot{w}$  in Rayleigh's system. As mentioned above, the displacement and velocity of the van der Pol system are the same as the velocity and acceleration of the Rayleigh system. This means, for example, that if the displacement of the van der Pol system is of interest, then the velocity of the Rayleigh system could be determined instead as this is identical to the displacement of the van der Pol system. The non-dimensional damping force of the Rayleigh system divided by  $\mu$  as a function of non-dimensional velocity is shown in figure 2. Note that if the mass and stiffness are removed from the Rayleigh system and the damper (combination of negative linear and positive cubic) has an imposed velocity, which cycles between positive and negative values, so that  $\max|F_d| = 1$  for example, then the resulting force will follow the curve in figure 2. When the product of the damping force and the velocity is positive (negative) then the damper will dissipate (supply) energy. This corresponds to the solid blue

lines when there is dissipation of energy ( $E_+$ ) and the dashed red lines when there is supply of energy ( $E_-$ ). Thus, it can be seen from figure 2, for  $|\dot{w}| > \sqrt{3}$  then the effective damper dissipates energy and for  $-\sqrt{3} < \dot{w} < \sqrt{3}$  the effective damper supplies energy. Referring to figure 2, D and B correspond to the minimum and maximum damping force respectively,  $\alpha$  and  $\beta$  correspond to the points in the cycle where the direction of energy flow changes, points D' and C correspond to  $F_d = -2/3$ , points A and B' correspond to  $F_d = 2/3$ , and 0 corresponds to point when both the velocity and the damping force are zero. Note the important feature, which is when the non-dimensional velocity has a value of  $\pm 2$ , the non-dimensional damping force is equal to  $\pm 2/3$  and corresponds points B' and D' respectively.

