Governing equations of multi-component rigid body-spring discrete element models of reinforced concrete columns

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Abstract. During the past decades, the complexity of conventional methods to perform seismic performance assessment of buildings led to the development of more effective approaches. The rigid body spring-discrete element method (RBS-DEM) is one of these approaches and has recently been applied to the study of the behavior of reinforced concrete (RC) buildings subjected to strong earthquakes. In this paper, the governing equations of RBS-DEM planar elements subjected to lateral loads and horizontal ground motion are presented and used to replicate the hysteretic behavior of experimental RC columns. The RBS-DEM models of columns are made up of rigid components connected by systems of springs that simulate axial, shear, and bending behavior of an RC section. The parameters of springs were obtained using Response-2000 software and the hysteretic response of the models of select columns from the Pacific Earthquake Engineering Research (PEER) Structural Performance Database were computed numerically. Numerical examples show that one-component models were able to simulate the initial stiffness reasonably, while the displacement capacity of actual columns undergoing large displacements were underestimated.

1. Introduction

The destruction brought about by earthquakes leads not only to economic loss, but also severe injuries and loss of lives due to collapse of buildings and civil infrastructure. In the Philippines, reinforced concrete (RC) structures are the most at risk due to the prevalent use of reinforced concrete as a building material. In order to prevent the collapse of these structures, their performance to strong ground shaking must be studied so as to improve the proposed design or to provide a retrofit scheme for existing structures.

During the past decades, the complexity of conventional methods such as the finite element method (FEM) to perform seismic performance assessment of buildings led to the development of more effective approaches. Most of these approaches are based on the discrete element method (DEM) originally proposed by Cundall [1] and the rigid body spring method (RBSM) proposed by Kawai [2]. DEM and RBSM models of structures make use of assembly of rigid elements connected by springs and are reported to be suitable for simulating the dynamic response of a structure up to collapse [3-6]. Compared with FEM, these methods are considered simpler and they allow separation of the components that make up the structure and, therefore, are capable of simulating the progressive collapse mechanisms [6-7].

The rigid body spring-discrete element method (RBS-DEM) is one of these approaches that have recently been applied to the study of the behavior of RC buildings subjected to strong earthquakes [8-12]. This method is based on the plastic hinge concept and the observed typical failure mechanism of beams and columns, showing large rotations at the ends and relatively small rotations in the middle of the elements. In the RBS-DEM, a structure is disintegrated into a number of rigid body elements connected by spring systems between them. The deformation of the structure is described by the centroidal displacements and rotations of the spring-connected rigid bodies, converting the problem of structural analysis into a problem of rigid body mechanics. For structures subjected to strong ground motions or loaded up to failure, material nonlinear behaviour of the elements is simulated using the nonlinear springs at the pre-identified possible location of plastic hinges in the element.

Time-stepping methods have been used to obtain the response of RBS-DEM models and results have been validated using available experiments [8-10, 12]. Although it was shown that the RBS-DEM yielded reasonable results, the development of alternative methods of validation and/or verification of results is also important.

This paper presents the governing equations of RBS-DEM models that may be used to obtain the dynamic properties and response to pseudo-static loads, and to analyze the hysteretic behavior of experimental RC columns. The response of RBS-DEM models of planar columns calculated using the analytical formulation will be compared to experimental data available in the Pacific Earthquake Engineering Research (PEER) Structural Performance Database [13].

2. Model formulation

2.1. Multi-component model

In the rigid body spring-discrete element method (RBS-DEM), a vertical column is modeled using a number, N, of rigid body components connected by spring systems that simulate the axial, shear, and flexural capacity as shown in figure 1. In plane problems, if vertical deformations are neglected and small rotations are assumed, the total degrees of freedom of the multi-component model shown will be two times the number of rigid components, i.e., the displacements of each n-th component may be derived from its relative horizontal displacement, u_n, and absolute rotation, θ_n, and the generalized displacements of components below it (see figure 1).

