

Two dimension magnetotelluric modeling using finite element method, incomplete lu preconditioner and biconjugate gradient stabilized technique

Muhammad Zukir¹ and Wahyu Srigutomo²

¹Student of Physics Departments, Bandung Institute of Technology, Indonesia

²Lecturer of Physics Departments, Bandung Institute of Technology, Indonesia

Email: zukir.m1@gmail.com

Abstract. Magnetotelluric (MT) method is a passive geophysical exploration technique utilizing natural electromagnetic source to obtain variation of the electric field and magnetic field on the surface of the earth. The frequency range used in this modeling is 10^{-4} Hz to 10^2 Hz. The two-dimensional (2D) magnetotelluric modeling is aimed to determine the value of electromagnetic field in the earth, the apparent resistivity, and the impedance phase. The relation between the geometrical and physical parameters used are governed by the Maxwell's equations. These equations are used in the case of Transverse Electric polarization (TE) and Transverse Magnetic polarization (TM). To calculate the solutions of electric and magnetic fields in the entire domain, the modeling domain is discretized into smaller elements using the finite element method, whereas the assembled matrix of equation system is solved using the Biconjugate Gradient Stabilized (BiCGStab) technique combined with the Incomplete Lower – Upper (*ILU*) preconditioner. This scheme can minimize the iteration process (computational cost) and is more effective than the Biconjugate Gradient (BiCG) technique with *LU* preconditions and Conjugate Gradient Square (CGS).

1. Introduction

Magnetotelluric method (MT) is a passive geophysical exploration method that utilizes natural electromagnetic fields generated by the solar wind and lightning activities. This method has wide spectrum of geomagnetic variations produced by magnetic induction from electric current between ionosphere and earth's magnetic field as well as electrical storm activity in the atmosphere. MT method is widely used to determine the electrical properties of rocks beneath the surface of earth at a relatively great depth (including the mantle). MT numerical modeling is easier to implement using finite difference method (FDM) than that of the finite element and integral equation methods ^[4]. However, compared to the former, the finite element method yield solutions with better accuracy. In this study we tried to compare the level of finite element solution errors for three different approaches in solving the MT forward problems: BiCGStab technique combined with *ILU* preconditioner; Biconjugate Gradient (BiCG) technique combined with *LU* preconditioner; and Conjugate Gradient Squared (CGS) technique.

2. Problem formulation

All electromagnetic phenomena can be solved using Maxwell's equations [5]. By using these equations, The electric field (\vec{E}) and magnetic field (\vec{H}) can be expressed in form of the Helmholtz equations,

$$\vec{\nabla}^2 \vec{E} + k^2 \vec{E} = 0 \quad (1)$$



$$\vec{\nabla}^2 \vec{H} + k^2 \vec{H} = 0 \quad (2)$$

$$k^2 = (\omega^2 \mu \varepsilon + i \omega \mu \sigma) \quad (3)$$

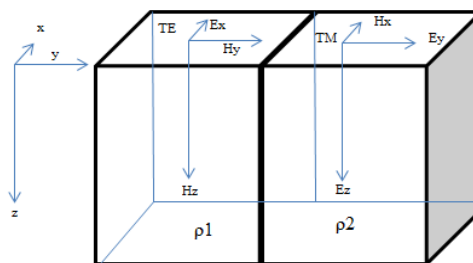
When the conduction current is much greater than the displacement current ($\sigma \gg \varepsilon \omega$), the components containing ε can be ignored, this is because the frequencies used in geophysical exploration is commonly lower than 10 Hz (consequently, $\omega \mu \sigma \gg \omega \mu \varepsilon$). Skin depth (δ) is the depth of electromagnetic wave decay in the subsurface when the amplitude becomes e^{-1} . The expression of skin depth can be written as

$$\delta = 503 \sqrt{\frac{\rho}{f}} \quad (4)$$

and it is controlled mainly by the resistivity of rocks and the operating frequency used.

3. Two Dimension magnetotelluric method

In 2D modeling, conductivity varies along the z -axis and one of the horizontal axis either the x -axis or y -axis, which is perpendicular to the strike. Magnetotelluric fields are divided into two polarization modes, Transverse Magnetic (TM) mode and Transverse Electric (TE) mode. TM and TE polarization modes are generally known as \vec{H} polarization (magnetic field polarized along strike of model) and \vec{E} polarization (electric field polarized along strike of model).



