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# Inferring unstable equilibrium configurations from experimental data

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**Abstract.**
This research considers the structural behavior of slender, mechanically buckled beams and panels of the type commonly found in aerospace structures. The specimens were deflected and then clamped in a rigid frame in order to exhibit snap-through. That is, the initial equilibrium and the buckled (snapped-through) equilibrium configurations both co-existed for the given clamped conditions.

In order to transit between these two stable equilibrium configurations (for example, under the action of an externally applied load), it is necessary for the structural component to pass through an intermediate unstable equilibrium configuration. A sequence of sudden impacts was imparted to the system, of various strengths and at various locations. The goal of this impact force was to induce relatively intermediate-sized transients that effectively slowed-down in the vicinity of the unstable equilibrium configuration. Thus, monitoring the velocity of the motion, and specifically its slowing down, should give an indication of the presence of an equilibrium configuration, even though it is unstable and not amenable to direct experimental observation. A digital image correlation (DIC) system was used in conjunction with an instrumented impact hammer to track trajectories and statistical methods used to infer the presence of unstable equilibria in both a beam and a panel.

## 1. Introduction

Although primary interest has historically been focused quite naturally on physically observable stable equilibria, in nonlinear systems it is the organizing role played by unstable equilibria that has a profoundly important influence on transient behavior and stability in-the-large [1]. The phenomenon of dynamic snap-through (a saddle-node bifurcation) is a key stability transition in this situation [2]. Consider the schematic shown in fig. 1. Part (a) shows a potential energy function, with a minimum, a position of stable equilibrium, at $x = 0$ and an adjacent hilltop, a position of unstable equilibrium at $x = 1$. If we then place this within the context of a dynamical system (a lightly damped oscillator) we can see that the unstable equilibrium acts as a barrier for confining phase trajectories - which eventually decay back to equilibrium given some positive

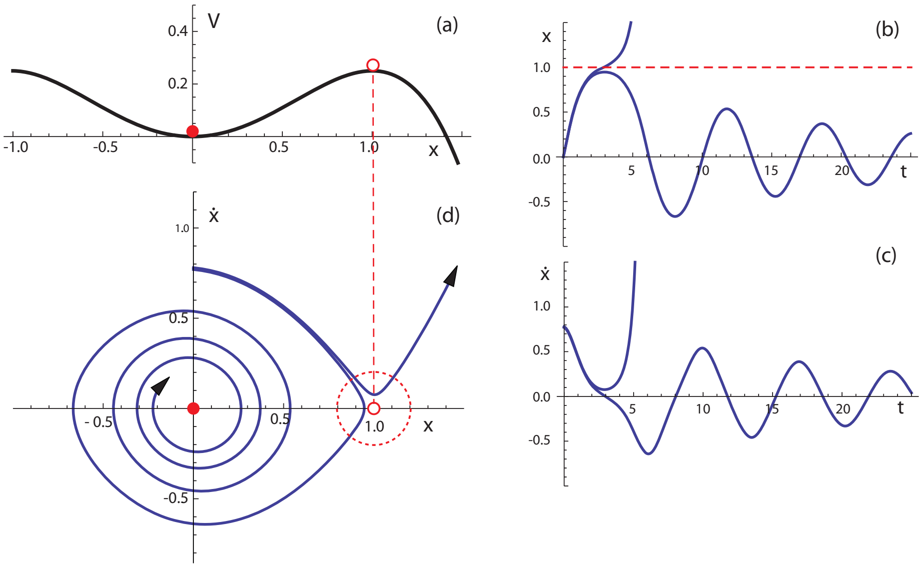


**Figure 1.** Adjacent trajectories spanning the seperatrix in a single-degree-of-freedom system.

energy dissipation, *provided the hilltop is not traversed*. For a one-dimensional conservative system it is trivial, for example, to determine the energy barrier, associated with a critical velocity say, that separates escaping and non-escaping trajectories, and trajectories have no choice but to pass directly over the hilltop if they become unbounded. However, with damping this becomes a little more involved. Parts (b-d) show two sample phase trajectories corresponding to the underlying potential from part (a), i.e., we have constructed a lightly-damped oscillator in which the restoring force (spring) is governed by the form of the underlying potential energy. The initial conditions closely span the critical velocity ($\dot{x}_{cr} = \sqrt{0.5}$ in the absence of damping).