When the column is subjected to horizontal and vertical concentrated forces acting at the top, P_x and P_y, the application of d'Alembert's principle to the dynamic equilibrium yields the expressions for the internal shear force, V_n, and bending moment, M_n, at the base of the component as:

$$V_n + \sum_{i=n}^{N}\left[m_i \sum_{j=1}^{i}\left(\tilde{l}_{ij}\ddot{\theta}_j + \ddot{u}_j\right)\right] = P_x(t) - P_y(t)\theta_{n-1} \tag{1}$$

$$M_n + \sum_{i=n}^{N}\left\{I_i\ddot{\theta}_i + \sum_{j=i}^{N}\tilde{l}_{ji}\left[m_j \sum_{k=1}^{j}\left(\tilde{l}_{jk}\ddot{\theta}_k + \ddot{u}_k\right)\right]\right\} = \sum_{i=n}^{N}\left\{P_x(t)l_i - P_y(t)l_i\theta_i\right\} \tag{2}$$

where $\tilde{l}_{ji} = \begin{cases} \frac{l_i}{2}\ if\ i = j \\ l_i\ if\ i \neq j \end{cases}$, and l_i , m_i, $I_i = m_i l_i^3\ /12$ respectively are the length, mass, and moment of inertia of the ith component.

The internal forces V_n and M_n are also the corresponding nonlinear restoring forces of the springs f_{Vn} and f_{Mn}. The $2N$ coupled differential equations (1)-(2) may be written in matrix form as:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{Bmatrix} \ddot{\boldsymbol{u}} \\ \ddot{\boldsymbol{\theta}} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{f_V}(\boldsymbol{u}) \\ \boldsymbol{f_M}(\boldsymbol{\theta}) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{M} \end{Bmatrix} \tag{3}$$

where the corresponding elements of the sub-matrices are

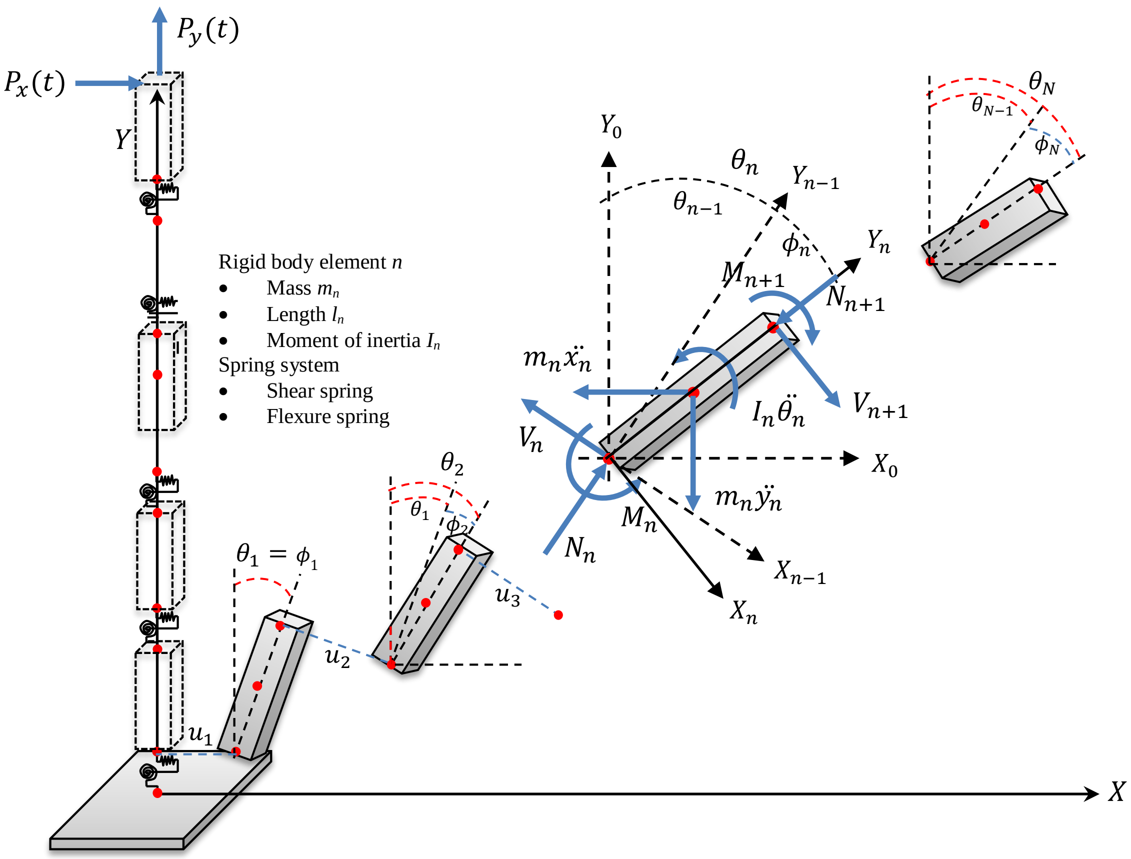

Figure 1. Multi-component RBS-DEM model of a column: discretization, degrees of freedom and dynamic equilibrium of the *n*th element