Picture 1. Illustration of TE and TM polarization modes in magnetotelluric method.

In 2D case, the formulations start from the Helmholtz equations. In TM polarization, the value of $\vec{H}_y = 0$ and $\vec{H}_z = 0$. The electric field components can be expressed as

$$\vec{E}_y = \frac{1}{\sigma} \frac{\partial \vec{H}_x}{\partial z}, \quad \vec{E}_z = -\frac{1}{\sigma} \frac{\partial \vec{H}_x}{\partial y} \quad (5)$$

Whereas, in TE polarisation the value of $\vec{E}_y = 0$ and $\vec{E}_z = 0$, and hence the magnetic field components are expressed by

$$\vec{H}_y = \frac{1}{i \omega \mu_0} \frac{\partial \vec{E}_x}{\partial z}, \quad \vec{H}_z = -\frac{1}{i \omega \mu_0} \frac{\partial \vec{E}_x}{\partial y} \quad (6)$$

Equation (5) and (6) can be formulated in term of impedance tensor matrix as shown below

$$\vec{E} = \vec{Z}\vec{H} \quad (7)$$

$$\begin{bmatrix} \vec{E}_x \\ \vec{E}_y \end{bmatrix} = \begin{bmatrix} 0 & \vec{Z}_{xy} \\ \vec{Z}_{yx} & 0 \end{bmatrix} \begin{bmatrix} \vec{H}_x \\ \vec{H}_y \end{bmatrix} \quad (8)$$

and hence equation (6) and (8) can be expressed in term of apparent resistivity (ρ_a)

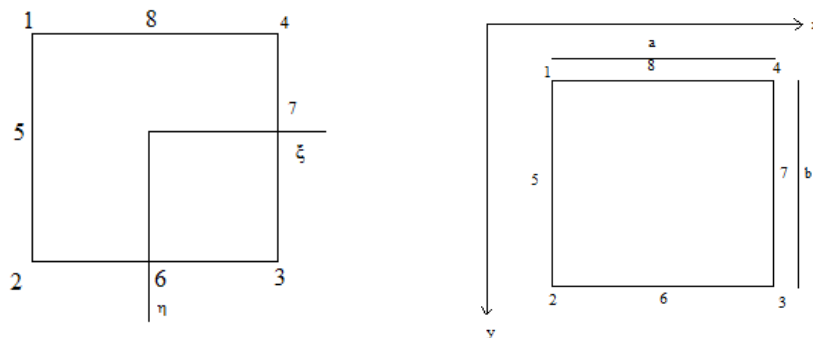
$$\rho_{xy} = \frac{1}{\omega\mu_0} |Z_{xy}|^2 = \frac{1}{\omega\mu_0} \left| \frac{\vec{E}_x}{\vec{H}_y} \right|^2, \rho_{yx} = \frac{1}{\omega\mu_0} |Z_{yx}|^2 = \frac{1}{\omega\mu_0} \left| \frac{\vec{E}_y}{\vec{H}_x} \right|^2 \quad (9)$$

and the impedance phase (ϕ)

$$\phi_{xy} = \tan^{-1} |Z_{xy}| = \tan^{-1} \left| \frac{\vec{E}_x}{\vec{H}_y} \right|, \phi_{yx} = \tan^{-1} |Z_{yx}| = \tan^{-1} \left| \frac{\vec{E}_y}{\vec{H}_x} \right| \quad (10)$$

4. Finite element method

In this study, the 2D modeling domain was discretized into square elements. Double quadratic interpolation unit aims to take eight points in each unit (4 knot). As shown below^[3]



Picture 2. Element unit of quadratic interpolation.