In both cases the trajectories slow down, bringing with them the kinetic energy, as they approach the unstable equilibrium, but then rapidly diverge from each other: one is bounded motion, the other is not. Their slowing down is most clearly seen in velocity time-series in part (d). Of course the trajectory also slows down, and actually passes through zero, each time there is a change in direction of the motion (even after snap-through has occurred) but here we see both trajectories dwell in the vicinity of the unstable equilibrium at $x \approx 1$. This simple case is really brought into focus if rather than identify the energy associated with a specific a critical initial velocity, we initiate motion with an impact force, and then follow the path of the resulting trajectory as it navigates the potential energy landscape. In general, it is possible to have a number of extreme in a potential energy curve. And it is when this landscape is greater than one-dimension that the motion explores a wide variety of paths and especially those associated with saddle points. For example, in two-dimensions the potential energy surface might comprise a number of maxima, minima and saddle points. Starting at a local minimum it is likely that a large-perturbation-initiated trajectory, rather than pass directly over a local maximum, would follow a path via a mountain pass, as it finds its way to another minima. And in such a higher-order system it is not necessary that all the components of a velocity vector reach zero at the same time.

The scenario under consideration concerns the situation in which the initial structural configuration is fixed, and then the system is subject to external perturbations, generating dynamics. That is, the potential energy surface is fixed, in contrast to many studies in buckling in which a slowly applied axial load might cause the shape of the potential energy to change, perhaps through thermal loading [3], as well as secondary mode jumping [4].

The central questions are as follows:

- can this slowing down infer the presence of unstable equilibria in a system which has more than a single-degree-of-freedom?
- how can we use this approach in a purely experimental context?

In order to answer them, we will present results generated form experimental testing of a beam and a panel. In both cases the systems are mechanically (elastically) buckled such that bi-stability results. Thus, a system at rest in a baseline equilibrium configuration may be 'snapped-through' to an alternative equilibrium configuration. The input of energy can be achieved with an instrumented impact hammer, and it is the resulting transient motion that is scrutinized for traces of characteristic behavior during its passage back to equilibrium. If the system is repeatedly subject to this kind of disturbance, given sufficient variety of disturbance magnitudes, then a fraction of the generated transients will, it is suggested, pass close-by any unstable equilibria, with a sluggish velocity revealing its presence. A related approach has been developed in which a controller is used to keep a system in the vicinity of unstable fixed point (a periodic orbit in their case) via a continuation [5].