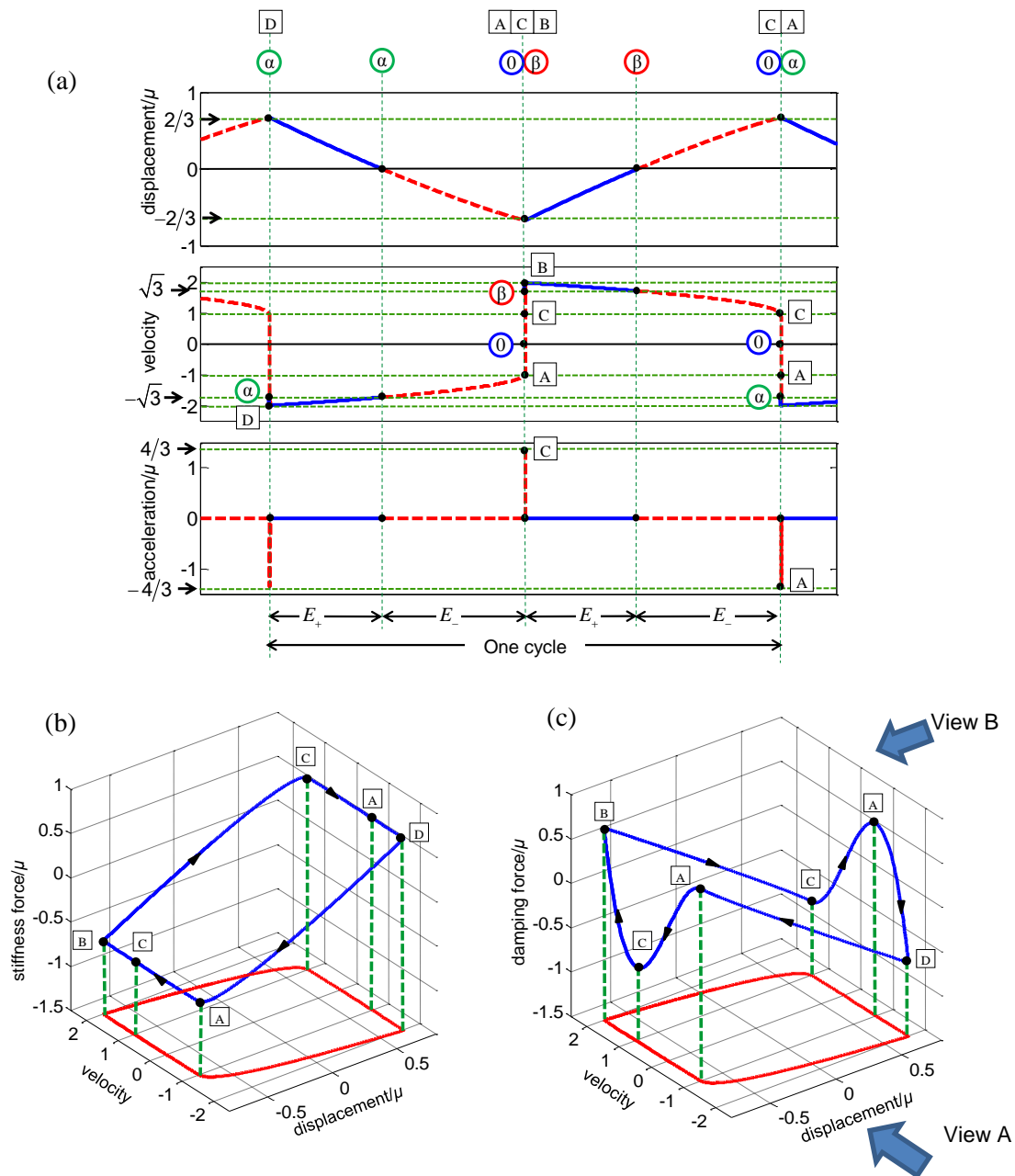


**Figure 2.** Plot of the non-dimensional damping force of the Rayleigh system as a function of non-dimensional velocity. The solid blue lines indicate that the energy is being dissipated by the damper and the dashed red lines indicate that energy is being supplied by the damper.

The graph in figure 2 is of fundamental importance in understanding the behaviour of the Rayleigh oscillator. It has a similar shape when the mass and spring are connected and is independent of the displacement; the only effect that the mass and spring have on the graph is to change the maximum values of the velocities and the corresponding damping forces, i.e., points B and D. Also note that close to points D' and B' the normalised force velocity relationship is given approximately by  $F_d/\mu \approx \mp 16/3 + 3\dot{w}$  respectively. The gradient of this approximation is of importance and is discussed later.

