Higher dimensional maximally symmetric stationary manifold with pure gauge condition and codimension one flat submanifold

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Abstract. An n dimensional flat manifold N is embedded into an $n+1$ dimensional stationary manifold M. The metric of M is derived from a general form of stationary manifold. By taking several assumption, such as 1) the ambient manifold M to be maximally symmetric space and satisfying a pure gauge condition, and 2) the submanifold is taken to be flat, then we find the solution that satisfies Ricci scalar of N. Moreover, we determine whether the solution is compatible with the Ricci and Riemann tensor of manifold N depending on the dimension n.

1. Introduction

Spacetime geometry has played an important part in theoretical physics since the advent of Einstein's relativity. Especially, in general relativity, gravitational interaction is described as a curved spacetime which is the part of Riemannian differential geometry. Besides, it is still unclear about the exact dimension of our universe, whether it is constructed by only four dimensional spacetime or more dimensions. Therefore, study of higher dimensional spacetime would be of interest.

In this paper, we discuss about higher dimensional stationary geometry with some constraints. The geometry is composed of two manifolds, those are an n dimensional manifold N which is embedded into the higher one, an $n+1$ dimensional manifold M. Manifold N is a Riemannian manifold which contains only spatial components. In the end, the submanifold N will be taken to be a flat manifold. On the other hand, the ambient manifold M is either a Riemannian or Lorentzian manifold, which contains both spatial and time components, and will be limited to some conditions. We assume that manifold M is a maximally symmetric space and satisfies a pure gauge like condition. Therefore, we will find the solution which satisfies the assumptions and what criteria should be fullfilled so that the solution is valid.

In order to find the solution, all the conditions mentioned above will not be implemented at once, but rather gradually. Firstly, we take a general form metric of stationary manifold, mentioned in equation (1), as the metric of manifold M. Then, the Riemann tensor of the manifold is calculated and compared to the Riemann tensor of manifold M if it is a maximally symmetric space. By applying a pure gauge like condition, the equation of the Riemann tensors of stationary and maximally symmetric manifold M are simplified in order to find the Riemann tensor of submanifold N. After that, the Ricci tensor and scalar of the submanifold is calculated. At last, submanifold N is taken to be a flat Euclidean manifold. The solution of these conditions is supposed to satisfy the submanifold's Ricci scalar. Furthermore, we will evaluate the solution's compatibility to submanifold's Ricci and Riemann tensor. This compatibility evaluation is limited only to the dimensional aspect.

2. Manifold M as a stationary manifold

As a stationary manifold, M has a time-like Killing vector. It means that the metric is invariant under time-like translation transformation. Generally, the time axis of the manifold is not orthogonal with its spatial hypersurface, so that the metric will not be invariant under time reversal transformation. The metric of the ambient M is taken to be

$$ds^2 = \frac{\epsilon}{\omega^2}(dt + \alpha_i dx^i)^2 + \omega^2 \hat{g}_{ij} dx^i dx^j. \tag{1}$$

In order to ensure the stationarity of the manifold, the α_i and ω should be the function of spatial parameter only. The ϵ in the metric has only two values, -1 or $+1$. If $\epsilon = 1$, then M is a Riemannian manifold, but if $\epsilon = -1$, then M is a Lorentzian manifold. [2] Into the metric of manifold M, a general form of metric N,

$$d\hat{s}^2 = \hat{g}_{ij} dx^i dx^j, \tag{2}$$

is embedded. Note that we take Latin index which runs from 1 to n and Greek index which runs from 0 to n. It is important to consider that Einstein's summation convention is used.