## 2. A buckled beam

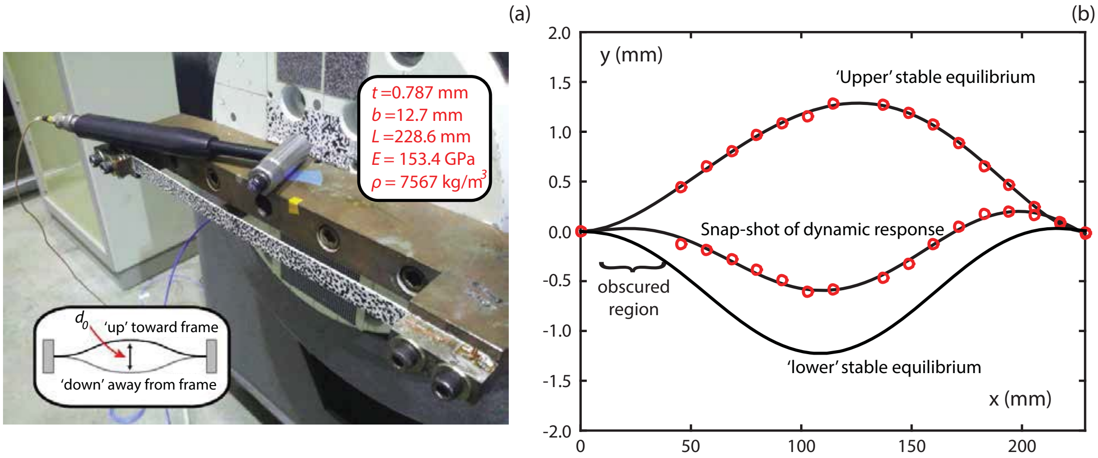


**Figure 2.** (a) the buckled beam clamped in position, with impact hammer, (b) beam shapes (and fit) indicating the acquired data. The curve without data points was obtained from a static DIC shot.

Figure 2(a) shows a thin steel-alloy beam (of dimensions $228 \times 12.7$ mm, and thickness 0.787 mm) clamped in a slightly buckled configuration. The initial location of the beam is entirely specified using the measurement capabilities of a dual camera DIC system (hence the speckled pattern on the beam). The beam can then be pushed-through to reveal the co-existing (snapped-through) equilibrium configuration. Part (b) shows these equilibria in terms of discrete measured locations along the beams length (the data points), together with a curve fit (the continuous lines), and a typical snap-shot extracted during motion. We note the slightly different shapes between the buckled equilibria as well as the slight asymmetry about the mid-point. Both of these effects are inevitable in experiments and underly the importance of including geometry imperfections in conventional buckling analysis. The small region without data points toward the left end of the beam resulted from the impact hammer interfering with the camera field of view. Due to the support structure it was not possible to strike the beam from the opposite side. Data was acquired at a sampling rate of 2 kHz. Due to data storage limitations the run-time of each of the (two) experimental test runs was about 100 seconds.

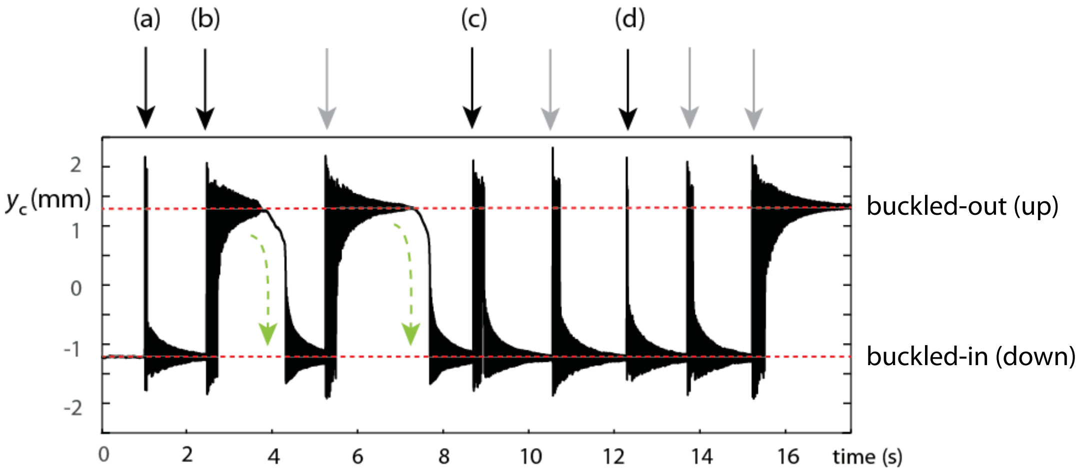


**Figure 3.** A typical time series indicating perturbations caused by hammer hits.

Consider the measured time series shown in figure 3, comprising a typical 18 second sample of acquired data. This is representative of the measured data from the center of the beam. The DIC provides full-field measurements, a crucial component in identifying unstable equilibria (shapes) to be described later. Initially it sits in the buckled 'buckled-out' configuration ($y_c \approx -1.2$mm). After about a second (labeled a) the beam is hit by the hammer and it snaps-through a few times before settling back onto the original equilibrium configuration. The relatively slow rate of settling is indicative of the light damping in the system. Another strike was imparted at a little after 2 seconds (labeled b) and this time there are quite a few snaps before the system settles onto the alternative 'buckled-in' configuration. It is then manually pulled through to the 'buckled-out' configuration (indicated by the green arrow), a necessity due to the single-sided constraint on the impact direction mentioned earlier. This whole process was repeated many times.