Based on Figure 2. can be formulated the equation as

$$x = x_0 + \frac{a}{2}\xi, \quad y = y_0 + \frac{b}{2}\eta \quad (11)$$

Where, x_0 and y_0 are the midpoint of the sub-units. Differential relationships between two units are:

$$dx = \frac{a}{2}\xi, \quad dy = \frac{b}{2}\eta, \quad dxdy = \frac{ab}{4}d\xi d\eta \quad (12)$$

The structures of the form functions are,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \quad N_2 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \quad N_4 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$N_5 = \frac{1}{2}(1-\eta^2)(1-\xi) \quad N_6 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_7 = \frac{1}{2}(1-\eta^2)(1+\xi), \quad N_8 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

Integration of elements is

$$\int_e \frac{1}{2} \tau (\nabla u)^2 d\Omega = \frac{1}{2} u_e^T K_{1e} u_e \quad (13)$$

By $K_{1e} = (k_{ij})$, $k_{ij} = k_{ji}$

$$k_{ij} = \int_e \tau \left[\left(\frac{dN_i}{d\xi} \frac{d\xi}{dx} \right) \left(\frac{dN_j}{d\xi} \frac{d\xi}{dx} \right) + \left(\frac{dN_i}{d\eta} \frac{d\eta}{dy} \right) \left(\frac{dN_j}{d\eta} \frac{d\eta}{dy} \right) \right] \frac{ab}{4} d\xi d\eta$$

Integral equation is

$$\int_e \frac{1}{2} \tau u^2 d\Omega = \frac{1}{2} u_e^T (k_{ij}) u_e = \frac{1}{2} u_e^T K_{2e} u_e \quad (14)$$

By, $K_{2e} = (k_{ij})$, $k_{ij} = k_{ji}$

$$k_{ij} = \int_e \lambda N_i N_j \frac{ab}{4} d\xi d\eta$$

Integral equation is,

$$\int_{ij} \frac{1}{2} \lambda u^2 d\Gamma = \frac{1}{2} u_e^T (k_{ij}) u_e = \frac{1}{2} u_e^T K_{3e} u_e \quad (15)$$

K_{1e} , K_{2e} , and K_{3e} are matrices for all nodes of all units. So that, they can be formulated by using the equation below,

$$F(u) = \sum F_e(u) = \sum \frac{1}{2} u_e^T (K_{1e} - K_{2e} + K_{3e}) u_e = \frac{1}{2} u^T \sum K_e u = \frac{1}{2} u^T K u \quad (16)$$

By, $K_e = K_{1e} - K_{2e} + K_{3e}$ $K = \sum K_e$

From the equation (16) can be applied the boundary conditions desired, and can be found the solutions for all values of u . When the partial derivatives u calculate the value for each node, it is possible to use the numerical method to find values along vertical cross-section $\frac{\partial u}{\partial z}$, by substituting into the formulation, the apparent resistivity and impedance phase can be calculated.

5. ILU preconditioner

An incomplete factorization replaces the search of triangle matrix L and U with the approach $A \approx LU$ rather than $A = LU$. Completion to $LU\mathbf{x} = \mathbf{b}$ can be solved faster but does not produce the exact solution for $A\mathbf{x} = \mathbf{b}$. So that, the matrix $M = LU$ is used as a preconditioner on the other iterations of algorithm solutions such as BiCGStab. Spread patterns of L and U are selected to be the same as spread patterns of the original matrix A . These are called *ILU* preconditioner. Error in *ILU* factorization of matrix A is defined by E . The formulation can be expressed as

$$A = LU + E$$

ILU Preconditioner factorization has been successful for many common cases that are non-symmetric and indefinite matrices^[2].

6. BiCGStab technique

There are several alternatives of Biconjugate Gradient (BiCG) that efficiently could be used in certain conditions. One is Biconjugate Gradient Stabilized (BiCGStab). BiCGStab is a technique that has ability to modify the matrix becomes convergent faster than BiCG (biconjugate gradient)^[1]. This method is also able to produce a smaller error. Suppose a matrix equation is formed to obtain the unknown value of unknown \vec{H} ,

$$\vec{E} = \vec{Z}\vec{H} \quad (17)$$

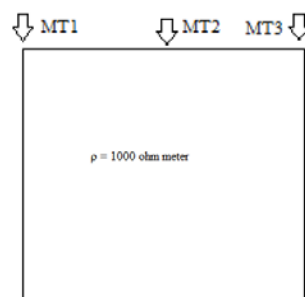
$$\mathbf{r}' = \vec{E} - \vec{Z} \bullet \mathbf{r}' \quad (18)$$

