Feynman path integral application on deriving black-scholes diffusion equation for european option pricing

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Abstract. Path Integral is a method to transform a function from its initial condition to final condition through multiplying its initial condition with the transition probability function, known as propagator. At the early development, several studies focused to apply this method for solving problems only in Quantum Mechanics. Nevertheless, Path Integral could also apply to other subjects with some modifications in the propagator function. In this study, we investigate the application of Path Integral method in financial derivatives, stock options. Black-Scholes Model (Nobel 1997) was a beginning anchor in Option Pricing study. Though this model did not successfully predict option price perfectly, especially because its sensitivity for the major changing on market, Black-Scholes Model still is a legitimate equation in pricing an option. The derivation of Black-Scholes has a high difficulty level because it is a stochastic partial differential equation. Black-Scholes equation has a similar principle with Path Integral, where in Black-Scholes the share's initial price is transformed to its final price. The Black-Scholes propagator function then derived by introducing a modified Lagrange based on Black-Scholes equation. Furthermore, we study the correlation between path integral analytical solution and Monte-Carlo numeric solution to find the similarity between this two methods.

1. INTRODUCTION

Option trading is spreading worldly since 1973 when CBOE (Chicago Board of Options Exchange) was formally established by American Government. As the trading volume increases, option brokers deal with a new rising problem, option pricing. Researches then carried out to find the most effective way to determine option price. The study was blooming when accurate model to predict option price, Black-Scholes Model was invented [1][2].

The model did not last long. It failed after the market crash occurred in The United States of America. A reason behind its failing is the number of requirements and restrictions in using the model. This issue motivated many mathematicians and physicists to enter the challenge of finding a new modified method in option pricing. A new way to approach Black-Scholes model and also has a huge potential to modify it is Path Integral. The path integral method was published by Richard P. Feynman to connect classical physics and quantum mechanics. Interestingly, this method could be applied not only in quantum mechanics, but also in finance realm[3].

Originally, a proper mathematical definition of the path integral formalism can be found in Wiener and Kac work on Stochastic Calculus[4]. While the formalism in quantum mechanics

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is an analogous that introduced by Feynman. The important of this formalism lies in the possibility of employing powerful analytical and numerical techniques. Nowadays, path integral method widely employed in many different subjects such as biology, chemistry, physics, and also in finance[5][6]. The main purpose of this study is to give a brief explication of how path integral works, particularly in option pricing.

1.1. Feynman's Derivation

Classical physics and quantum physics are as two very different things. Classical physics work on the system at the macroscopic world, where changes in the state of a system can be represented analytically. Quantum physics while working on a microscopic world with circumstances that can not be determined and measured, so that the result of the equation of state is expressed in terms of probability. One basic equation which becomes an analogy between the quantum and the mechanics is Newton's Second Law which corresponds to the equation Schrödinger:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{1}$$

Feynman started his derivation by taking P.A.M. Dirac's paper as a reference[8]. In his work, Dirac states that there is an analogy to the "kernel" or "propagator" from Huygens' principle of optic with classical action. Huygens states : "*Every point on a wave-front may be considered a source of secondary spherical wavelets which spread out in the forward direction at the speed of light. The new wave-front is the tangential surface to all of these secondary wavelets.*" Mathematically, Huygens principle described in the following integral equation:

$$\psi(x_f, t_f) = \int_{-\infty}^{\infty} G(x_i, x_f)\psi(x_i, t_i)dx\ , \tag{2}$$

where $G(x, y)$ function is called "kernel" or "propagator" function. The propagator represents the weighting for each secondary wavelets' function that produced by the primary wavefront of light. In Feynman's theory of quantum state, he deducted :

$$G(x_i, x_f) = exp(iS/\hbar) \tag{3}$$

with S as classical action and $\hbar$, the Planck's constant, is to make the argument of the exponential dimensionless.

Classical action expressed with integral of Lagrange function, $L = K - U$ with K as kinetic energy and U as potential energy, with the respect of time from initial to final of the time-frame.

$$S = \int_{t_i}^{t_f} L\ dt \approx L_{avg}\ \epsilon, \qquad \epsilon = \Delta t \tag{4}$$

L_{avg} is a numeric value of Lagrange, with the kinetic and potential energy in their simplest case represented in approximation $K = \frac{1}{2}m\left(\frac{x_f - x_i}{\epsilon}\right)^2$ and $U = U\left(\frac{1}{2}(x_f + x_i)\right)$. After performing the integration, we shall get Schrödinger Equation as described in Equation (1)[7].

