A Structured Model Reduction Method for Linear Interconnected Systems

Ryo Sato[1], Masaki Inoue[2] and Shuichi Adachi[3]

Faculty of Science and Technology, Keio University,
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan.

E-mail: [1]`ryo_sato@keio.jp`, [2]`minoue@appi.keio.ac.jp`, [3]`adachi@appi.keio.ac.jp`

Abstract. This paper develops a model reduction method for a large-scale interconnected system that consists of linear dynamic components. In the model reduction, we aim to preserve physical characteristics of each component. To this end, we formulate a structured model reduction problem that reduces the model order of components while preserving the feedback structure. Although there are a few conventional methods for such structured model reduction to preserve stability, they do not explicitly consider performance of the reduced-order feedback system. One of the difficulties in the problem with performance guarantee comes from nonlinearity of a feedback system to each component. The problem is essentially in a class of nonlinear optimization problems, and therefore it cannot be efficiently solved even in numerical computation. In this paper, application of an equivalent transformation and a proper approximation reduces this nonlinear problem to a problem of the weighted linear model reduction. Then, by using the weighted balanced truncation technique, we construct a reduced-order model with preserving the feedback structure to ensure small modeling error. Finally, we verify the effectiveness of the proposed method through numerical experiments.

1. Introduction

Many practical dynamic systems are large-scale and contain specific internal structure in their components. Since the model of the large-scale system tends to be very complicated, it is difficult to directly apply the control and analysis method to the model. As a preliminary stage for control and analysis, we need to extract the essence from the complicated model and construct a reduced-order model.

It is desirable for reduce-order models to preserve the physical characteristics such as the internal structure[1][2]. Although there are many works on model reduction methods for linear dynamic systems, most of them do not explicitly consider structure in the target model. Furthermore, the reduced-modeling error should be minimized under some theoretical guarantee.

In this paper, we focus on the feedback structure in a linear large-scale system. Then, we propose a model reduction method that preserves the structure and guarantee the performance of the reduced-order model.

The constitution of this paper is as follows. First, we introduce the definition of H_2 norm in Section 2 and formulate the model reduction problem in Section 3. Second, we propose a new model reduction method in Section 4 and an accuracy of the reduced-order model is evaluated by an accuracy index in Section 5. Finally we verify the effectiveness of the proposed method by numerical experiments in Section 6 and conclude this paper in Section 7.

2. Notation and Definition

In this paper, we evaluate the accuracy of a reduced-order model by using H_2 norm. We consider a continuous-time system described by

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Cx + Du, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}$ are constant matrices. Define the transfer function of the system as

$$G(s) := C(sI - A)^{-1}B + D \tag{3}$$

and let the realization of $G(s)$ described by A, B, C and D be

$$G(s) := (A, B, C, D). \tag{4}$$

Then, the H_2 norm of $G(s)$ is defined as

$$\|G(s)\|_{H_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}(G(j\omega)^* G(j\omega))d\omega}. \tag{5}$$

3. Problem Formulation

In this paper, we consider a single-input/single-output feedback system illustrated in Fig. 1. The system consists of two linear dynamic components, and they are connected in a feedback form. These components are described by m-th and n-th order models, and their transfer functions are $F(s)$ and $G(s)$, respectively. Let a realization of $G(s)$ be

$$G(s) = (A, B, C, D), \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}$ are constant matrices. Then, we define the transfer function of the feedback system that consists of $F(s)$ and $G(s)$ as

$$\mathrm{FB}(F, G) := \frac{G(s)F(s)}{1 + G(s)F(s)}. \tag{7}$$

The aim of this paper is to construct a reduced-order feedback model that approximately expresses $\mathrm{FB}(F, G)$. In particular, we design a reduced-order model of $G(s)$, which is denoted by $G_r(s)$, such that $\mathrm{FB}(F, G_r)$ properly approximates $\mathrm{FB}(F, G)$. Let a realization of $G_r(s)$ be

$$G_r(s) = (A_r, B_r, C_r, D_r), \tag{8}$$

where $A_r \in \mathbb{R}^{r\times r}$, $B_r \in \mathbb{R}^{r\times 1}$, $C_r \in \mathbb{R}^{1\times r}$ and $D_r \in \mathbb{R}$ are constant matrices. Then, we define the transfer function of the error system between $\mathrm{FB}(F, G_r)$ and $\mathrm{FB}(F, G)$ as

$$E(s) := \frac{G(s)F(s)}{1 + G(s)F(s)} - \frac{G_r(s)F(s)}{1 + G_r(s)F(s)}. \tag{9}$$

We aim to construct a reduced-order model with preserving feedback structure and minimizing the H_2 norm of $E(s)$.