By using the metric, we are able to calculate the Riemann tensor [1] of manifold M. The components of the Riemann tensor are written as follow.

$$R_{0i0j} = \frac{\epsilon}{\omega^3}\nabla_i\nabla_j\omega - \frac{4\epsilon}{\omega^4}\nabla_i\omega\nabla_j\omega + \frac{\epsilon}{\omega^4}\nabla^k\omega\nabla_k\omega\hat{g}_{ij} + \frac{1}{4\omega^6}F_i{}^k F_{jk} \tag{3}$$

$$\begin{aligned} R_{0ijk} = {} & \frac{\epsilon}{2\omega^2}\nabla_i F_{jk} - \frac{\epsilon}{\omega^3}\nabla_i\nabla_{[j}\omega\alpha_{k]} - \frac{2\epsilon}{\omega^3}\nabla_i\omega F_{jk} + \frac{\epsilon}{\omega^3}F_{i[j}\nabla_{k]}\omega \\ & -\frac{1}{4\omega^6}F_i{}^l\alpha_{[j}F_{lk]} - \frac{\epsilon}{2\omega^3}\hat{g}_{i[j}F_{k]l}\nabla^l\omega + \frac{4\epsilon}{\omega^4}\nabla_i\omega\nabla_{[j}\omega\alpha_{k]} \\ & -\frac{\epsilon}{\omega^4}\hat{g}_{i[j}\alpha_{k]}\nabla^l\omega\nabla_l\omega \end{aligned} \tag{4}$$

$$\begin{aligned} R_{ijkl} = {} & \omega^2\hat{R}_{ijkl} + \omega(\hat{g}_{i[l}\nabla_j\nabla_{k]}\omega - \hat{g}_{j[l}\nabla_i\nabla_{k]}\omega) - 2\nabla_{[i}\omega\hat{g}_{j][k}\nabla_{l]}\omega \\ & -\hat{g}_{i[k}\hat{g}_{jl]}\nabla^m\omega\nabla_m\omega + \frac{\epsilon}{2\omega^2}(\alpha_{[i}\nabla_{j]}F_{kl} + \alpha_{[k}\nabla_{l]}F_{ij}) \\ & -\frac{\epsilon}{4\omega^2}F_{i[k}F_{jl]} - \frac{\epsilon}{2\omega^2}F_{ij}F_{kl} - \frac{2\epsilon}{\omega^3}(\alpha_{[i}\nabla_{j]}\omega F_{kl} + \alpha_{[k}\nabla_{l]}\omega F_{ij}) \\ & -\frac{\epsilon}{\omega^3}(\nabla_{[i}\omega F_{j][k}\alpha_{l]} + \nabla_{[k}\omega F_{l][i}\alpha_{j]}) - \frac{\epsilon}{\omega^3}\alpha_{[i}\nabla_{j]}\nabla_{[k}\omega\alpha_{l]} \\ & -\frac{\epsilon}{2\omega^3}(\alpha_{[i}\hat{g}_{j][k}F_{l]m}\nabla^m\omega + \alpha_{[k}\hat{g}_{l][i}F_{j]m}\nabla^m\omega) \\ & -\frac{\epsilon}{\omega^4}\alpha_{[i}\hat{g}_{j][k}\alpha_{l]}\nabla^m\omega\nabla_m\omega + \frac{4\epsilon}{\omega^4}\alpha_{[i}\nabla_{j]}\omega\nabla_{[k}\omega\alpha_{l]} \\ & +\frac{1}{4\omega^6}\alpha_{[i}F_{j]}{}^m\alpha_{[k}F_{l]m} \end{aligned} \tag{5}$$

In the equations above, there is tensor which can be associated with field strength tensor, F_{ij}, and defined as

$$F_{ij} \equiv \partial_{[i}\alpha_{j]}. \tag{6}$$

Notation ∇ denotes covariant derivative operator and ∂ denotes partial derivative operator. The parenthesis or bracket symbol written before or after an index represents a symmetrical or antisymmetrical operation which is defined as

$$A_{(a}B_{b)} \equiv A_aB_b + A_bB_a \tag{7}$$
$$A_{[a}B_{b]} \equiv A_aB_b - A_bB_a \tag{8}$$

Note that the $\hat{R}_{ijkl}$ in equation (5) is the Riemann tensor of submanifold N which is defined as

$$\hat{R}_{ijkl} = \hat{g}_{im}(\partial_k\hat{\Gamma}^m_{jl} - \partial_l\hat{\Gamma}^m_{jk} + \hat{\Gamma}^m_{kn}\hat{\Gamma}^n_{jl} - \hat{\Gamma}^m_{ln}\hat{\Gamma}^n_{jk}). \tag{9}$$

where $\hat{\Gamma}^i_{jk}$ denotes the Christoffel symbold of the submanifold.