In this study, we use Feynman's path integral process to derive Black-Scholes equation. Black-Scholes equation is a stock option model to define a option price in certain time from its initial condition. A share on its initial price and time has an infinite "path" that it can take to reach a certain price at a certain time. Mathematically, we can form this transformation as Equation (2) given by :

$$C(S_f, t_f) = \int_{-\infty}^{\infty} G(S_i, S_f) C(S_i, t_i) dS \tag{5}$$

with stock price S as analogy to particle position, and $C(.)$ as analogy to wave function of the particle in certain position and time. $C(.)$ in economics signify option's price, a function with stock price and time as the dependent variables.

1.2. Stock Options
Before further discussion about this method's process in option market, we shall clarify some important notes about the stock options. Option is a contract that gives the buyer the right, but not the obligation to buy or sell an underlying asset at a specific price on or before a certain date. Every option has a price that reflects its intrinsic value. The option's intrinsic value represents the economic value if it was exercised immediately, therefore the holder get the underlying asset's value minus its exercise price or zero if it was less than zero. Mathematically,

$$C(t) = max(S(t) - K, 0) \tag{6}$$

for call options, and

$$P(t) = max(K - S(t), 0) \tag{7}$$

for put options. Both of Equation (6) and Equation (7) are called payoff equation.

There are several factors which could affect the European option price fluctuation, that is :

- Underlying Asset Price. A call option price is worth more if the underlying asset price increased, otherwise put option is worth less if the underlying asset price increased.
- Volatility. Volatility describes the uncertainty level regarding the underlying asset's fluctuation. The more volatility an underlying asset has, the riskier it gets.
- Risk-free interest rate. increasing risk-free interest rate increase the valuation of an underlying asset.
- Dividends. In the moment a firm records its shareholders for dividend payment, volatility of the firm's share increases rapidly.

2. MATERIAL AND METHODS
Equation (6) and (7) explicitly tells us that we need the underlying asset's price in valuating the option price. Therefore, we need to define a share price model. According to its characteristics, the model must contain stochastic and deterministic function.

$$f(t) = \text{deterministic} + \text{stochastic} \tag{8}$$

According to Equation (8), we conduct a stochastic model to determine a share price :

$$dS = \mu_s S\ dt + \sigma_0 S dZ \tag{9}$$

with μ_s as the risk-free interest rate, and σ_0 as volatility from the underlying asset. $\mu_s S\ dt$ in Equation (9) is a deterministic differential and $\sigma_0 S dZ$ part has dZ as the random number generator. If we performed the integration on Equation (9), the solution shall be

$$S(t) = S_0\, exp\left(\left(\mu_s - \frac{1}{2}\sigma^2\right) t + \sigma z(t)\right) \tag{10}$$

with $z(t)$ as a random variable. Equation (9) is considered appropriate as the share price model. Equation (9) fits the characteristic of shares, such as it could not be zero and follows log-normal distribution. This stochastic model is going to be used to define the underlying asset's price on this study.

2.1. Black-Scholes Partial Differential Equation

Black-Scholes Equation is one of many models that frequently used by option brokers for pricing the stock option. Basic principle which used to derive the Black-Scholes equation is diversifying the portfolio in order to make it risk-free.

The value of a portfolio that consists stocks and options described as follows :

$$\Pi = N_s S(t) + N_c C(S,t) \tag{11}$$

with Π as portfolio value, N_s is the numbers of stocks held, N_c is the numbers of call option held, $S(t)$ as the value of the stocks at a certain time, and $C(S,t)$ as the value of call option on stocks at a certain time t. Risk-free portfolio means the portfolio change to time is following risk-free interest rate.

$$d\Pi = r_0 \Pi dt \tag{12}$$

On working with Equation (12), we need to derive Equation (11). We already have the derivation of share price (dS) from Equation (9). To gather derivation of option price(dC), we do total differential on $C(S,t)$ as follows