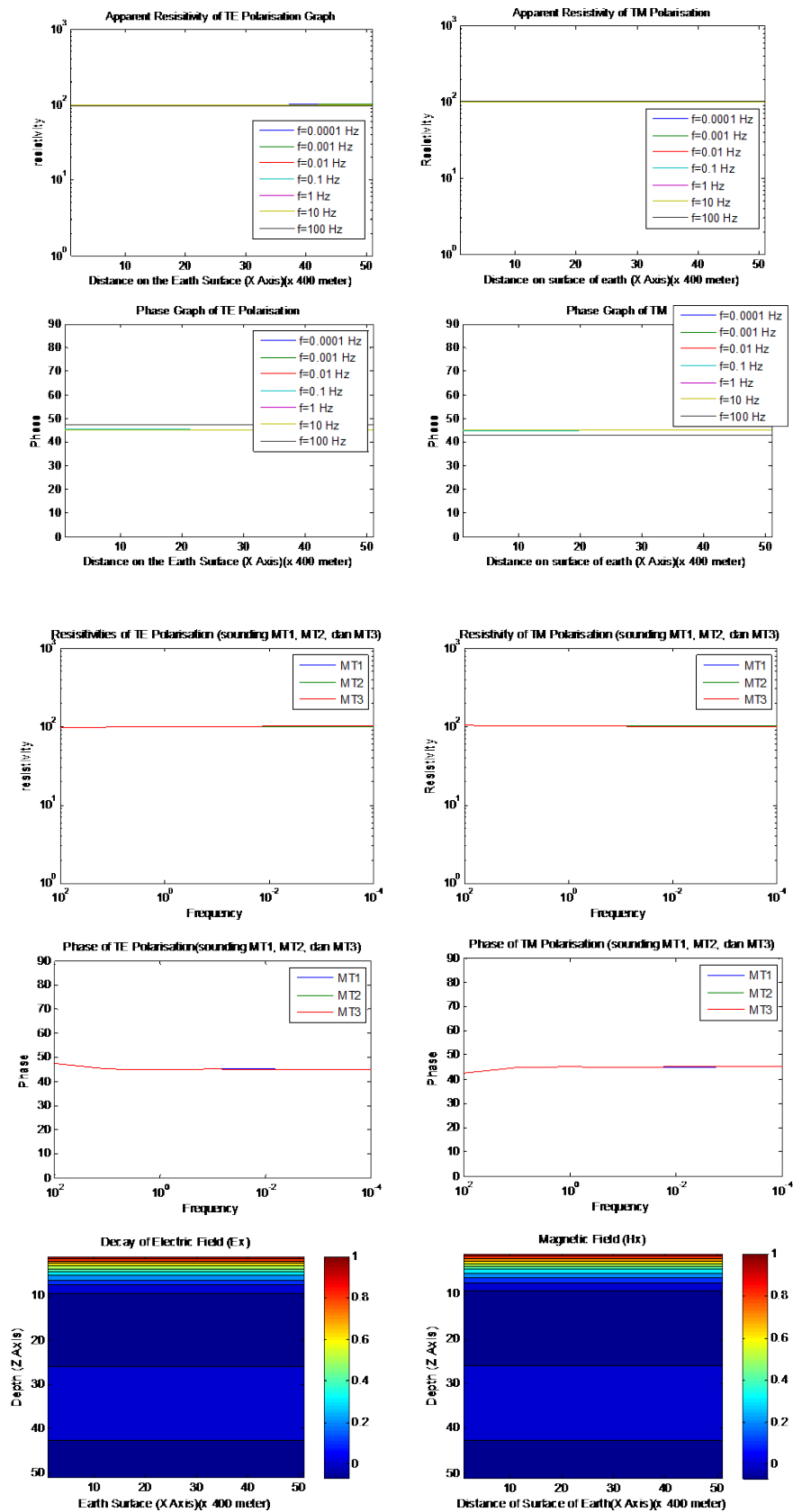
Where $\|\bullet\|$ indicates the norm vector. Stopping criteria of BiCGStab technique in this article is limited by the minimum error 10^{-15} . The residual error can be defined as