3. Manifold M as a maximally symmetric space

As an $n+1$ dimensional maximally symmetric space, manifold M has $\frac{1}{2}(n+1)(n+2)$ Killing vectors. By definition [3], the Riemann tensor of manifold M should satisfy the following equation.

$$R_{\mu\alpha\nu\beta} = \kappa(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}) \tag{10}$$

By using the metric components of M, the Riemann tensor of maximally symmetric manifold M can be calculated and satisfies the following equation.

$$R_{0i0j} = \epsilon\kappa\hat{g}_{ij} \tag{11}$$

$$R_{0ijk} = -\epsilon\kappa\hat{g}_{i[j}\alpha_{k]} \tag{12}$$

$$R_{ijkl} = -\epsilon\kappa\alpha_{[i}\hat{g}_{j][k}\alpha_{l]} + \kappa\omega^4\hat{g}_{i[k}\hat{g}_{jl]} \tag{13}$$

These results should be equated to the Riemann tensor which has been obtained in equation (3), (4), and (5) in order to maintain the consistency of manifold M.

4. Manifold N with pure gauge condition

Gauge formulation is related to the transformation of field or gauge. In the metric, we can assume that α_i is a vector field and ω is a scalar field. Pure gauge condition is a situation when gauge transformation is applied to a null field. [4] In an abelian case, where gauge transformation can be written as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu f(x), \tag{14}$$

then, if the transformation is applied to a null field, we obtain

$$A'_\mu(x) = \partial_\mu f(x). \tag{15}$$

Therefore, if the "field strength tensor" F_{ij} has vector field α_i which is abelian, then, by applying a pure gauge like condition, we can suppose that

$$\alpha_i(x) = \partial_i\phi(x). \tag{16}$$

Consequently, under this condition, the F_{ij} tensor vanishes.

After equating the Riemann tensor of stationary manifold M, which is shown in equation (3) to (5), to the maximally symmetric one, which is shown in equation (11) to (13), and applying the pure gauge like condition, we could obtain

$$\epsilon\kappa\hat{g}_{ij} = \frac{\epsilon}{\omega^3}\nabla_i\nabla_j\omega - \frac{4\epsilon}{\omega^4}\nabla_i\omega\nabla_j\omega + \frac{\epsilon}{\omega^4}\nabla^k\omega\nabla_k\omega\hat{g}_{ij} \tag{17}$$

$$\begin{aligned}\epsilon\kappa\hat{g}_{i[j}\alpha_{k]} &= \frac{\epsilon}{\omega^3}\nabla_i\nabla_{[j}\omega\alpha_{k]} - \frac{4\epsilon}{\omega^4}\nabla_i\omega\nabla_{[j}\omega\alpha_{k]} \\ &\quad + \frac{\epsilon}{\omega^4}\hat{g}_{i[j}\alpha_{k]}\nabla^l\omega\nabla_l\omega\end{aligned} \tag{18}$$

$$\begin{aligned}\kappa\omega^4\hat{g}_{i[k}\hat{g}_{jl]} - \epsilon\kappa\alpha_{[i}\hat{g}_{j][k}\alpha_{l]} &= \omega^2\hat{R}_{ijkl} + \omega(\hat{g}_{i[l}\nabla_j\nabla_{k]}\omega - \hat{g}_{j[l}\nabla_i\nabla_{k]}\omega) - 2\nabla_{[i}\omega\hat{g}_{j][k}\nabla_{l]}\omega \\ &\quad -\hat{g}_{i[k}\hat{g}_{jl]}\nabla^m\omega\nabla_m\omega - \frac{\epsilon}{\omega^3}\alpha_{[i}\nabla_{j]}\nabla_{[k}\omega\alpha_{l]} \\ &\quad -\frac{\epsilon}{\omega^4}\alpha_{[i}\hat{g}_{j][k}\alpha_{l]}\nabla^m\omega\nabla_m\omega + \frac{4\epsilon}{\omega^4}\alpha_{[i}\nabla_{j]}\omega\nabla_{[k}\omega\alpha_{l]}\end{aligned} \tag{19}$$