In terms of the usable data the following effects came in to play. First, some of the strikes were double-hits and transients associated with these occurrences were eliminated, since it was difficult to accurately establish the loss of contact. Additionally, any relatively light strikes (those not even close to causing snap-through) were also eliminated from the usable data. On the other hand, some of the transients involved multiple snap-through events. Thus, in the 200 seconds of acquired data, 24 useful hits (resulting in many more snap-through events) were identified for subsequent scrutiny, and especially the last snap-through event in a sequence since this would have the lowest velocity.

It is well established that moderately buckled beams of the type under consideration are effectively modeled using two modes: symmetric and asymmetric [6, 7]. For mildly buckled beams, or very shallow arches, a single (symmetric) mode will usually suffice, especially if the forcing is applied directly at the center [8]. In fact there may be many unstable equilibrium paths in the behavior of thin arches [9].

Although this paper is focused on experimental data, knowledge of the dominant two-DOF behavior encourages a plot of the data in terms of a configuration space using the central lateral deflection ($y_c$) versus the difference between the left and right quarter points of the beam: $(y_L - y_R)/2$, i.e., a measure of the angle, or asymmetry, in the response. This is shown in figure 4 for some representative transients. They are transients generated by strikes (a)-(d) from figure 3. The

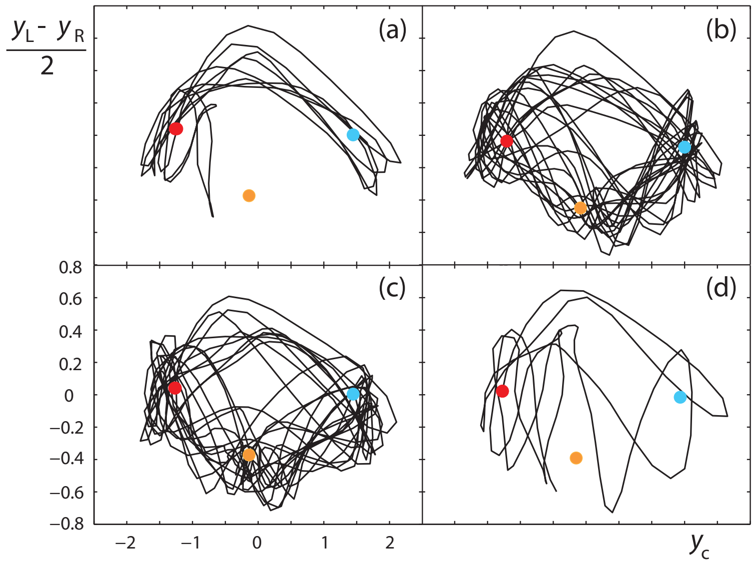


**Figure 4.** Some typical transient time series plotted as phase projections. The individual panels (a-d) correspond to the labels appearing in figure 3. The axes units are mm.

red dots correspond to the starting point., the blue correspond to the alternate stable equilibrium. In parts (a), (c) and (d) the trajectory ends up back on the original equilibrium. In case (b) the trajectory makes a relatively high number of snap-throughs before it settles into the opposite equilibrium (the blue data point). The path taken by the trajectory during snap-through typically involves an appreciable asymmetric component. This is suggestive of a saddle point in the potential energy surface, since the path will follow the path of least energy. As mentioned in the introduction, this is unambiguous in the one-dimensional case but requires more consideration in higher order systems.