$$\text{Residual error} = \frac{\|\mathbf{r}'\|}{\|\mathbf{Z}\|} \quad (19)$$

7. Discussion

7.1. Modeling for homogeneous earth





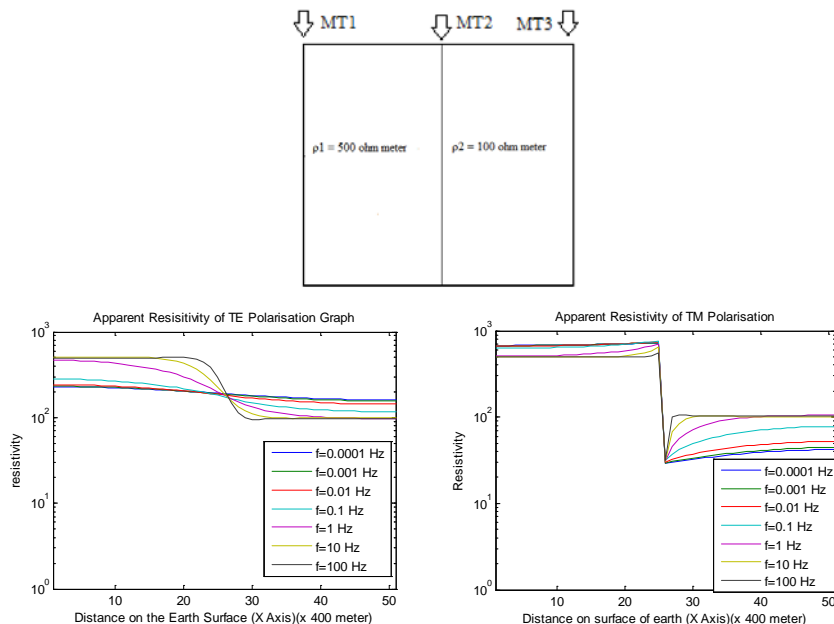
Picture 3. Modeling results for homogeneous earth