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2$$

and by using ($\frac{N_s}{N_c} = -\frac{\partial C}{\partial S}$) which means the numbers of investing multiplied by the investing value of the stock and call option is fixed [9], therefore

$$r_0(-\frac{\partial C}{\partial S}S + C)dt = \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma_0^2 S^2 dt$$

$$-r_0 S\frac{\partial C}{\partial S} + r_0 C = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma_0^2 S^2 \frac{\partial^2 C}{\partial S^2}$$

and by separating differential and non-differential part,

$$r_0 C = \frac{\partial C}{\partial t} + r_0 S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma_0^2 S^2 \frac{\partial^2 C}{\partial S^2} \tag{13}$$

Equation (13) is called Black-Scholes Partial Differential Equation. By taking $x = ln(S)$, we simplify Equation (13) to

$$r_0 C = \frac{\partial C}{\partial t} + \mu\frac{\partial C}{\partial x} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 C}{\partial x^2} \tag{14}$$

with $\mu = r_0 - \frac{1}{2}\sigma_0^2$, therefore the Black-Scholes partial differential equation has no random variable S which has been described in Equation (10).

Interestingly, we have an equation in Physics that has similarity with Black-Scholes equation. The equation is called one-dimensional diffusion equation :

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

Both Black-Scholes partial differential equation and one-dimensional diffusion equation have first order time derivation and second order space (or in finance, stock price) derivation.

We use path integral to solve Black-Scholes partial differential equation. In solving Equation (14), we do not apply path integral to transform option price from final to initial, but otherwise from initial to final. Although the aim of Black-Scholes equation is to find initial option price, however we use other method to perform the final to initial transformation, called reverse discounted method. Reverse discounted method is economical formula to define time value of money in risk-free condition.

3. RESULTS AND DISCUSSION

As has been described by Equation (2), we need the propagator function to perform path integral. Propagator is derived from exponential of Action where Action formula is built of the integration of Lagrange function. Therefore, a Lagrange equation is a key function to perform the path integral process in solving the Black-Scholes diffusion equation. By using a quantum analogy to the Black-Scholes equation, there is a new Hamiltonian equation that is corresponds to Schrödinger's Hamiltonian, namely Black-Scholes Hamiltonian :

$$\hat{H}_{BS} = \left(\frac{1}{2}\sigma_0^2 - r_0\right)\frac{\partial}{\partial x} - \frac{1}{2}\sigma_0^2\frac{\partial^2}{\partial x^2} \tag{15}$$

Knowing the Black-Scholes Hamiltonian, the Lagrangian function is derived as follows

$$\mathcal{L}_{BS} = \frac{1}{2\sigma^2}\left(\dot{x}(t') - \mu\right)^2 \tag{16}$$

The Black-Scholes Lagrange then used to perform path integral process in Feynman-Kac equation.

Equation (14) and Equation (6) are known as parabolic partial differential equation in mathematics, and according to [11], those equations have an unique solution by using Feynman-Kac equation

$$C(S_0, t) = e^{-r\tau}E_{(S_0,t)}[C(S_T)], \quad \tau = T - t \tag{17}$$

with $E_{(t,S)}[.]$ shows expected value, or in simplest form called averages, from every possible option price at $t = T$ which produced by stock with initial price S_0 and time t. The $e^{-r\tau}$ part shows reverse discounted process, where the expected value of Option price brought from its expiration date T to initial t. We use reverse discounted because the risk-free assumption used in reverse discounted is the same as Black-Scholes.

The expected value is represented by doing path integral[12]. Path integral method is appropriated as the expected value because this method's principal is doing the same with expected value. With modified $x_0 = ln(S_0), x_T = ln(S_T)$ Equation (17) is written as

$$C(S_0, t) = e^{-r\tau}\int_{-\infty}^{\infty}\int_{x(t)=x_0}^{x(T)=x_T} C(e^{x_T})e^{-A_{BS}[x(t')]}\,\mathcal{D}x(t')\,dx_T \tag{18}$$

The key point in performing the integration in the Feynman-Kac equation is completing the Black-Scholes Action that shown on $A_{BS}[x(t')]$ function. The value of $A_{BS}[x(t')]$ shown with time integration of $t \leq t' \leq T$ from Black-Scholes Lagrange.