Note that the equation (18) is similar to equation (17). Substitute this equation into equation (19), then rearrange and simplify the equation, so that we obtain

$$\hat{R}_{ijkl} = 2\kappa\omega^2\hat{g}_{i[k}\hat{g}_{jl]} + \frac{1}{2\omega}(\hat{g}_{i[k}\nabla_{l]}\nabla_j\omega - \hat{g}_{j[k}\nabla_{l]}\nabla_i\omega) \tag{20}$$

which will be used as the Riemann tensor of submanifold N for the next calculation. Ricci tensor and scalar of the submanifold are calculated and obtained as follow.

$$\hat{R}_{ij} = 2(n-1)\kappa\omega^2\hat{g}_{ij} + \frac{1}{2\omega}(\hat{g}_{ij}\nabla^k\nabla_k\omega + (n-2)\nabla_i\nabla_j\omega) \tag{21}$$

$$\hat{R} = 2n(n-1)\kappa\omega^2 + \frac{1}{\omega}(n-1)\nabla^i\nabla_i\omega \tag{22}$$

5. Manifold N as a flat manifold

In this part, we specify the submanifold N as a flat Euclidean manifold. As a result, the submanifold's Ricci scalar $\hat{R}$ vanishes and equation (22) becomes

$$\Delta\omega + 2n\kappa\omega^3 = 0 \tag{23}$$

where $\Delta \equiv \nabla^i\nabla_i$ is a Laplace-Beltrami operator. This equation is a non-linear differential equation which cannot be solved easily. However, we attempt a trial to solve the differential equation and find the ω which satisfies the equation. We assume that ω is a function of distance r in manifold N .

For a flat Euclidean geometry, the metric of the manifold has form similar to Kronecker delta,

$$\hat{g}_{ij} = \delta_{ij}, \tag{24}$$

so that the Laplace-Beltrami operator of the manifold would satisfy the following equation.

$$\Delta = \nabla^i\nabla_i = \partial^i\partial_i \tag{25}$$

This flat manifold also has zero Ricci and Riemann tensor. Because ω is taken to be the function of distance r, it is convenient to choose coordinates where the corresponding distance is radially directed. Therefore, we could take the calculation in spherical coordinates in n dimensional flat space. In this coordinates, the Laplacian operator can be written as

$$\Delta = \partial^i \partial_i = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}(r^{n-1}\frac{\partial}{\partial r}) + \frac{1}{r^2}\Delta_{S^{n-1}}, \tag{26}$$

where $\Delta_{S^{n-1}}$ is a Laplace-Beltrami operator on hypersurface of n-dimensional ball, $(n-1)$-sphere (S^{n-1}), known as $(n-1)$ dimensional spherical Laplacian that consists of angular parameters. Thus, the differential equation (23) becomes

$$\frac{1}{r^{n-1}} \frac{d}{dr}(r^{n-1}\frac{d\omega}{dr}) + 2n\kappa\omega^3 = 0. \tag{27}$$

See that the n-dimensional differential equation has become a 1-dimensional differential equation problem.

Although the dimension has been reduced, the problem is still difficult to be solved because the equation (27) is still non-linear. The equation still has a second derivative term, a first derivative term, and an ω^3 term. Then, in order to simplify the problem, we should eliminate the first derivative term. It can be done by defining variable u as follow.

$$u \equiv r^{2-n} \tag{28}$$

By this definition, the differential equation (27) can be written in a simpler form as

$$\frac{d^2\omega}{du^2} + \frac{2n\kappa}{(2-n)^2} u^{\frac{2(n-1)}{2-n}} \omega^3 = 0. \tag{29}$$