#### 4. System consisting of a spring and damper only

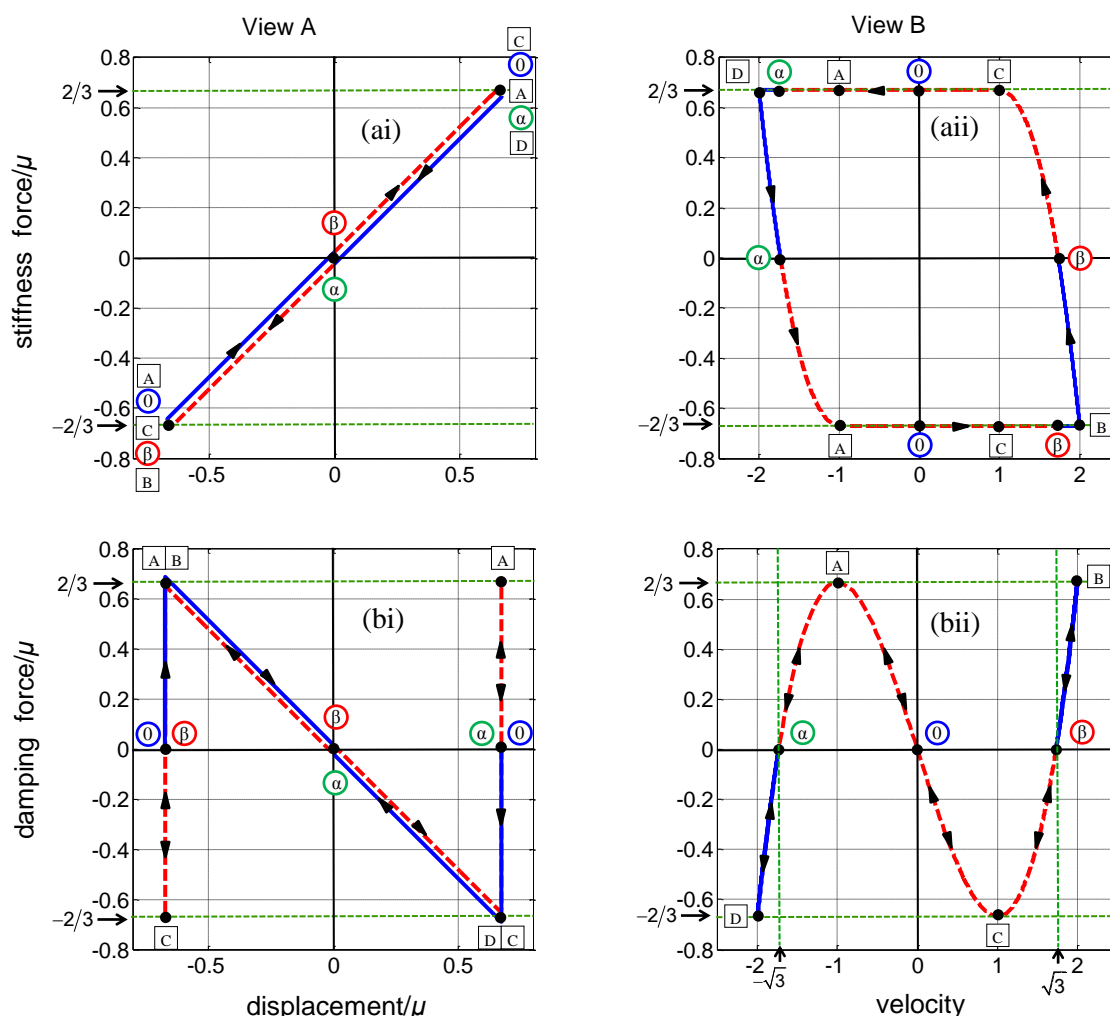
It is instructive to investigate the case when the mass in the Rayleigh system is negligible, i.e., the system can be considered to be simply a spring in parallel with the damper. In this case, the damping force and the stiffness force have to balance at every point in the cycle (note, that only steady-state motion is considered). Setting  $\mu = 100$ , which renders the inertia forces to be very small for most of the cycle, time domain responses for this system are determined by numerical integration of equation (4). The time histories of displacement/ $\mu$ , velocity and acceleration/ $\mu$  are shown in figure 3(a). The points that are marked on figure 2 are also marked in figure 3(a) for ease of reference. In figures 3(b) and (c) are shown the three dimensional normalised stiffness and damping forces as a function of both displacement and velocity. Also plotted in these figures is the phase portrait.



**Figure 3.** Features of the responses of the Rayleigh system for  $\mu = 100$  (the inertia force is negligible). (a) Time histories of non-dimensional displacement, velocity and acceleration. The displacement and acceleration are normalised by  $\mu$ . The solid blue lines indicate that the stored energy is decreasing with time and the dashed red lines indicate that the stored energy is increasing with time. (b) stiffness force normalised by  $\mu$  as a function of displacement and velocity (c) damping force normalised by  $\mu$  as a function of displacement and velocity.

Note first that the maximum/minimum velocity is  $\pm 2$  corresponding to  $F_d = \pm 2/3$ , which means that points D' and D in figure 2 are coincident, as are B and B'. This can also be deduced by the following argument. Consider a cycle that starts at point D, the velocity is at a maximum, which coincides with maximum displacement. Between points D and A, the spring compresses, passing

through point  $\alpha$  at which the effective damper changes from dissipating energy to supplying energy. This energy supply causes the spring to move from the equilibrium position (zero displacement) at point  $\alpha$  to maximum compression at point A. At this point the velocity changes instantaneously from a value of -1 to a value of +2 (also see figure 2) and the damping force passes through points 0, C,  $\beta$  arriving at point B, taking an infinitesimal amount of time to do so. At point B the spring is still fully compressed. This completes half a cycle, which is then repeated, but with the spring moving from full compression to full extension at points C and D. The three-dimensional plots show the stiffness and damping forces during a cycle, but do not show the dimension of time, and must, therefore be interpreted with care. For example, as mentioned above, the system moves from point A to B and from C to D instantaneously, but the time taken for the system to move from point D to A and from point B to C each takes half a period. This system essentially behaves as a relaxation oscillator, which has a period given approximately by  $1.61\mu$  [9].