We summarize the problem setting described above to formulate the following optimization problem. Define a cost function J:

$$J(G_r) := \|E(s)\|_{H_2} = \left\| \frac{G(s)F(s)}{1 + G(s)F(s)} - \frac{G_r(s)F(s)}{1 + G_r(s)F(s)} \right\|_{H_2}. \tag{10}$$

Then, the optimization problem is defined as

Problem 1 Find A_r^*, B_r^*, C_r^* and D_r^* minimizing J.

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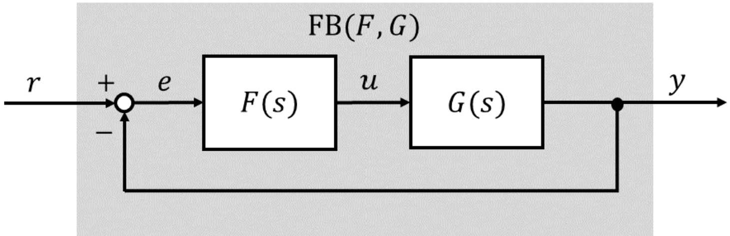

Fig. 1. Block diagram of feedback system

4. Model Reduction Method

Since Eq. (10) is nonlinear in parameters A_r, B_r, C_r and D_r, it is difficult to solve the optimization problem defined as Problem 1 in a computationally efficient way. This is one of the difficulties appear in the structure-preserving model reduction problems.

A simple approach to the problem is to apply a general linear model reduction method directly to $G(s)$. We expect that if the error in $G(s)$ and $G_r(s)$ is sufficiently small, $J(G_r)$ is small as well. For example, we can apply the well-known balanced truncation method[3] to $G(s)$. The resulting reduced-order model of $G(s)$ is denoted as $G_r^{\mathrm{bt}}(s)$. In this approach, the feedback structure is not considered. Obviously, this $G_r^{\mathrm{bt}}(s)$ is not the optimal solution to Problem 1.

4.1. Structure-Preserving Model Reduction

First, we apply an approximation technique to Problem 1 in order to reduce it to a convex optimization problem. Then, we propose a new model reduction method that consists of two stages, which are briefly introduced here. In Stage 1, we construct an intermediate reduced-order model by applying the weighted balanced truncation method[3]. In Stage 2, we further transform the intermediate model to a reduced order model $\mathrm{FB}(F, G_r)$ by solving the proposed optimization problem.

4.1.1. Problem Approximation

First, we transform Eq. (9) to

$$E(s) = W(s)G(s) - W_r(s)G_r(s), \tag{11}$$

where $W(s)$ and $W_r(s)$ are described by

$$W(s) := \frac{F(s)}{1 + G(s)F(s)}, \qquad W_r(s) := \frac{F(s)}{1 + G_r(s)F(s)}, \tag{12}$$

respectively. By this transformation, $E(s)$ is expressed as the error between $G(s)$ and $G_r(s)$ with the frequency weights $W(s)$ and $W_r(s)$. Now, supposing that

$$W(s) \approx W_r(s) \tag{13}$$

we can reduce Eq. (11) into

$$E(s) \approx E'(s) := W(s)(G(s) - G_r(s)). \tag{14}$$

Let a realization of $W(s)$ and $E'(s)$ be

$$W(s) = (A_w, B_w, C_w, D_w), \qquad E'(s) = \left(\tilde{A}, \tilde{B}, \tilde{C}, 0\right), \tag{15}$$

respectively, where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are described by

$$\tilde{A} = \begin{bmatrix} A & 0 & BC_w \\ 0 & A_r & B_rC_w \\ 0 & 0 & A_w \end{bmatrix},\ \tilde{B} = \begin{bmatrix} BD_w \\ B_rD_w \\ B_w \end{bmatrix},\ \tilde{C}^\top = \begin{bmatrix} C \\ -C_r \\ 0 \end{bmatrix},$$

respectively. Defining

$$J'(G_r) := \left\|E'(s)\right\|_{H_2} = \|W(s)(G(s) - G_r(s))\|_{H_2}, \tag{16}$$

from Eq. (13), we have $J'(G_r) \approx J(G_r)$. We obtain the approximated optimization problem as

Problem 2 Find A_r^*, B_r^*, C_r^* and D_r^* minimizing J'.

