Numerical Approach of Hamilton Equations on Double Pendulum Motion with Axial Forcing Constraint

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Abstract. Double pendulum with axial forcing constraint is considered by using Hamilton equations. In this case, the total Hamiltonian is complicated because of its constraint. There is additional terms which is add to the usual Hamiltonian. Four equations of motion is obtained from the Hamilton equations since the degree of freedom is four. Solutions of the equations are solved numerically by Runge-Kutta method. The results are plotted in poincare maps. In this case, the maps is displayed in various initial value. The chaotic behavior can be indicated which depends on given time function forcing constraint.

1. Introduction

The purpose of this article is to solve Hamilton equations on constrained double pendulum motion. This was initialized by given usual Hamiltonian on double pendulum system after the generalized coordinates and momenta had been determined. Total Hamiltonian on constrained system was constructed by axial forcing given on second pendulum. The equations of motion were decomposed from thus Hamiltonian. They were solved by numerical approach. The Runge-Kutta method which is finite difference method was used to solve the differential equations of the equations motion. This method is sophisticated approach in various case. The results of this approach were possed in poincare maps. The chaotic motion was seen clearly from the map since the constraint was time function.

The double pendulum system had been considered by [1] in fractional order Langrange equations. This consideration was about constrained system given on second pendulum. In few years ago, [2] introduced the equations of motion on constrained Hamiltonian system. It will be interesting to try to solve constrained pendulum on [1] using equation of motion on [2]. The equations of motion of double pendulum system had been decompossed by [3, 4] without constraint. They include differential equations which consist of four equations. Runge-kutta method is uslly used on many work of periodic motion as numerical approach. So this approach is used in this work. Otherwise, in this article numerical approach is choosen to solve the equations since constrained system is not easy to solve analitically. This approach usually uses to solve physics system, as shown in [5, 6]. Some people will try to detect chaos behavior on motion which looks like complicated periodic motion. One of method on the detection is a poincare map. It simplifies the complicated system and useful for stability analysis. Chaotic and other motion can be distinguished visually by distinct point on the poincare map [7, 8, 9].

2. Double Pendulum

2.1. The System

Figure 1 shows the double pendulum model which consists of simple pendulum system attached to the end of another simple pendulum. First simple pendulum is initialized by mass m_1, thin rod wire length l_1 and the second one by m_2 and l_2. Axial forcing Q is choosen as constraint of the system.

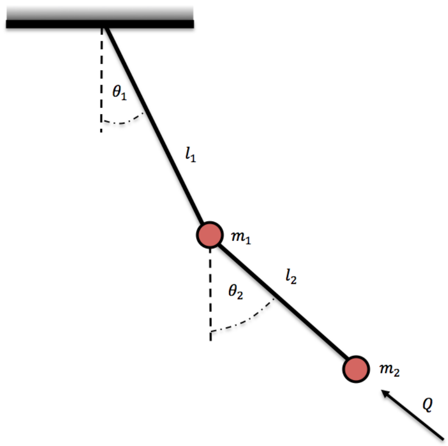

Figure 1. Double pendulum model.

2.2. Equation of Motion

Usual Hamiltonian on double pendulum system is

$$H = \frac{m_2 l_2^2 p_{\theta_1}^2 + (m_1 + m_2) l_1^2 p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]} - (m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2, \tag{1}$$

where θ_1, θ_2 are generalized coordinates and $p_{\theta_1}, p_{\theta_2}$ are generalized momenta. The axial forcing constraint is assumed

$$Q = A \sin(\theta_2 - \theta_1), \tag{2}$$

where A is a constant. According to the [2] equation of motion for the constrained system can be derived as

$$\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}} - \lambda \frac{\partial Q}{\partial p_{\theta_i}}, \qquad \dot{p}_{\theta_i} = -\frac{\partial H}{\partial \theta_i} + \lambda \frac{\partial Q}{\partial \theta_i}, \tag{3}$$

where λ can be unknown parameter which can be a function which depends on t, p_{θ_i}, and θ_i, $i = 1, 2$. By a comparison to the Hamiltonian of unconstrained systems, a more general definition for the Hamiltonian of the system can be defined as

$$H_{total} = H - \lambda Q. \tag{4}$$

In this case, the equations of motion are decomposed from the terms of total Hamiltonian H_{total}. Actually, some steps are needed to obtained the equation of motion in terms of H_{total}, but for this numerical approach, it is assumed that

$$\dot{\theta}_i \approx \frac{\partial H_{total}}{\partial p_{\theta_i}}, \qquad \dot{p}_{\theta_i} \approx -\frac{\partial H_{total}}{\partial \theta_i}. \tag{5}$$