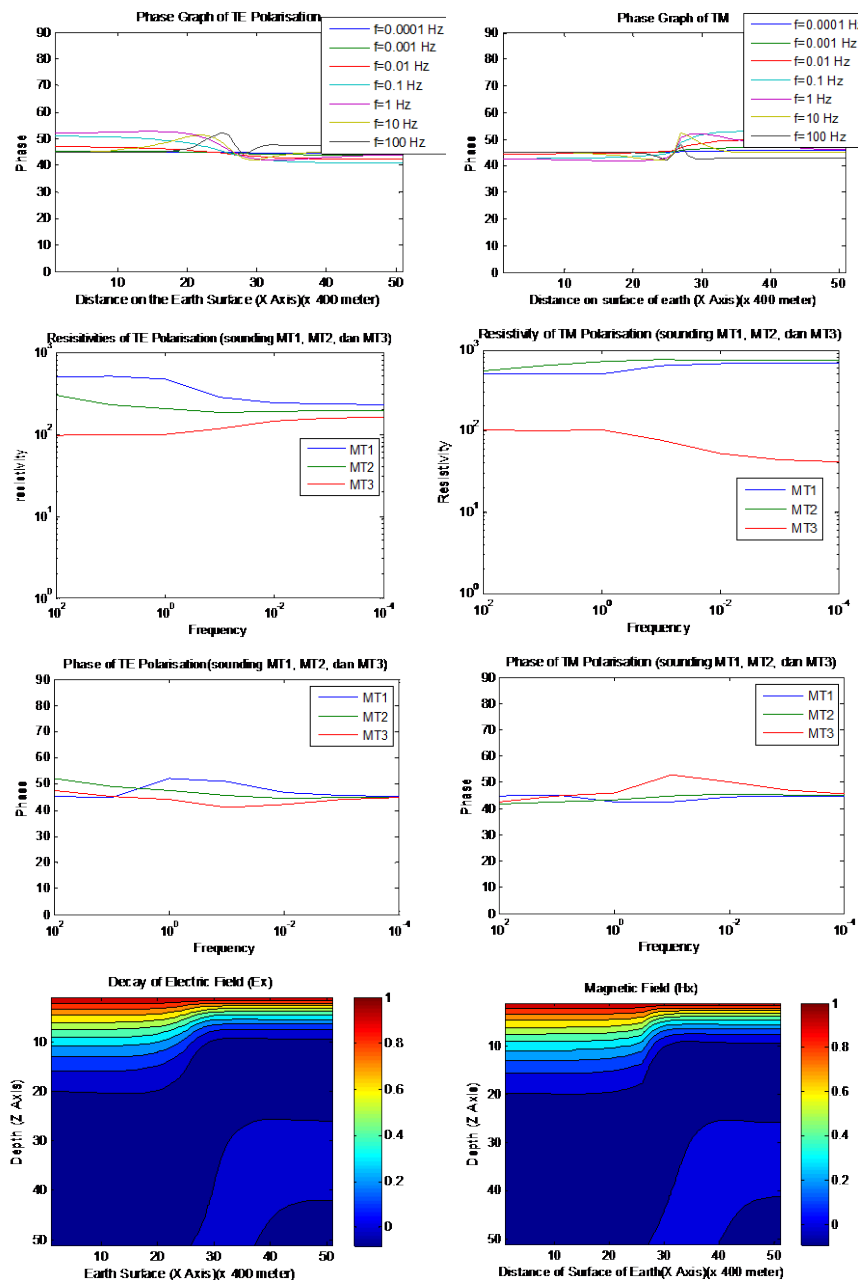
Apparent resistivities in this homogeneous earth for TE and TM polarisation modes for all frequencies indicate 100 ohm-meter, for a high frequencies show that the error is quite striking, this caused by skin depth that is more shallow. Phase for homogeneous layer is about 45° for all frequencies. The decay of the electric field and magnetic field looks homogeneous along each nodes in z -axis. So that, the homogeneous earth model can be seen more clearly. Resistivity for all frequencies in the sounding of MT1, MT2 and MT3 sounding is 100 ohm-meter, this illustrates the homogeneous layer of earth. Subsurface phase is 45° which indicates homogeneity of the earth. The high-frequency shows the deflection, the causes are the shallow skin depth and the influence of a half space condition.

Table 1. Comparison techniques combined by preconditioner in the case of homogeneous earth.

Methods	Residual error for TE polarisation (min-max)	Residual error for TM polarisation (min-max)
<i>ILU</i> +BiCGStab	6.8×10^{-17} - 4.6×10^{-16}	1.1×10^{-16} - 5.4×10^{-16}
LU+BiCG	3.8×10^{-15} - 9.9×10^{-15}	2×10^{-16} - 5.6×10^{-16}
CGS	4.9×10^{-15} - 1×10^{-14}	8.3×10^{-15} - 1×10^{-14}