$$a_{ij} = \begin{cases} \sum_{k=i}^{N} m_k \,, & i \geq j \\ \sum_{k=j}^{N} m_k \,, & i < j \end{cases} \qquad b_{ij} = \begin{cases} \sum_{k=i}^{N} m_k \tilde{l}_{kj} \,, & i \geq j \\ \sum_{k=j}^{N} m_k \tilde{l}_{kj} \,, & i < j \end{cases}$$

$$c_{ij} = \begin{cases} \sum_{h=i}^{N}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh} \,, & i \geq j \\ \sum_{h=i}^{j-1}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh} + \sum_{h=i}^{N}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh} \,, & i < j \end{cases} \qquad d_{ij} = \begin{cases} \sum_{h=i}^{N}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh}\tilde{l}_{kj} \,, & i > j \\ I_j + \sum_{h=i}^{N}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh}\tilde{l}_{kj} \,, & i = j \\ I_j + \sum_{h=i}^{j-1}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh}\tilde{l}_{kj} + \sum_{h=i}^{N}\sum_{k=j}^{N} m_k \,\tilde{l}_{kh}\tilde{l}_{kj} \,, & i < j \end{cases}$$

$$f_{V_i} = f_V(u_i) \qquad p_i = P_x(t) - P_y(t)\theta_{i-1}$$

$$f_{M_i} = f_M(\theta_i) \qquad m_i = \sum_{i=n}^{N} P_x(t) l_i - P_y(t) l_i \theta_i$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$.

When the column is modeled using only one rigid component and vertical deformations are neglected, the equation governing the response under the assumptions of large and small rotations may be written as in equation (4) and equation (5), respectively

$$\begin{bmatrix} m & \frac{ml}{2}cos\theta \\ \frac{ml}{2}cos\theta & I+\frac{ml^2}{4} \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix}+\begin{Bmatrix} f_V(u)-\frac{1}{2}ml\dot{\theta}^2 sin\theta \\ f_M(\theta)-\frac{1}{4}ml\dot{\theta}^2 sin\theta cos\theta \end{Bmatrix}=\begin{Bmatrix} P_x(t)-m\ddot{x}_g \\ P_x(t)l\,cos\theta-\frac{1}{2}m\ddot{x}_g lcos\theta \end{Bmatrix} \quad (4)$$

$$\begin{bmatrix} m & \frac{ml}{2} \\ \frac{ml}{2} & I+\frac{ml^2}{4} \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} k_V & 0 \\ 0 & k_M \end{bmatrix}\begin{Bmatrix} u \\ \theta \end{Bmatrix}=\begin{Bmatrix} P_x(t)-m\ddot{x}_g \\ P_x(t)l-\frac{1}{2}m\ddot{x}_g l \end{Bmatrix} \quad (5)$$

where k_V and k_M are the elastic stiffness constants of the shear and flexural springs, respectively.

2.2. *Solution method*
With the above governing equations, in the assumption of small rotations and linear springs, the closed-form exact solution for the response of the model when subjected to, for example, harmonic loads may be obtained and dynamic properties such as frequencies and mode shapes may also be easily derived. For example, the natural frequency of vibration of the single-component model (in equation 5) may be expressed as

$$\omega^2=\frac{k_M}{2l}\left[1+\kappa\pm\sqrt{1+\kappa+\kappa^2}\right] \quad (6)$$

where $\kappa=l^2k_V/3k_M$ is the stiffness ratio. In the general analysis of RC columns subjected to pseudo-dynamic loads, however, a numerical integration scheme is required. We propose to use a time-stepping classical method such as the fourth-order Runge-Kutta method (RK4) to solve equations (3)-(5).