For the simulation, assume that two masses are equal $m_1 = m_2 = m$, two length rods are aqual $l_1 = l_2 = l$ and $\lambda = qp_{\theta_2} + f \cos \omega t$, where q, f, ω are constant, so equation (5) reduces to

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{p}_{\theta_1} \\ \dot{p}_{\theta_2} \end{pmatrix} = \begin{pmatrix} \frac{p_{\theta_1} - p_{\theta_2} \cos(\theta_1 - \theta_2)}{ml^2 \left[1 + \sin^2(\theta_1 - \theta_2)\right]} \\ \frac{2p_{\theta_2} - p_{\theta_1} \cos(\theta_1 - \theta_2)}{ml^2 \left[1 + \sin^2(\theta_1 - \theta_2)\right]} \\ -2mgl \sin\theta_1 - c_1 + c_2 + k \\ -mgl \sin\theta_2 + c_1 - c_2 - k \end{pmatrix}, \tag{6}$$

where

$$c_1 = \frac{p_{\theta_1} p_{\theta_2} \sin(\theta_1 - \theta_2)}{ml^2 \left[1 + \sin^2(\theta_1 - \theta_2)\right]}, \tag{7}$$

$$c_2 = \frac{p_{\theta_1}^2 + p_{\theta_2}^2 - 2p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{ml^2 \left[1 + \sin^2(\theta_1 - \theta_2)\right]} \sin(\theta_1 - \theta_2), \tag{8}$$

and $k = (qp_{\theta_2} + f \cos \omega t) \cos(\theta_2 - \theta_1)$.

2.3. Numerical Approach
Numerical approach used is fourth order Runge-Kutta's method. The solution is decomposed according to

$$\theta_{i_{n+1}} = \theta_{i_n} + \frac{1}{6}(j_{i1} + 2j_{i2} + 2j_{i3} + j_{i4}), \tag{9}$$

$$p_{\theta_{i_{n+1}}} = p_{\theta_{i_n}} + \frac{1}{6}(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}), \tag{10}$$

where

$$\begin{array}{ll} j_{i1} = h\dot{\theta}_{i_n}\left(\theta_i, p_{\theta_i}\right), & k_{i1} = h\dot{p}_{\theta_{i_n}}\left(\theta_i, p_{\theta_i}, t\right), \\ j_{i2} = h\dot{\theta}_{i_n}\left(\theta_i + \frac{j_{i1}}{2}, p_{\theta_i} + \frac{k_{i1}}{2}\right), & k_{i2} = h\dot{p}_{\theta_{i_n}}\left(\theta_i + \frac{j_{i1}}{2}, p_{\theta_i} + \frac{k_{i1}}{2}, t + \frac{h}{2}\right), \\ j_{i3} = h\dot{\theta}_{i_n}\left(\theta_i + \frac{j_{i2}}{2}, p_{\theta_i} + \frac{k_{i2}}{2}\right), & k_{i3} = h\dot{p}_{\theta_{i_n}}\left(\theta_i + \frac{j_{i2}}{2}, p_{\theta_i} + \frac{k_{i2}}{2}, t + \frac{h}{2}\right), \\ j_{i4} = h\dot{\theta}_{i_n}\left(\theta_i + j_{i3}, p_{\theta_i} + k_{i3}\right), & k_{i4} = h\dot{p}_{\theta_{i_n}}\left(\theta_i + j_{i3}, p_{\theta_i} + k_{i3}, t + h\right), \end{array} \tag{11}$$

and h is time steps on numerical grid.