7.2. Earth modeling for vertical contact





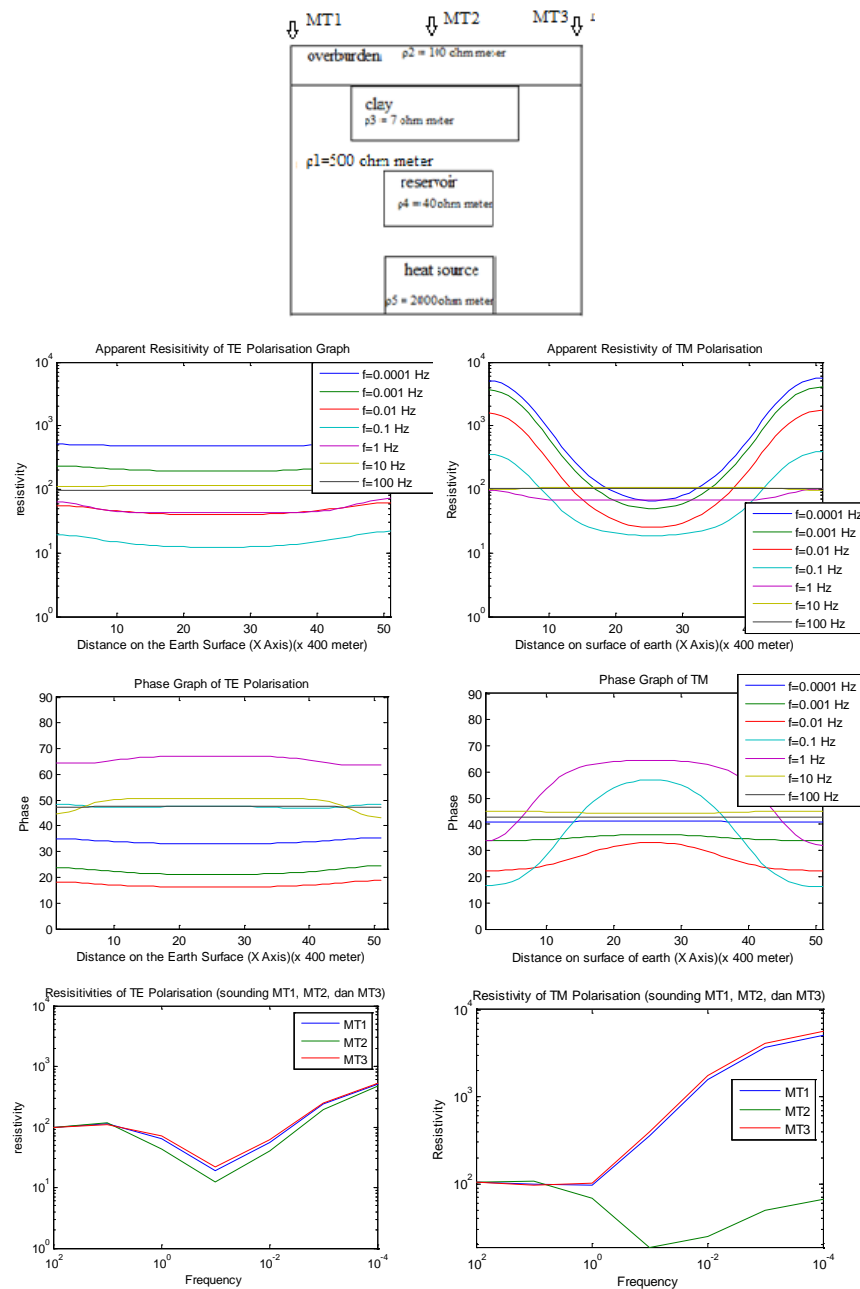
Picture 4. Modeling results for the earth vertical contact

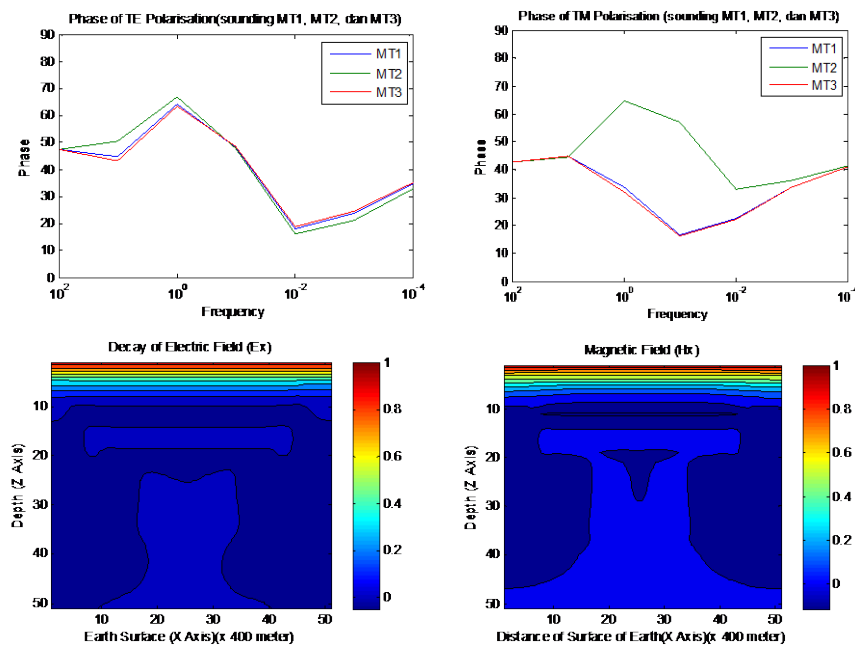
The modeling results show the apparent resistivity on the earth's surface for TE and TM polarisation modes, the resistivity change varies from a higher to a lower direction, these graphs provide a clear picture of the layers of the earth in the form of vertical contact. The decay of the magnetic field and electric field appear the difference in color, for the layer with high resistivity show that the magnetic field and electric field decay more slowly than the layer with a low resistivity. This is caused by the layer with low resistivity more tends to be capacitive. Resistivity and phase for all frequencies in sounding of MT1, MT2 and MT3 are affected by the value of the frequencies.