Focusing on the velocity, and hence, kinetic energy associated with the trajectories we can numerically differentiate the transient behavior. This is shown for a single trajectory in figure 5, and corresponds to a portion of case (b) shown in figures 3 and 4. The duration shown here includes the last five snap-through events, and these are labeled 1 through 5, with 6 corresponding to a near-miss snap-through, i.e., by this stage the free vibration energy has dissipated sufficiently that the beam doesn't quite snap-through and subsequent motion is contained in the 'buckled-in' shape. The kinetic energy reaches local maxima every time it passes a stable equilibrium, and reaches zero when the trajectory changes direction (with the shapes labeled $M$). However we also see cases in which the kinetic energy reaches a relative minimum (labeled by $S$). From this plot we can infer that the slowing down of the trajectories in the vicinity of the shapes given by $S1$ to $S5$ reflect the presence of an unstable equilibrium. The approximate location of these shapes are indicated by the orange circles in figure 4(b). There is another saddle point locating for positive angles (not shown). Depending on the hammer strike, it possible to snap-through with the right half of the beam leading the way. The dashed red line in figure 5 represents an approximate location of an energy barrier associated with snap-through.

## 3. A buckled panel

We next present the results from an analogous study on a buckled rectangular steel panel of dimensions 228 × 127mm, and thickness 0.5mm. This study is still *work-in-progress*, so we just

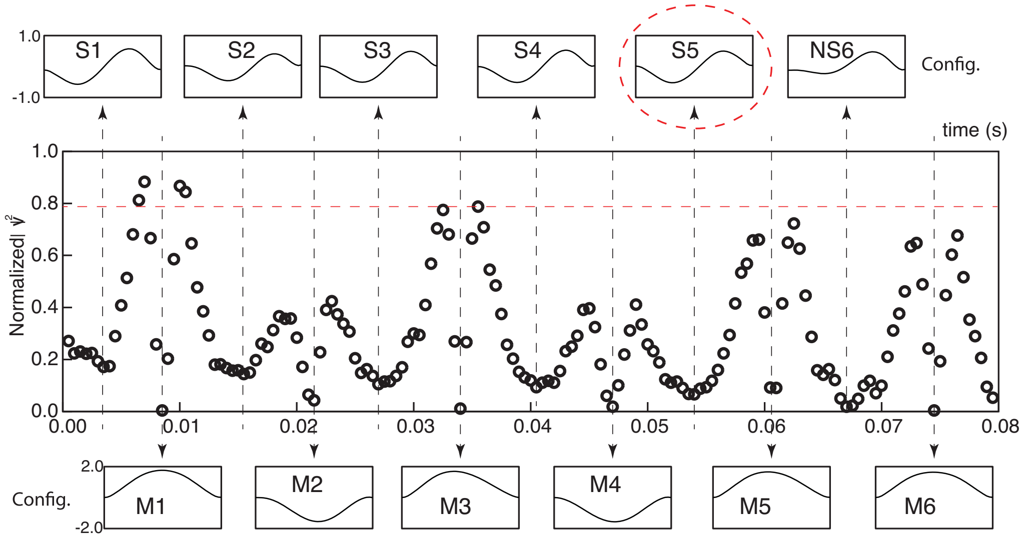


**Figure 5.** The normalized velocity time series corresponding to a trajectory resulting from a single hammer hit.

present some very preliminary results here. The experimental approach is essentially the same as for the buckled beam. An initially (almost) flat panel is bent and then clamped such that two co-existing stable equilibria are present. Figure 6(a) shows an image of the main components of the