$$A_{BS}[x(t')] = \int_t^T \mathcal{L}_{BS}\,dt' \tag{19}$$

$$\begin{aligned}\mathcal{L}_{BS} &= \frac{1}{2\sigma^2}\left(\dot{x}(t') - \mu\right)^2 \\ &= \frac{1}{2\sigma^2}\left(\dot{x}^2(t') - 2\mu\dot{x}(t') + \mu^2\right) \\ &= \frac{1}{2\sigma^2}\left(\left(\frac{dx}{dt'}\right)^2 - 2\mu\frac{dx}{dt'} + \mu^2\right)\end{aligned}$$

Then, substituting $\mathcal{L}_{BS}$ into $A_{BS}[x(t')]$,

$$A_{BS}[x(t')] = A_0[x(t')] - \frac{\mu}{\sigma^2}(x_T - x_0) + \frac{\mu^2\tau}{2\sigma^2} \tag{20}$$

with $A_0[x(t')]$ as Black-Scholes Action process without drift (μ), or called *zero drift process.*

$$A_0[x(t')] = \int_t^T \frac{1}{2\sigma^2}\left(\frac{dx}{dt'}\right)^2 dt' \tag{21}$$

$A_0[x(t')]$ could be transformed into discrete form with $N \to \infty$,

$$A_0[x(t')] = \sum_{i=0}^{N-1} \frac{1}{2\sigma^2}\left(\frac{x_{i+1} - x_i}{\Delta t'}\right)^2 \Delta t'$$

Using discrete form of $A_0[x(t')]$ to Equation (17), we have

$$C(S_0, t) = e^{-r\tau} \int_{-\infty}^{\infty} exp\left(\frac{\mu}{\sigma^2}(x_T - x_0) - \frac{\mu^2\tau}{2\sigma^2}\right) C(e^{x_T})\, \mathcal{K}(x,t)\, dx_T \tag{22}$$

with $\mathcal{K}(x,t)$ as a transition probability equation to brownian motion without drift condition. The value of $\mathcal{K}(x,t)$ is given by

$$\begin{aligned}\mathcal{K}(x,t) &= \int_{x(t)=x_0}^{x(T)=x_T} e^{-A_0[x(t')]}\mathcal{D}x(t') \\ &:= \lim_{N\to\infty} \underbrace{\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty}}_{N-1} exp\left(-\frac{\sum_{i=0}^{N-1}(x_{i+1} - x_i)^2}{2\sigma^2\Delta t}\right) \times \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \times \frac{dx_1}{\sqrt{2\pi\sigma^2\Delta t}} \times ... \times \frac{dx_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}\end{aligned} \tag{23}$$

The multiple integral in Equation (23) is Gaussian and we calculate this equation using the following identity

$$\int_{-\infty}^{\infty} e^{-a(p-x)^2 - b(x-q)^2} dx = \sqrt{\frac{\pi}{a+b}} exp\left[-\frac{ab}{a+b}(p-q)^2\right] \tag{24}$$

Therefore by integrating Equation (23) one by one from dx_1 to dx_N,

$$\mathcal{K}(x,t) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi\sigma^2(N\Delta t)}}\, exp\left[-\frac{(x_N - x_0)^2}{2\sigma^2(N\Delta t)}\right] \tag{25}$$

For $N \to \infty$, then $N(\Delta t) = \tau$ and $x_N = x_T$. In this condition, the transition probability function $\mathcal{K}(x,t)$ then have no dependent variable t'

$$\mathcal{K}(x,t) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\, exp\left[-\frac{(x_T - x_0)^2}{2\sigma^2\tau}\right] \tag{26}$$