Since the equation has been simplified, this equation is still a non-linear differential equation. However, we would try to solve the differential equation as follow. Firstly, the equation is rewritten as

$$\begin{aligned} (\frac{d^2}{du^2} + C^2 u^{\frac{2(n-1)}{2-n}} \omega^2)\omega &= 0 \\ (\frac{d}{du} + iCu^{\frac{n-1}{2-n}}\omega)(\frac{d}{du} - iCu^{\frac{n-1}{2-n}}\omega)\omega &= 0 \end{aligned} \tag{30}$$

where C is a dimension depended coefficient $C(n)$. If ω is a non-zero function, then

$$\frac{d\omega}{du} \pm iCu^{\frac{n-1}{2-n}}\omega^2 = 0 \tag{31}$$

At this moment, we should not concern about the sign $\pm$ because both of them will produce similar results. Solving this equation will result as follow.

$$\begin{aligned} \frac{d\omega}{\omega^2} &= iCu^{\frac{n-1}{2-n}} du \\ -\frac{1}{\omega} &= iC'u^{\frac{1}{2-n}}, \qquad C' \equiv C'(n) \\ \omega &= iC''u^{\frac{1}{n-2}}, \qquad C'' \equiv C''(n) \end{aligned} \tag{32}$$

This equation still has an arbitary coefficient C'' which has to be determined. Substitute back this solution into its differential equation (29) and we obtain

$$\frac{iC''}{(n-2)^2} u^{\frac{5-2n}{n-2}} ((3-n) - 2n\kappa C''^2) = 0. \tag{33}$$

This equation is satisfied if $C'' = 0$ or

$$(3-n) - 2n\kappa C''^2 = 0 \leftrightarrow C'' = \pm\sqrt{\frac{3-n}{2n\kappa}}. \tag{34}$$

The $C'' = 0$ should be excluded because it produces a trivial solution ω. Hence, the solution ω is obtained as

$$\omega(u) = \pm\sqrt{\frac{n-3}{2n\kappa}}u^{\frac{1}{n-2}}. \tag{35}$$

Rewrite the ω as a function of r and we obtain

$$\omega(r) = \pm\sqrt{\frac{n-3}{2n\kappa}}r^{-1}. \tag{36}$$

The $+$ or $-$ solution must be stated separately because they cannot be combined linearly. Moreover, any other possible solution of this differential equation is also not able to be combined linearly. It is because of the non-linearity of the differential equation. The solutions may be able to be combined in a special way, but it is not guaranteed.

Note that the solution gives a real value only if $n > 3$. When $n = 3$, the solution ω is trivial, while $n < 3$ will give an imaginary value of ω. Therefore, the ω gives a value which is physicaly well defined only for $n > 3$. However, the solution is still mathematically permitted for all n except for $n = 0$. It is important to consider that the trivial value of ω for $n = 3$ will ruin the calculation. When $\omega = 0$, the metric of manifold M becomes singular. Moreover, the singularity of the manifold would disable any further calculation. Therefore, the solution ω for $n = 3$ is not physically well defined and should be excluded.

6. Examination of ω as the solution of submanifold N

In this part, we examine the compatibility of ω as the solution of the submanifold. Firstly, we would examine its compatibility as the solution of the Ricci scalar. After applying the ω, the Ricci scalar in equation (22) can be rewritten as follow.

$$\frac{\omega\hat{R}}{n-1} = -\sqrt{\frac{n-3}{2n\kappa}}r^{-3}(n-3) + 2n\kappa\left(\frac{n-3}{2n\kappa}\sqrt{\frac{n-3}{2n\kappa}}r^{-3}\right) \quad = \quad 0 \tag{37}$$

This equation is satisfied if and only if $\hat{R} = 0$, so that we believe that N with ω stated in equation (36) is a flat manifold. Note that the Ricci scalar always vanishes for any ω for $n = 1$, however it is actually a certainty.