Table 2. Comparison techniques with preconditioner in the case of vertical contact Earth ($\rho_1 > \rho_2$).

Methods	Residual error for TE polarisation (min-max)	Residual error for TM polarisation (min-max)
<i>ILU</i> +BiCGStab	1.6×10^{-16} - 4.8×10^{-16}	1.8×10^{-16} - 4.7×10^{-16}
LU+BiCG	4.2×10^{-15} - 8.2×10^{-15}	1.8×10^{-16} - 4.7×10^{-16}
CGS	4.6×10^{-15} - 9.6×10^{-15}	8×10^{-15} - 9.9×10^{-15}

7.3. Modeling of Heat Source Case





Picture 5. Modeling results in the case of heat source

One case in geophysics is heat source, namely the existing sources of hot rock in subsurface. The apparent resistivity of TM polarisation is clearly significant in changes, it describes the position of heat source below the surface of earth. Phase that change in the TE and TM polarisation modes indicates the other layers under the earth's surface. The decay of the magnetic field and electric field gives a more real picture of the subsurface conditions, it seems the overburden layer, a layer of clay, reservoir, and a heat source. Resistivity and phase for all frequencies on the sounding of MT1, MT2 and MT3. frequency domain provides an overview of the distribution of the layers that exist below the surface of the earth.

Table 3. Comparison technique with preconditioner in heat source case.

Metode	Residual Error modulus TE min-max	Residual Error modulus TM min-max
<i>ILU</i> +BiCGStab	2.6×10^{-16} - 4.2×10^{-16}	2×10^{-16} - 5.6×10^{-16}
LU+BiCG	5.3×10^{-15} - 9.3×10^{-15}	2.7×10^{-16} - 6.1×10^{-65}
CGS	3×10^{-15} - 1×10^{-13}	5.6×10^{-12} - 8.8×10^{-12}

8. Conclusion

Conclusions of 2D magnetotelluric modeling using finite element with *ILU* preconditioner and BiCGStab technique are BiCGStab technique with *ILU* preconditioner is good enough to be used for modeling when compared with BiCG technique with preconditioner LU and CGS technique. And, finite element method with preconditioner *ILU* and BiCGStab technique have responses that almost corresponds to the actual conditions. These responses can be observed in the relation between apparent resistivity and phase against distance, resistivity and phase against the frequency, as well as the shape of the decay of the magnetic field and electric field in various cases.

9. Acknowledgments

The author would like to thanks Wahyu Srigutomo, M.Si, Ph.D for the guidances and the advices, so I can finish this modeling. Thanks are also due to the members of modelling and complex system laboratory.

10. References

- [1] Babaoğlu, Barış. 2003. *Application of biconjugate Gradient Stabilized Method with Spectral Acceleration for Propagation over terrain Profiles*. Bilkent University.
- [2] Chow, Edmond. 1997. Experimentaly study of preconditioner for indefinite matrices. USA: university if Minnesota.
- [3] Jian-xing, LIU. 2009. *Aplication of BICGSTab Algorithm with Incomplete LU Decomposition Precondetioning to Two-Dimentional Magnetotelluric Forward Modeling*. Changsha: Central South University.
- [4] Kumar, Krishna. 2011. *Efficient two-dimensional magnetotellurics modeling using implicitly restarted Lanczos method*. India: Indian Institute of Technology Roorkee.
- [5] Nabighian, M. N. 1981. *Extensions of the magnetometric resistivity (MMR) method*. Geophysics.
- [6] Simpson, Fiona. 2005. *Praktical Magnetotellurics*. UK: Cambridge Unversity Press.