3. Model parameters for reinforced concrete columns and experimental validation

3.1. *Nonlinear spring parameters*
In order to simulate the hysteretic behavior of RC columns, nonlinear shear and flexural (or rotational) springs have to be used. In this paper, the spring parameters are estimated using the moment-curvature and shear force-strain diagrams of the RC section obtained using the software Response-2000 [14]. The cross-section parameters were inputted, then various combination of linear curves were fitted onto the resulting shear-shear strain and moment-curvature diagrams.

As shown in figure 2 for a sample column section, the bilinear curve was fitted to the shear-strain diagram and a trilinear curve to the moment-curvature diagram. The bilinear curve will be the backbone curve of the bilinear shear restoring force-displacement hysteresis model of shear springs and the trilinear curve for the Takeda [15] hysteresis model used for the flexural spring.

The spring parameters to be estimated and used for the RBS-DEM model of RC columns are shown in table 1. The control points that signify cracking, yielding, and failure of a given column section are estimated as shown in figure 2 and the other parameters are estimated for the hysteresis models used. For the flexural spring, the unloading stiffness degradation factors B_0 and B_1 dictates the fatness of the hysteresis loop of the Takeda model. A value of 0.4 is assumed for parameter B_0 since it cannot be determined by using the geometric and material properties of the RC column [16]. The degradation parameter B_1 that describes the loss of rigidity of a structure during cyclic loading is estimated by taking the ratio of K_r (given in equation (7)) with the initial stiffness of the structure [16].

$$K_r=\frac{F_c+F_y}{D_c+D_y}\left|\frac{D_m}{D_y}\right|^{-0.4} \quad (7)$$

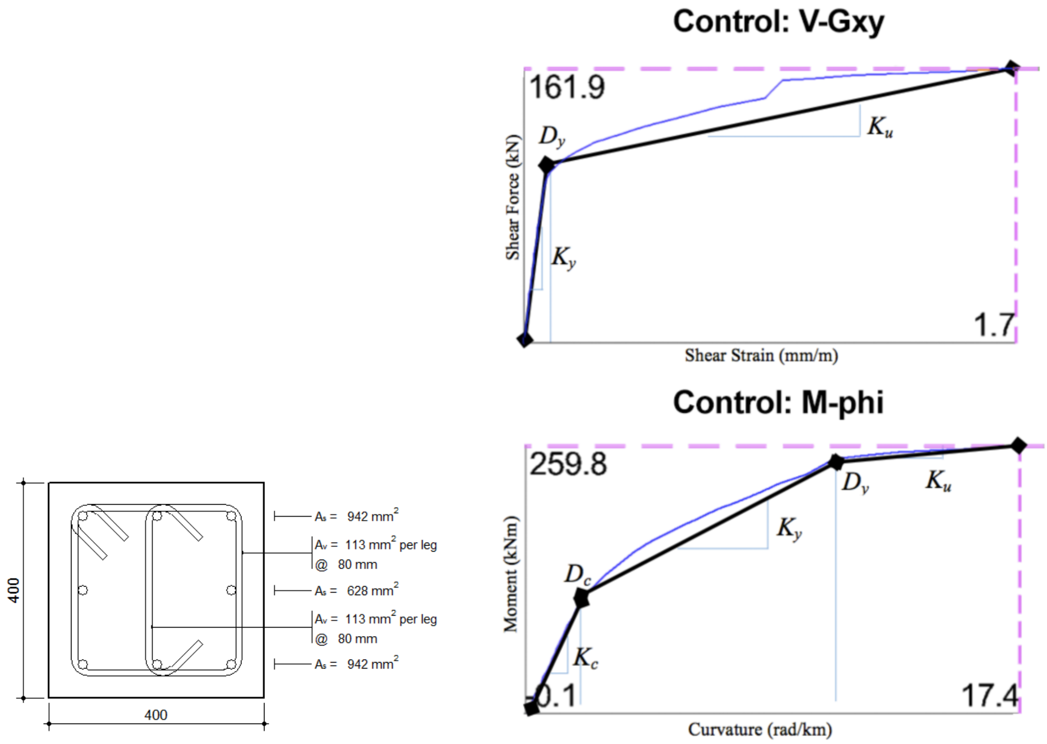

Figure 2. Derivation of the nonlinear spring parameters of RBS-DEM model from the shear-strain and moment-curvature diagrams of the RC section