**Figure 4.** Stiffness and damping forces corresponding to View A and View B in Fig. 3. The forces are non-dimensional and are normalised by  $\mu$ . The solid blue lines indicate that the stored energy is decreasing with time and the dashed red lines indicate that the stored energy is increasing with time. The arrows indicate the direction of positive time.

To help interpretation of the three dimensional plots of the stiffness and damping forces in figures 3(a) and (b), views A and B from these, corresponding to plotting the forces as a function of displacement/  $\mu$  and as a velocity, respectively, are plotted in figure 4. Also shown on these figures

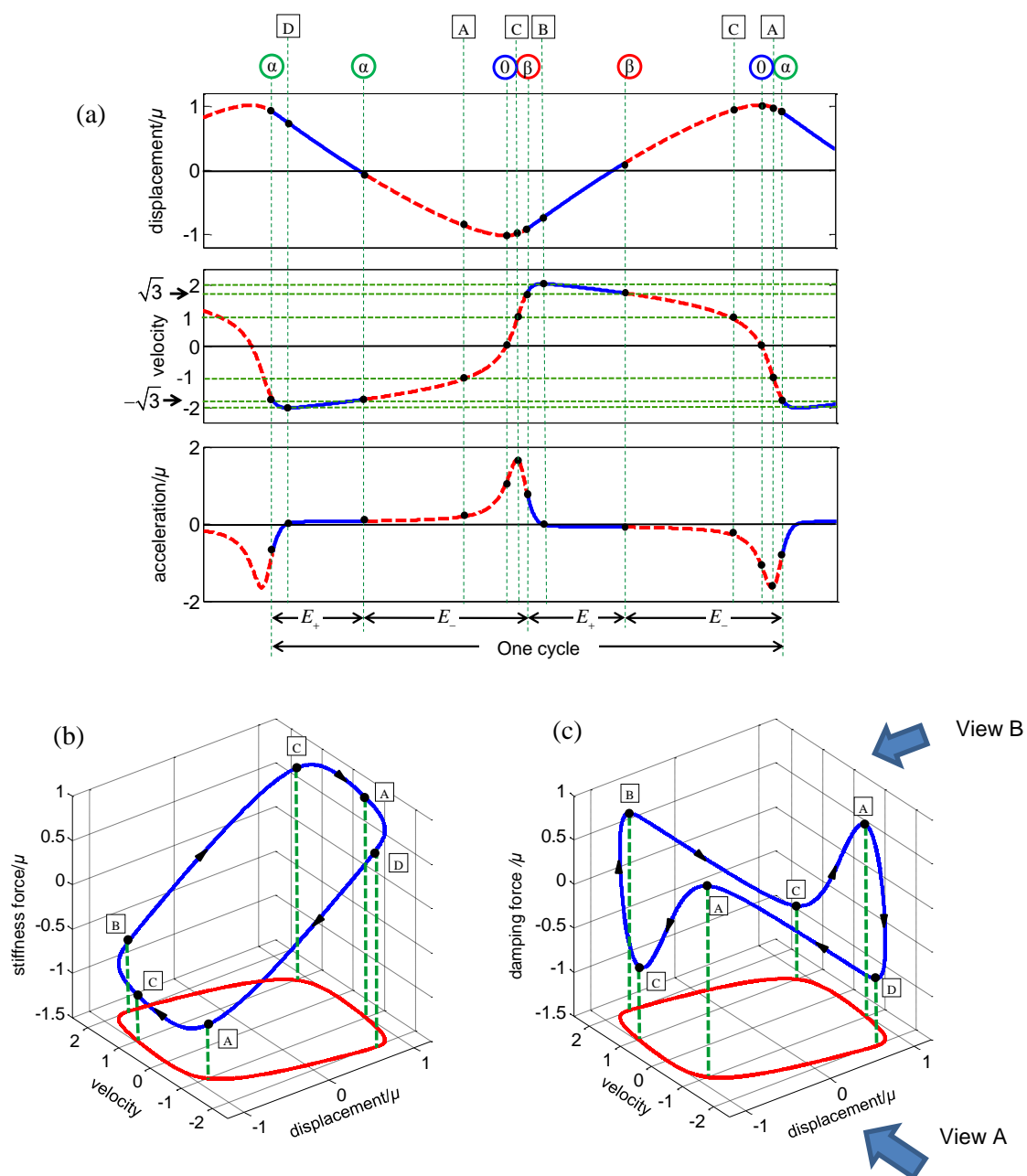
are the points shown in the time histories and the regions in the cycle where energy is dissipated or supplied. Note that the shapes of the stiffness force as a function of displacement and the damping force as a function of velocity do not change even when the inertia force become significant (as is shown later); the only effect that the inertia force has is to increase the maximum displacement and velocities, and the corresponding stiffness and damping forces. In figures 4(ai) the solid blue lines and the dashed red lines overlay in reality but have been separated here for clarity (they have also been separated in figure 4(bi)). By examining the stiffness and damping forces as a function of displacement, it can be seen that are equal and opposite between the minimum and maximum displacements/  $\mu = \mp 2/3$ . The discontinuities in the damping force at the maximum and minimum displacements (vertical lines) are equal and opposite to the inertia forces (not shown for brevity), which are delta functions (as shown in Fig. 3a), and correspond to the instantaneous change in velocity. Examining the stiffness force as a function of velocity, it can be seen that at the maximum and minimum velocities of  $\pm 2$ , there is an abrupt change. This is only possible because the inertia in the system is negligible. Note also, that because of the way in which the equation of motion is normalised, the stiffness force as a function of velocity is also the phase portrait which is shown in figures 3(a) and (b).