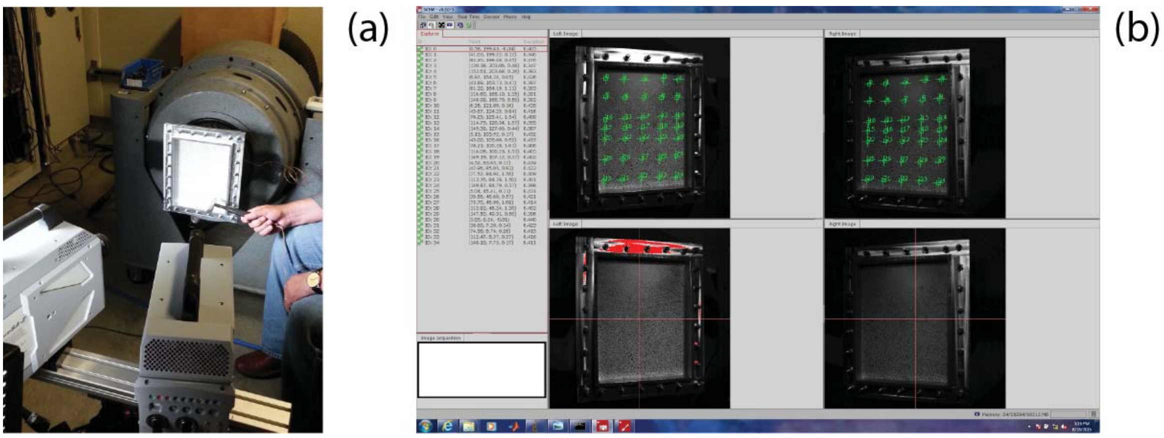


**Figure 6.** (a) the buckled panel clamped in position, with impact hammer, (b) discrete measurement points as seen by the cameras.

experimental set-up: clamped panel, impact hammer, and DIC measurement system. In similarity with the previous section, although the DIC provides full-field data, we choose a finite number of discrete locations: a $7 \times 5$ rectangular grid giving 35 measurement points (figure 6(b)), counting from 0 in the upper left corner. Four tests of about 30 seconds each were conducted. Tests #1-3 involved striking the panel in the bottom right, test #4 involved striking the panel near the upper right corner.

Figure 7(d) shows a typical time series obtained by repeated strikes from the hammer. Again the 'buckled-out' configuration is the fundamental equilibrium and we use this to zero the displacement. The measurement point for this time series is #17, i.e., the center of the panel. The 'buckled-in' configuration is approximately 1.6 mm from the other equilibrium at this location, and appears to involve lighter damping (a degree of asymmetry is inevitable in practice). Figure 7(a - c) shows

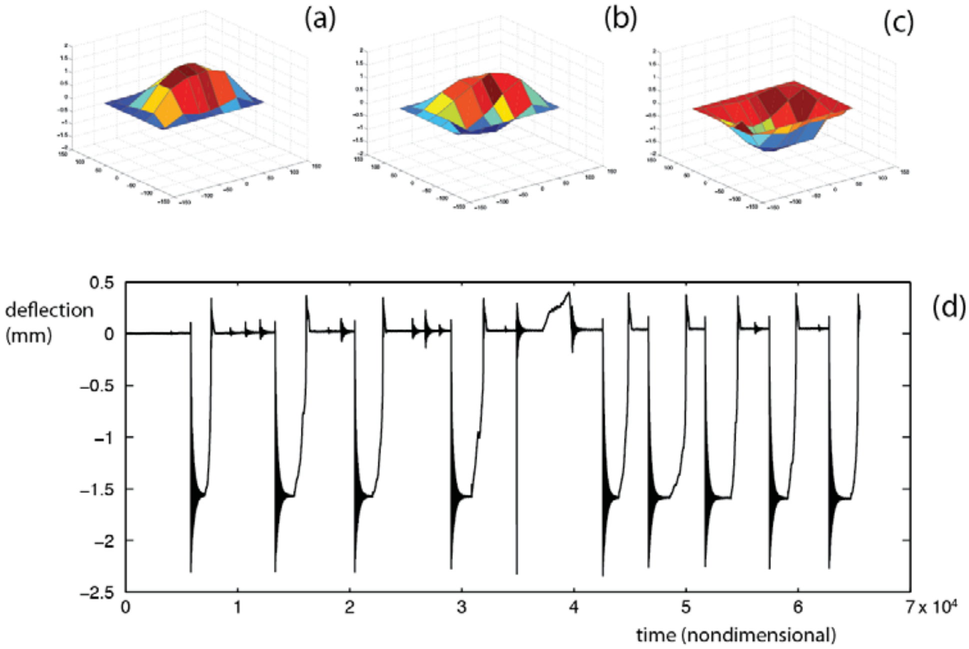


**Figure 7.** (a-c) snap-shots of the panel before, during, and after a snap-through event, (d) a time series showing repeated perturbations. Test #1, location #17.

data from all 35 measurements points at specific time instants: before, during, and after the snap-through event. Unlike for the beam, the points were not fit to give a smooth function. In this figure the time axis is in terms of sampling rate (2000 Hz). The snap-shot shown in part (b) is very close to the snap-through event in which the trajectory slows down.