Table 1. Nonlinear spring parameters required for RBS-DEM model of RC columns

	Parameters
Shear spring	*Ky* (N/m) Initial stiffness *Dy* (m) Yield point shear displacement *Ku* (N/m) Post-yielding stiffness
Flexural spring	*Kc* (Nm/θ) Initial stiffness *Dc* (θ) Cracking angle *Ky* (Nm/θ) Cracked stiffness *Dy* (θ) Yield point rotation angle *Ku* (Nm/θ) Post-yielding stiffness B_0 Degradation parameter B_1 Degradation parameter

3.2. Pseudo-dynamic loading of RC columns
In order to validate the proposed RBS-DEM model, models of actual RC columns subjected to pseudo-dynamic loads will be used and the load-displacement curves will be compared with experimental results available in the PEER Structural Performance Database [13]. In the experiment, RC columns that support constant axial loads are subjected to cyclic lateral loads. Four columns were selected and their section and material properties are listed in table 2.

Table 2. Properties of select RC columns from PEER database

	Properties	Soesianawati No 1	Tanaka & Park No 1	Tanaka & Park No 5	Zahn No 7
	Span-depth ratio	4	4	3	4
	Length (mm)	1600	1600	1650	1600
	Axial load (kN)	744	819	968	1010
	Concrete strength (MPa)	46.5	25.6	32	28.3
	Width (mm)	400	400	550	400
	Depth (mm)	400	400	550	400
	Concrete cover (mm)	13	40	40	13
Longitudinal Reinforcement	#	12	8	12	12
	Diameter (mm)	16	20	20	16
	Yield stress (MPa)	446	474	511	440
	Ultimate stress (MPa)	702	721	675	674
	Area (mm^2)	201.06	314.16	314.16	201.06
Transverse Reinforcement	Yield stress (MPa)	364	333	325	466
	Ultimate stress (MPa)	521	481	429	688
	Diameter (mm)	7	12	12	10
	s (mm)	85	80	110	117
	Area (mm^2)	38.48	113.10	113.10	78.54
	Stirrups Type	Rec.	Rec. w/ jhook	Rec. w/ jhook	Rec.
	Shear Legs	4	3	4	4

(The four specimen columns are grouped under the heading "Specimen".)

Table 3. Computed RBS-DEM spring parameters of selected RC columns using Response-2000

			Soesianawati No 1	Tanaka & Park No 1	Tanaka & Park No 5	Zahn No 7
Spring Parameters	Shear	K_y (N/m)	7.0387E+08	6.3089E+08	9.6302E+08	6.2110E+08
		D_y (m)	1.2960E-04	1.3280E-04	1.7325E-04	1.6800E-04
		K_u (N/m)	8.3185E+06	2.5114E+07	4.0628E+07	6.3231E+06
	Flexure	K_c $(\frac{Nm}{\theta})$	2.5428E+08	1.7426E+08	6.2258E+08	2.1371E+08
		D_c (θ)	3.1891E-04	4.5449E-04	2.4423E-04	4.4191E-04
		K_y $(\frac{Nm}{\theta})$	6.2937E+07	4.7797E+07	1.3001E+08	6.2188E+07
		D_y (θ)	2.7183E-03	3.7852E-03	2.5311E-03	2.6367E-03
		K_u $(\frac{Nm}{\theta})$	7.5772E+06	4.5919E+06	3.7331E+07	1.4052E+07
		B_0	0.40	0.40	0.40	0.40
		B_1	0.22	0.42	0.68	0.15

(The four specimen columns are grouped under the heading "Specimen".)

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The experimental results for the four selected columns are shown as broken gray lines in figure 3, where the horizontal applied load is plotted against the displacement of the column top. The one-component RBS-DEM models of the same columns with the spring parameters tabulated in table 3 were subjected to the same loading history and the response were computed by solving equation (4) using RK4. The corresponding nonlinear parameters of the springs are computed from the section properties listed in table 2 and by using Response-2000. The load-deformation curves are shown in figure 3 as solid black lines.