3. Result and Discussion

The results of this paper display in the poincare maps. They are shown in Figure 2 which are displayed for some different initial values. The some initial values which are equal are $\theta_1 = 30^o, \theta_2 = -30^o, p_{\theta_1} = 0$, and $p_{\theta_1} = 0$. Figure 2(a) is a motion without constraint which is identified by $q = 0$ and $f = 0$. Figure 2(b) shows constrained system which is identified by $q = 0.5$ and $f = 0$. The constraint is a damped function which depends on p_{θ_2}. Figure 2(c) includes constrained system which is identified by $q = 0$ and $f = 40$. But, in this case, the constraint is a axial forcing which depends on t. Figure 2(d) also shows constrained system which is identified by $q = 0.5$ and $f = 40$. In addition, in this case, the constraint consist of a damped function depending on p_{θ_2} and a axial forcing depending on t

Figure 2(a) and (b) display the poincare map in closed path. It means that the motions are quasi-periodic [7]. Meanwhile, distinct point are found on the poincare map in Figure 2(c) and (d). This means that the motion is chaotic behavior. While the given constraint depends on time, chaotic behavior can be found in determined initial value.

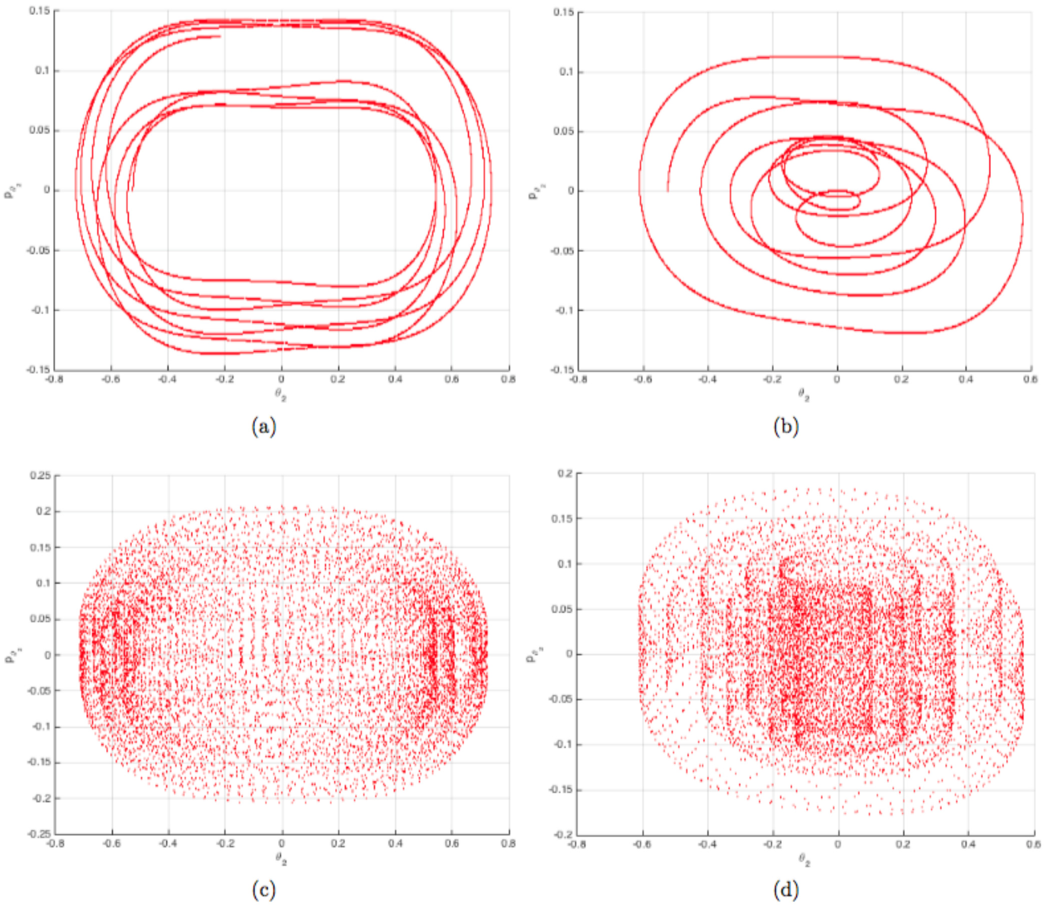

Figure 2. Poincare maps in various initial values on double pendulum system .

4. Conclusion

Constrained Hamilton equation on the double pendulum system has been solved using fourth order Runge-Kutta method. The chaotic motion can be identified on double pendulum system which depend on the given constraint related to time function.

Acknowledgments

This article is dedicated to fulfill of "IPA Terapan" subject on Doctoral Program at Pascasarjana Unnes.

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