A close-up view of a typical transient initiated by a hammer strike is shown in figure 8(a), and now the time scale is in seconds. The corresponding velocity is shown in part (b). This data came from the location point #15, i.e., closer to the center edge, and hence the smaller displacements. There isn't a clear slowing down of the trajectory during the snap-through process, as seen in the phase projection of part (c). This is indicative of an impact strike that it too large. The lightly damped nature of the system is again clearly evident by the number of oscillations it takes to decay to equilibrium. For other snap-through events there is a clearer slowing down effect, and these will be scrutinized in much the same way as for the beam in the previous section. Multiple sets of data were generated for the 35 spatial measurement locations, and will be analyzed for the presence of unstable equilibrium configurations.

## 4. Conclusions

This paper has described some initial research concerned with using transient dynamic data to estimate the location (shapes) of unstable equilibria in nonlinear structural systems in which multiple equilibria exist, for example, in post-buckled structures. Trajectories initiated by a sudden

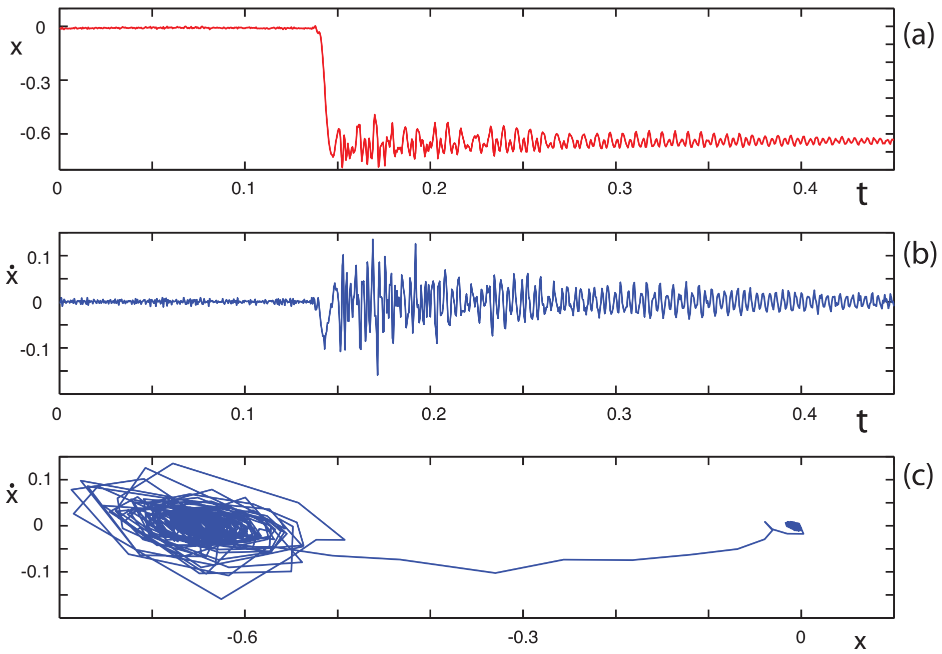

**Figure 8.** The response of the panel resulting from a single hammer hit, (a) position time series, (b) corresponding velocity time series, (c) phase projection. Note - test #2, location point 15.

input of energy may escape the confines of the local potential energy minimum and pass through to an adjacent minima (if one exists). A small percentage of these arbitrary disturbances will initiate transients that slow down as they pass an unstable equilibrium (a saddle point) since this will typically be the energetically advantageous path. The location of unstable equilibrium paths is useful information in the large deflection analysis of nonlinear structures.

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