Minimization of J' is rewritten as

$$\text{minimize}\ \gamma\ \ \text{subject to}\ \ J' = \left\|E'(s)\right\|_{H_2} < \gamma. \tag{17}$$

Then, the constraint on the H_2 norm is reduced to linear matrix inequalities (LMIs) as follows. We introduce the following lemma.

Lemma : LMI description of H_2 characteristic for continuous-time systems[4]

Consider a continuous-time system represented by $E'(s) := (\tilde{A}, \tilde{B}, \tilde{C}, 0)$.
Then, for a $\gamma > 0$, the following statements are equivalent.
(i) $\|E'(s)\|_{H_2} < \gamma$
(ii) There exist positive definite symmetrical matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{l\times l}$ such that the following LMIs hold

$$\begin{pmatrix} P\tilde{A} + \tilde{A}^\top P & P\tilde{B} \\ \tilde{B}^\top P & -I \end{pmatrix} < 0, \tag{18}$$

$$\begin{pmatrix} Q & \tilde{C} \\ \tilde{C}^\top & P \end{pmatrix} > 0, \tag{19}$$

$$\mathrm{trace}(Q) < \gamma^2. \tag{20}$$

By using these LMI description of the H_2 constraint, we further reduce Problem 2 into the following problem:

Problem 3 Find A_r^*, B_r^* and C_r^* minimizing γ subject to Eqs. (18)–(20).

By the transformation above, we reduce Problem 2, in which Eqs. (18)–(20) are imposed on a functional space, into a problem with matrix inequality constraints. However, there is still difficulty to solve the problem. Since the inequalities in Eqs. (18)–(20) are bilinear to decision matrix variables A_r, B_r, C_r, P and Q, the problem cannot be solved in an effective way. Therefore, in the following method, we first obtain an intermediate approximated model $G_r^{\mathrm{wbt}}(s) = (A_r^{\mathrm{wbt}}, B_r^{\mathrm{wbt}}, C_r^{\mathrm{wbt}}, D_r^{\mathrm{wbt}})$ to reduce the bilinear matrix inequalities (BMIs) into LMIs for fixed A_r^{wbt} and B_r^{wbt}. Then, we solve Problem 3 with $A_r = A_r^{\mathrm{wbt}}$ and $B_r = B_r^{\mathrm{wbt}}$ to obtain C_r^{lmi} and D_r^{lmi}. Then, by the resulting linear reduced-order model $G_r^{\mathrm{lmi}}(s) = (A_r^{\mathrm{wbt}}, B_r^{\mathrm{wbt}}, C_r^{\mathrm{lmi}}, D_r^{\mathrm{lmi}})$, we construct $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$, which has the same feedback structure as $\mathrm{FB}(F, G)$ and approximately minimizes J in Problem 1.

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4.1.2. Proposed Model Reduction Method

We summarize the discussion above to the following model reduction method. The method is composed of two stages as follows.

Proposed Model Reduction Method

Stage 1:Model Reduction for $G(s)$

With $W(s)$ of Eq. (12), apply the weighted balanced truncation method to $G(s)$, which is one of the components consist $\mathrm{FB}(F, G)$ of Eq. (7), and construct a linear reduced-order model

$$G_r^{\mathrm{wbt}}(s) = \left(A_r^{\mathrm{wbt}}, B_r^{\mathrm{wbt}}, C_r^{\mathrm{wbt}}, D_r^{\mathrm{wbt}}\right). \tag{21}$$

Stage 2:Model Reduction for $\mathrm{FB}(F, G)$

Solve Problem 3 with $A_r = A_r^{\mathrm{wbt}}$ and $B_r = B_r^{\mathrm{wbt}}$ to construct

$$G_r^{\mathrm{lmi}}(s) = \left(A_r^{\mathrm{wbt}}, B_r^{\mathrm{wbt}}, C_r^{\mathrm{lmi}}, D_r^{\mathrm{lmi}}\right). \tag{22}$$

By the resulting $G_r^{\mathrm{lmi}}(s)$, we obtain $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$, which has the same feedback structure as $\mathrm{FB}(F, G)$.

5. Evaluation of Model Reduction Method

In this section, we evaluate the accuracy of the proposed reduced-order model $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$. The accuracy of $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ is compared with $\mathrm{FB}(F, G_r^{\mathrm{bt}})$, where G_r^{bt} is a linear reduced-order model that is constructed by the balanced truncation method.