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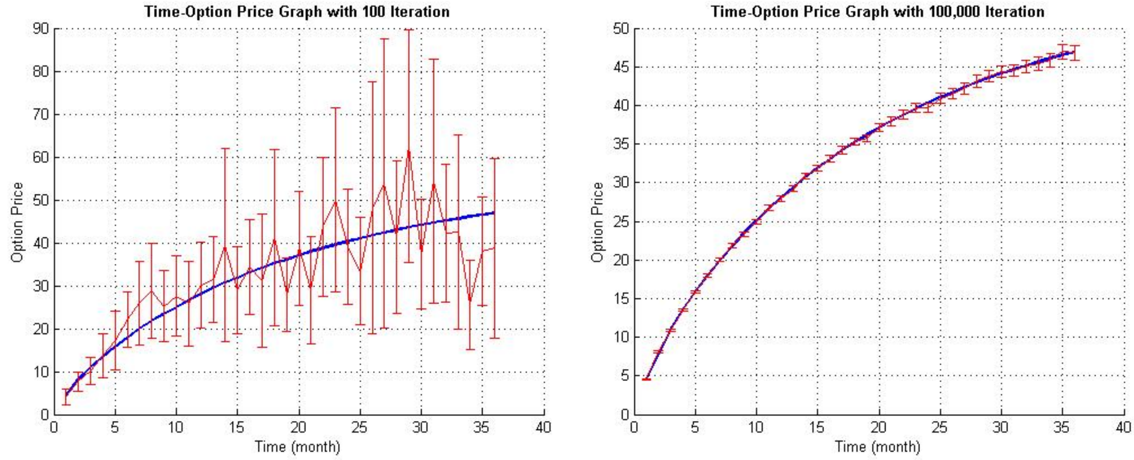

Figure 1. Application of Monte-Carlo methods in solving Black-Scholes equation. Monte-Carlo computation results drawn in red line and analytical result drawn in blue line. The vertical red line shows standard error of Monte-Carlo computing.

Using Equation (26), we can rewrite Equation (22) into following equation.

$$C(S_0,t) = e^{-r\tau} \int_{-\infty}^{\infty} C(e^{x_T}) \, \frac{1}{\sqrt{2\pi\sigma^2\tau}} exp\left[-\frac{(x_T - x_0 - \mu\tau)^2}{2\sigma^2\tau}\right] \, dx_T \tag{27}$$

Integral at Equation (27) is the path integral on solving Black-Scholes diffusion equation. Propagator, as the weighting factor for calculating the expected value written as

$$\mathcal{G}(x_T, x_0) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} exp\left[-\frac{(x_T - x_0 - \mu\tau)^2}{2\sigma^2\tau}\right] \tag{28}$$

The Black-Scholes propagator in Equation (28) is a Gaussian Function. Therefore, with $C(e^{x_T}) = max(e^{x_T} - K, 0)$, integration results following equation

$$C(S_0,t) = S_0 N(d_2) - e^{-r\tau} K N(d_1) \tag{29}$$

with $N(.)$ as Normal Distribution Function, and the d_1 & d_2 described as follows

$$d_1 = \frac{ln\,(S/K) + \mu\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 + \sigma\sqrt{\tau} \tag{30}$$

Besides doing analytical derivation, there are computational methods on deriving Black-Scholes equation. One of several computational methods is Monte-Carlo method. This method does a similar way in solving Black-Scholes equation, that is by using Feynman-Kac equation. Different than derivation by path integral, we use the simplest method to determine the expected value. That is by averaging all option's final price that generated using computational generator. By using this method, we need more iteration for a better solution.

Computation results are shown in Figure (3). These results shows us the more Monte-Carlo iteration predict more accurate results. The imprecision of smaller iteration is shown by the bigger errors on 100 iteration in Figure (3). The results implied in using Monte-Carlo methods, we have to use huge iteration in order to find the precision result.

4. CONCLUSION

Black-Scholes Model with its diffusion equation have similarities with schrödinger equation form in Quantum Mechanics. The differences are shown in Hamiltonian conformation from each of these equation. The different Hamiltonian conformations produces different Lagrange. The Black-Scholes Lagrange function form is similar to Classical Lagrange of free particle system which has no potential on the system. The free particle classical Lagrange is $\mathcal{L} = mv^2/2$ and by using $m = 1/\sigma^2$ and $v = \dot{x}(t') - \mu$, we have a Black-Scholes Lagrange. A path integral method on deriving the solution of Black-Scholes equation was described.

5. FURTHER DIRECTION

We only use European Option which is a path-independent option on this study. There are many other option types with different characteristics that could be applied using path integral methods. The interesting thing about path integral is it use Lagrange which could be easily modified. In this study, we use zero potential condition Lagrange function. However, one could add potential condition on this Lagrange function to level derivation condition to real condition. The same as in quantum mechanics, zero potential condition is a simplest condition and there are several potential variations, such as potential step, infinite well, potential barrier, etc. For example, the volatility of a share is unstable in real finance world. Therefore, there are potential models which conducted by volatility and other financial variables.

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