## 5. Effects of inertia

The simulations presented in the previous section repeated for a system with a non-negligible inertia force. Two cases are considered. The first when the system parameters are such that the maximum non-dimensional velocity of 2.0234 is achieved, and the other is when the damping forces are small in comparison with the stiffness and inertia forces. These correspond to values of  $\mu = 3.3$  and  $\mu = 0.1$  respectively.

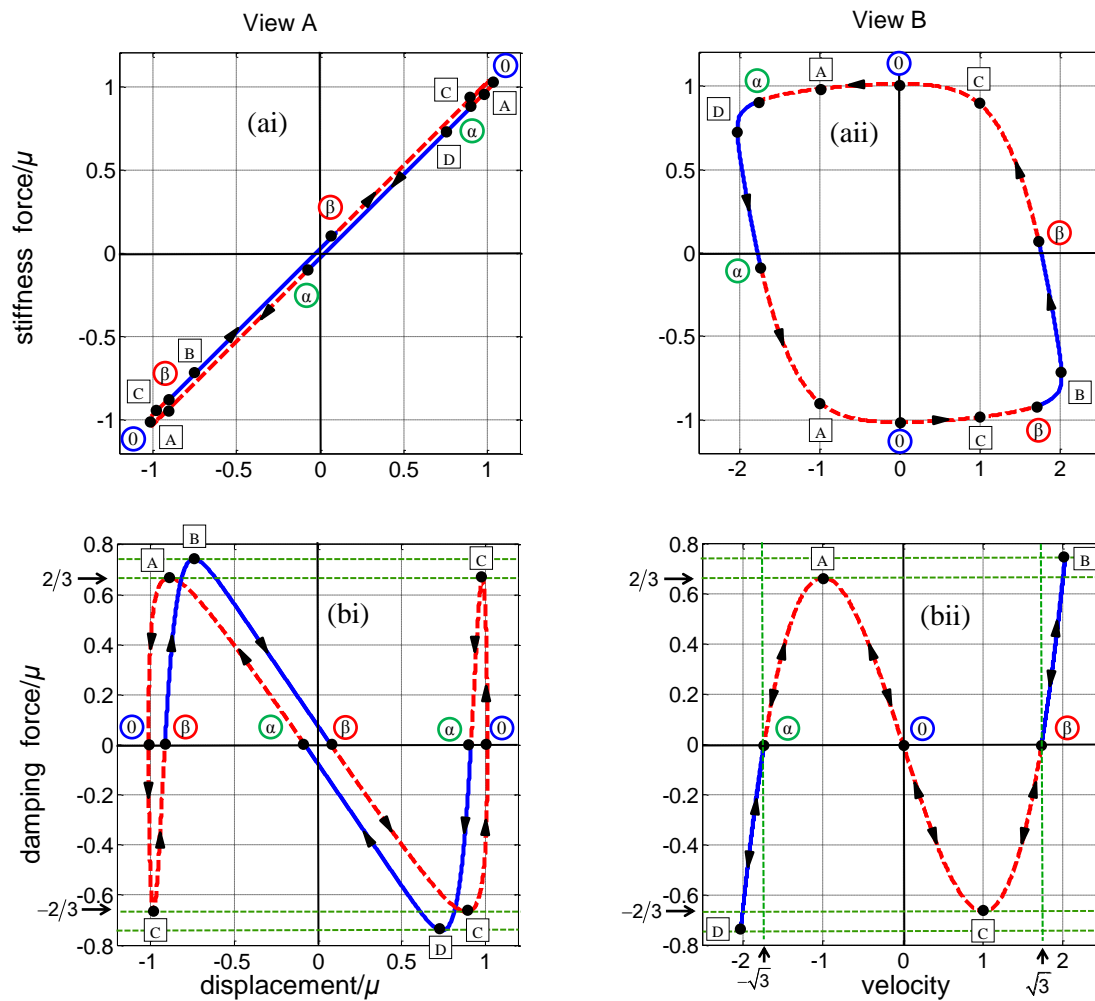
### 5.1 $\mu = 3.3$

The figures for this case are shown in figure 5 (corresponding to figure 3) and figure 6 (corresponding to figure 4). Comparing the time histories in figure 5 with those in figure 3, it can be seen that the main effect of the inertia is to change the abrupt reversal in the sign of the velocity, i.e., for example points A, 0, C,  $\beta$ , B, are no longer coincident, so that it takes a finite time to reverse the direction of the moving mass. Because of this, the displacement of the mass increases, as does the maximum velocity. The effect of the inertia on the stiffness and damping forces can be seen by examining the three dimensional plots in figures 5(b) and (c), and in figure 6. Although the maximum velocity is only increased marginally from a value of 2 to about 2.0234 [15], an increase of about 1.2%, the damping force increases from  $2/3$  to about 0.74, an increase of about 10.5%. This is clearly seen in figures 4b(i) and 4b(ii), and is due to the large gradient of the damping force as a function of the velocity as discussed previously.