By the proposed method, we construct the reduced-order feedback model $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$. It should be noted that $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ decreases the value of J', which is an *approximated* cost function of J of Problem 1. Therefore, we need to evaluate the accuracy of the reduced-order models for Problem 1. This can be evaluated by the value of J. Define a new index:

$$J_e := J(G_r^{\mathrm{bt}}) - J(G_r^{\mathrm{lmi}}). \tag{23}$$

$J_e > 0$ implies that the proposed reduced order model $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ is a more accurate solution to Problem 1 than the model $\mathrm{FB}(F, G_r^{\mathrm{bt}})$ by the conventional balanced truncation method. Since we cannot guarantee $J_e > 0$ theoretically, we verify the effectiveness of the proposed method by numerical experiments in many trials.

6. Numerical Experiment

6.1. Objective System

In this section, we consider a feedback system shown in Fig. 2. In this figure, $F(s)$ is a controller that is designed in advance. $G(s)$ is a coupled spring-mass-damper system, which expresses distributed parameter systems such as vibration behavior in buildings. The system is described by the equation of motion as

$$\mathcal{M}\ddot{x} + \mathcal{C}\dot{x} + \mathcal{K}x = bu, \tag{24}$$

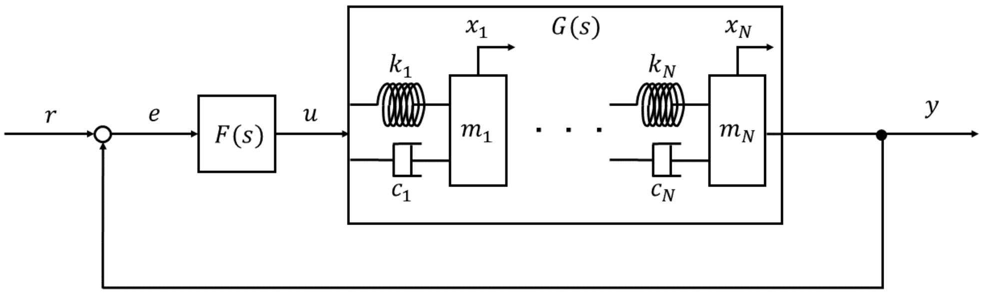

Fig. 2. Block diagram of feedback system with a connected spring-mass-damper system

where

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^\top \\ \mathcal{M} &= \mathrm{diag}(m_1, m_2, \cdots, m_N) \\ \mathcal{C} &= \mathrm{diag}(c_1, c_2, \cdots, c_N) \\ \mathcal{K} &= \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -k_{N-1} & k_{N-1}+k_N & -k_N \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & -k_N & k_N \end{bmatrix} \\ b &= \begin{bmatrix} b_1 & b_2 \cdots & b_N \end{bmatrix}^\top \end{aligned}$$

and x_i is the displacement of each mass, m_i the mass, k_i the spring constant and c_i the viscous friction coefficient, respectively, where $i \in \{1, \cdots, N\}$ is the index number. Letting a new state variable as

$$\tilde{x} = \begin{bmatrix} \dot{x}^\top & x^\top \end{bmatrix}^\top,$$

we obtain the first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathcal{M} & 0 \\ 0 & I_{N\times N} \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} -\mathcal{C} & -\mathcal{K} \\ I_{N\times N} & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u. \tag{25}$$

Then, defining

$$A = \begin{bmatrix} \mathcal{M} & 0 \\ 0 & I_{N\times N} \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{C} & -\mathcal{K} \\ I_{N\times N} & 0 \end{bmatrix},\ B = \begin{bmatrix} \mathcal{M} & 0 \\ 0 & I_{N\times N} \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix},$$

we can describe the system by state space representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = A\tilde{x} + Bu, \tag{26}$$
$$y = C\tilde{x}. \tag{27}$$

6.2. Conditions of Experiment

In Fig. 2, $F(s)$ is described by

$$F(s) = \frac{10^{-5}s + 1}{10^{-3}s + 1}.$$

The values of the parameters in the system Eqs. (26)–(27) are given as follows.