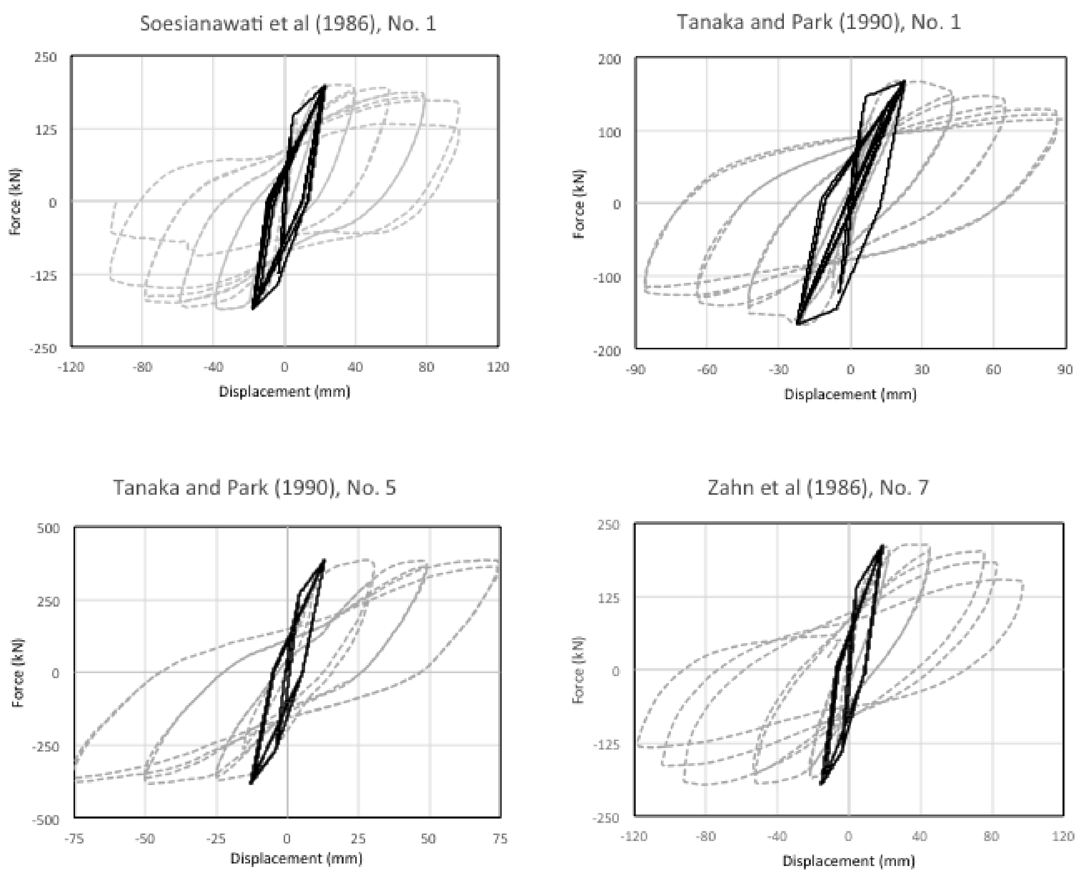

Figure 3. Force-displacement plots of select experimental RC columns (shown as broken gray lines) and RBS-DEM models (shown as solid black lines).

A general trend may be observed from the load-deformation plots of the RBS-DEM models of the selected RC columns. The column models were able to simulate to some extent the initial behaviour of the columns but fail to cover the entire force-displacement plot. The result is expected since the models lumped the elasticity using a spring system located only at the base of the columns.

4. Concluding remarks

The governing equations of multi-component rigid body spring-discrete element models of reinforced concrete columns subjected to lateral loads were presented. The $2N$ coupled ODEs in terms of shear and flexural deformations of the springs of the model can be useful to study the hysteretic behavior of actual columns and to validate nonlinear time-history analyses of RC frames.

Response of a single component RBS-DEM model of a RC column was compared with experimental results from the PEER database. Nonlinearity of the analytical model were evaluated using Response-2000 software. The bilinear and Takeda models were used to account for the hysteretic behaviour of RC under shear and flexural restoring forces, respectively. Examples show that the model is able to simulate the elastic stiffness, but underestimates the column displacement capacity due to lumped stiffness at the base of the column. The response of the column model may be improved by investigating the use of multi-component RBS-DEM models taking into account vertical displacement, and the use of more appropriate hysteresis rules for the effective shear and flexural springs to be used. Future work will also investigate the use of energy methods to derive governing equations in terms of relative inter-component rotations to obtain symmetric mass and stiffness matrices.

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