After that, we would examine the ω as the solution of the Ricci tensor. By applying the ω, we are able to rewritten the Ricci tensor mentioned in equation (21) as follow.

$$2\omega\hat{R}_{ij} \quad = \quad -3(n-2)\sqrt{\frac{n-3}{2n\kappa}}r^{-3}\left(\frac{\delta_{ij}}{n} - \frac{x_i x_j}{r^2}\right) \tag{38}$$

Generally, equation (38) shows that $\hat{R}_{ij} \neq 0$. This equation vanishes if the following equation is satisfied.

$$\frac{\delta_{ij}}{n} - \frac{x_i x_j}{r^2} = 0 \quad \leftrightarrow \quad x_i x_j = \frac{r^2}{n}\delta_{ij} \tag{39}$$

This equation cannot be satisfied generally. Multiply the equation by vector x^j, then we obtain

$$r^2 x_i \neq \frac{r^2}{n}x_i. \tag{40}$$

However, the Ricci tensor in equation (38) vanishes for $n = 2$ or $n = 3$, and also if it satisfies equation (39). Note that this equation is satisfied only if $n = 1$. Moreover, we should remember that $n = 3$ should be excluded from the solution. Therefore, we can say, that ω obtained in equation (36), generally, is not the solution that gives $\hat{R}_{ij} = 0$, except for $n = 1$ and $n = 2$.

Finally, we would examine the ω's compatibility as the solution of the submanifold's Riemann tensor. By applying the ω, the Riemann tensor stated in equation (20) can be rewritten as follow.

$$2\omega\hat{R}_{ijkl} = 3\sqrt{\frac{n-3}{2n\kappa}}r^{-3}(\frac{2}{n}\delta_{i[l}\delta_{jk]} - \frac{1}{r^2}x_{[i}\delta_{j][k}x_{l]}) \quad (41)$$

Generally, equation (41) shows that $\hat{R}_{ijkl} \neq 0$. This equation vanishes if the following equation is satisfied.

$$\begin{aligned} \frac{2}{n}\delta_{i[l}\delta_{jk]} - \frac{1}{r^2}x_{[i}\delta_{j][k}x_{l]} &= 0 \\ (\frac{\delta_{il}}{n} - \frac{x_i x_l}{r^2})\delta_{jk} + (\frac{\delta_{jk}}{n} - \frac{x_j x_k}{r^2})\delta_{il} - (\frac{\delta_{ik}}{n} - \frac{x_i x_k}{r^2})\delta_{jl} - (\frac{\delta_{jl}}{n} - \frac{x_j x_l}{r^2})\delta_{ik} &= 0 \end{aligned} \quad (42)$$

This equation is similar to equation (39) and by the same argument, we can say that this equation is also satisfied only for $n = 1$. Note that the Riemann tensor in equation (41) also vanishes for $n = 3$. However, we should remember that $n = 3$ should be excluded from the solution. Therefore, we can say, that ω obtained in equation (36), generally, is not the solution that gives $\hat{R}_{ijkl} = 0$, except for $n = 1$.

7. Conclusion

A stationary geometry is constructed of 2 manifolds where an n-dimensional manifold N is embedded into an $(n+1)$-dimensional manifold M. The ambient is assumed to be maximally symmetric and satisfy the pure gauge like condition, while the submanifold is taken to be a flat Euclidean manifold. The ω is supposed to be the function of distance and satisfy the submanifold's Ricci scalar. This solution is obtained as stated in equation (36) and should be examined as the solution of the submanifold. By taking the examination, there are several conclusions corresponding to the solution compatibility of ω as follow.

- ω gives a real value for $n > 3$, an imaginary value for $n < 3$, and a trivial value for $n = 3$.
- The triviality of ω for $n = 3$ makes the metric of M singular and should be excluded from the solution.
- ω is the solution that gives $\hat{R} = 0$ for all n except for $n = 3$.
- ω is not the solution that gives $\hat{R}_{ij} = 0$ except for $n = 1$ and $n = 2$.
- ω is not the solution that gives $\hat{R}_{ijkl} = 0$ except for $n = 1$.

By this consideration, it is convenient to say that ω is not a general solution that satisfy flat submanifold condition, but may be only a special solution or a subsolution of a more general and complete solution.

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