- Number of mass : $N = 10$
- Model order : $n = 20$
- Each constant : Setting numbers in random such that m_i is between 1 and 10, k_i between 1 and 10^5, and c_i between 0 and 1.
- Matrix B and C :

$$B = \begin{bmatrix} 1 \\ 0_{19} \end{bmatrix}, \; C = \begin{bmatrix} 1_{10} & 0_{10} \end{bmatrix}.$$

By this matrix choice of B and C, we represent that an actuator input from $F(s)$ controls the velocity $\dot{x}_1$ of the first-mass m_1, and the information on the summation of the velocity of all masses is measurable.

In this experiment, we will construct a linear reduced-order model $G_r(s)$, in particular, a 6-dimentional state space model. Under the conditions above, we evaluate the model reduction methods in 200 trials by the index introduced in Section 5. To describe and solve LMIs, we use YALMIP[5] and SeDuMi[6], respectively.

6.3. Experiment Result

The result for numerical experiments in 200 trials is summarized in Table 1. In the table, every trial is classified into two categories depending on the sign of J_e. As defined in Section 5, $J_e > 0$ means the effectiveness of the proposed method for Problem 1. From the table, we see that the proposed method generates more accurate reduced-order feedback models than the conventional method for almost 75 percent of all trials.

We note here that the values of J and J' are small enough in almost all trials, which means that the proposed approximation given in Eq. (13) is proper. The details are omitted in this paper.

The gain plots of the original feedback model $\mathrm{FB}(F, G)$ and the reduced-order feedback models $\mathrm{FB}(F, G_r^{\mathrm{bt}})$ and $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ for a trial are shown in Fig. 3. From the figure, the peak preserved in $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ is different from that in $\mathrm{FB}(F, G_r^{\mathrm{bt}})$. The proposed $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ successfully preserves the maximum peak, which is the most important property in the original $\mathrm{FB}(F, G)$, while the conventional $\mathrm{FB}(F, G_r^{\mathrm{bt}})$ does not. Although the proposed reduced-order model is not always the best as illustrated in Table 1, we showed that it can be a better model at least for some case. The proposed method is practically useful for constructing a reduced-order model with preserving the feedback structure.

7. Conclusion

In this study, we proposed a new model reduction method for a feedback system. Then, we verified the effectiveness of the proposed method by numerical experiments on a large-scale interconnection spring-mass-damper systems.

In future works, we will theoretically show the effectiveness of the proposed method and apply to a real system.

Table 1. The result for numerical experiments in 200 trials. Every trial is classified into two categories depending on the signs of J.

	$J_e > 0$	$J_e < 0$
Trial number	147	53

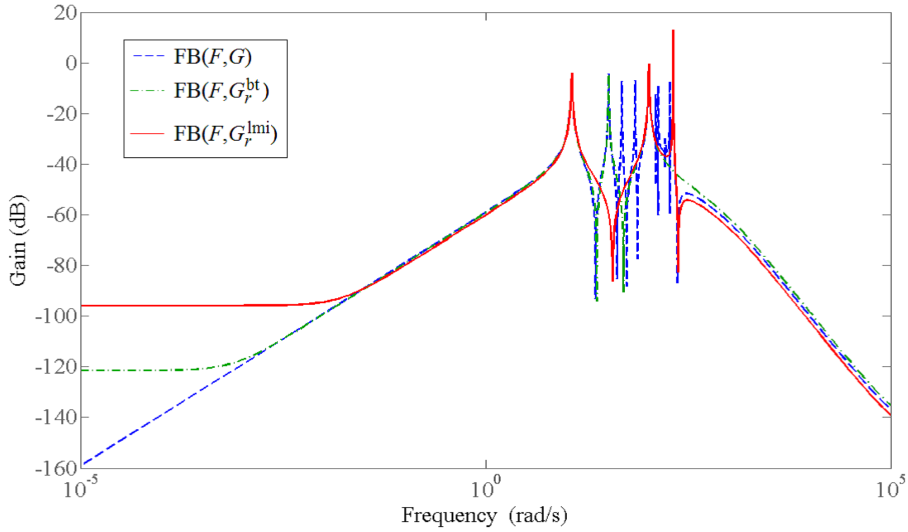

Fig. 3. Gain plot of a trial. The blue dashed line represents the original feedback model $\mathrm{FB}(F, G)$. The green chain line represents the reduced-order feedback model $\mathrm{FB}(F, G_r^{\mathrm{bt}})$ by the conventional method. The red solid line represents the reduced-order feedback model $\mathrm{FB}(F, G_r^{\mathrm{lmi}})$ by the proposed method.

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