**Figure 5.** Similar to figure 3 but for  $\mu = 3.3$ .

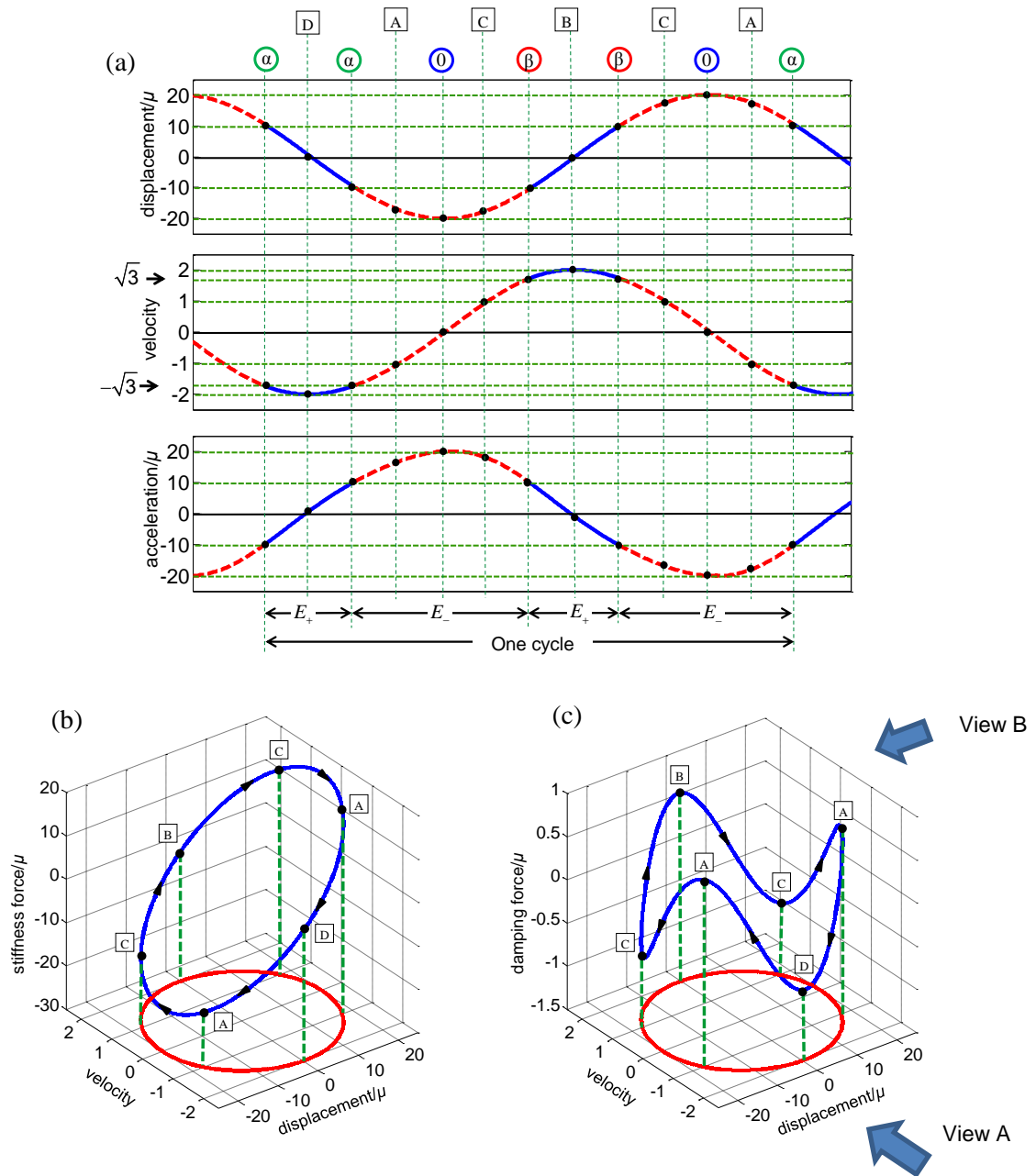




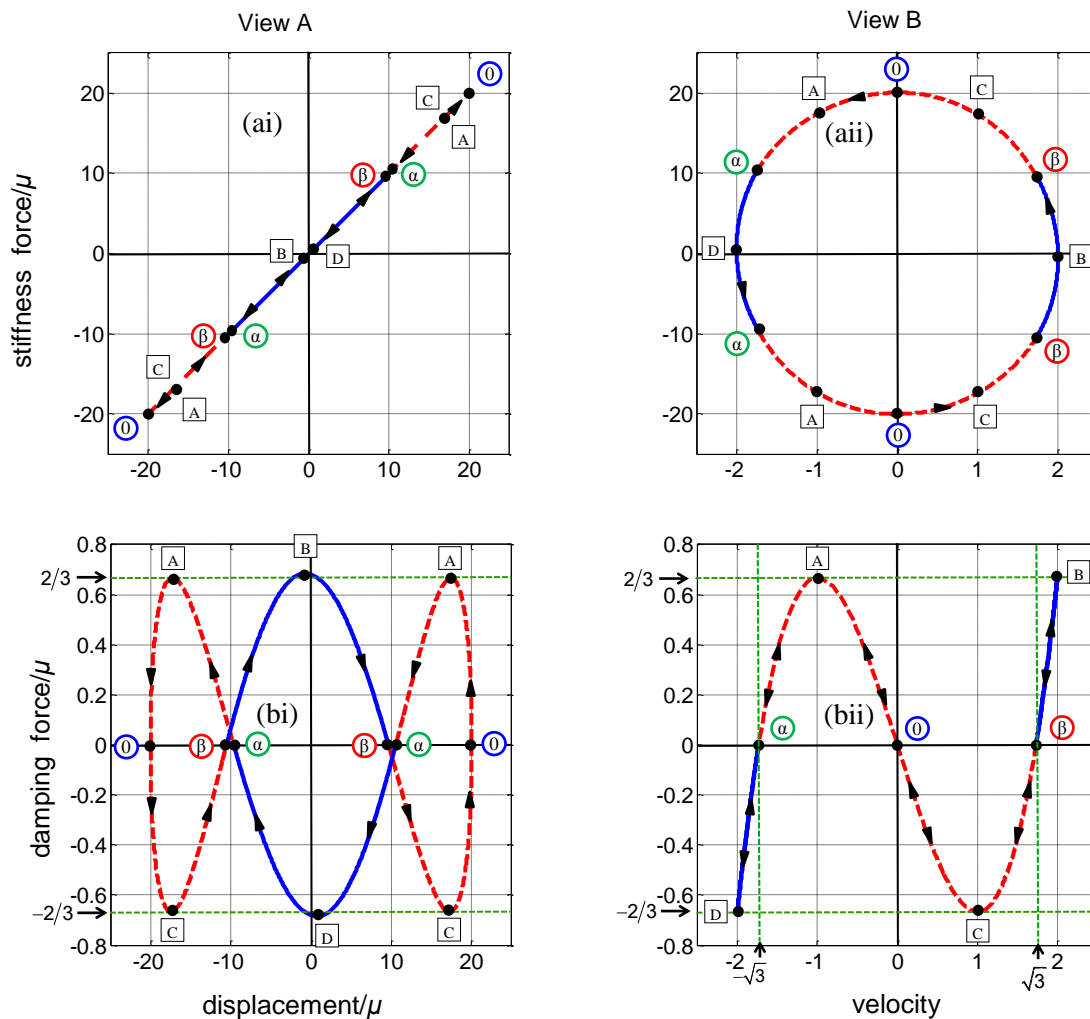
**Figure 6.** Similar to figure 4 but for  $\mu = 3.3$ .

### 5.2 $\mu = 0.1$

When  $\mu$  is set to 0.1, the maximum damping force is much smaller than either the maximum stiffness or the inertia forces. Moreover the maximum stiffness force is similar to the maximum inertia force. This means that the system behaves in a similar way to the free vibration of an undamped system, as which the system oscillates closely to the undamped natural frequency  $\omega_n$  and the displacement and velocity are almost sinusoidal. This can be seen in the time histories of the system shown in figure 7(a). The stiffness and damping forces are shown in figures 7(b) and (c) in three-dimensional form and separately as functions of displacement and velocity in figure 8. In this case the minimum and maximum non-dimensional velocities are  $\approx \pm 2$ . This has been shown by several researchers for example [16] and can be determined analytically simply by assuming harmonic motion and considering the balance of energy supplied and dissipated in a cycle.



**Figure 7.** Similar to figure 3 but with  $\mu = 0.1$ .



**Figure 8.** Similar to figure 4 but with  $\mu = 0.1$ .

## 6. Conclusions

This paper has presented an alternative way of investigating the dynamic behaviour of a van der Pol or Rayleigh oscillator. Simple mechanical systems that behave as such oscillators have been shown, and it has been suggested that there are advantages in focusing on the Rayleigh oscillator. This is because the damping force is only a function of velocity, which is not the case with the van der Pol oscillator. Simulations have been presented which show clearly when energy is supplied to the system and when energy is dissipated, which help in the interpretation of the dynamic behaviour. To gain physical insight into the effect of the damping force on the dynamic behaviour, a system was first studied in which the inertia was negligible (which represents a system undergoing relaxation oscillations). By examining the stiffness and damping forces, it was easy to show in this case that the maximum amplitude of the steady-state limit cycle non-dimensional velocity is 2. A system in which the inertia was non-negligible was also considered, by reducing the damping forces. It was shown that this is responsible for marginally increasing the maximum velocity to a maximum value 2.0234 when  $\mu = 3.3$ , which then reduces to about 2 when  $\mu$  is very small. For  $\mu = 0.1$  the motion of the mass is almost sinusoidal, as are the stiffness and damping forces, which is in stark contrast to the case when  $\mu$  is very large.

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