

# Multi-point quasi-rational approximants for the modified Bessel function $I_1(x)$

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**Abstract.** Approximants for the modified Bessel function  $I_1(x)$  has been found using the multi-point quasi-rational technique. The approximations here determined has good accuracy for any positive value of the variable, and it seems to be adequate for most of the works where this function are used. Furthermore, the approximants are simple to calculate numerically in a direct way or using any usual MAPLE or MATLAB software.

## 1. Introduction

The modified Bessel function is a very important tool in several areas of Physics and Mathematics [1, 2, 3, 4]. The power series is an entire function, that is, it has an infinite radius of convergence. However precise calculations for regular values of  $x$  requires sometimes a lot of terms from the power series (several hundreds for high vales of  $x$ ), which limits their use. The asymptotic expansions of these functions are also well-known [1, 2, 3, 4, 5], but its use is restricted to very high values of the independent variable  $x$ , besides the asymptotic expansions are semi-convergent series (convergent only in the Poincaré sense), which is also a problem, because to obtain the best accuracy different numbers of terms have to be taken depending of the values of  $x$ . There are some polynomials approximations [3], but there are good only in small range of the  $x$ -variable.

On the other way recently has been published a technique to obtain precise approximants to functions and eigenvalues [6, 7, 8, 9, 10, 11] using simultaneously power series and asymptotic expansions and building a bridge between both expansions using rational approximants, as in Padé method, but adding also some non-rational auxiliary functions.

The way to obtain the approximant with the present technique will be presented in section 2. The last section will be left for the conclusion.

## 2. Theoretical treatment

The power series of the modified Bessel function  $I_1(x)$  is

$$I_1(x) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{x^2}{4}\right)^k. \quad (1)$$



The asymptotic expansion for this function is also well known

$$I(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{3}{8x} - \frac{15}{128x^2} - \dots \right] \quad (2)$$

A bridge between both expansions has to be built by using the MPQA (multi-point quasi-rational approximant) technique. This requires to build a bridge using both powers simultaneously and using simple functions combined with rational ones.

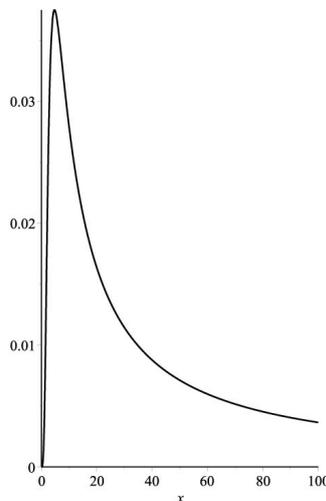
It is clear from the asymptotic expansion that there is a fractional power at the infinite, but a simple auxiliary function as  $\left(\frac{1}{\sqrt{x}}\right)$  can not be used because this kind of functions has the good behavior at the infinite, but at the same time appears a singularity as ramification at  $x = 0$ . However this singularity at  $x = 0$  does not correspond to  $I_1(x)$ , and in this will be put at the approximant not convenient singularity.

To avoid this problem we should use an auxiliary function as  $\left(\frac{1}{x+\lambda}\right)$  with  $\lambda > 0$ . In this way the inconvenient singularity will be in the negative axis, that is, outside of the region of approximation, which in our case will be  $x \geq 0$  or right half complex plane. There is also a new reason to discuss this auxiliary function, since the power series for  $I_1(x)$  has only odd powers of  $x$ , then in order to obtain a more efficient and adequate auxiliary function this has to be chosen as  $\left(\frac{x}{(1+\lambda x^2)^{\frac{3}{4}}}\right)$ . The simplest of these approximants will be with  $\lambda = \frac{1}{4}$ . Similar analysis to the previous one can me performed for the exponential behavior at the infinite and in this way the simplest approximant found for the function  $I_1(x)$  is written as

$$\tilde{I}_1(x) = x (e^x + e^{-x}) \left(1 + \frac{x^2}{4}\right)^{-3/4} \cdot \frac{\left(\frac{1}{2} + \frac{x^2}{32(\sqrt{\pi}-1)}\right)}{\left(2 + \frac{3\sqrt{\pi}x^2}{8(\sqrt{\pi}-1)}\right)}, \quad (3)$$

where the coefficients of the polynomial numerator and denominator has been chosen in such a way that the two first power series of the functions  $I_1(x)$  and  $\tilde{I}_1(x)$  are incidents as well as the leading term of the asymptotic expansion.

The relative error from  $0 < x < 100$  is also shown in figure 1.



**Figure 1.** In this figure, the relative error of the appoximant is shown for  $0 < x < 100$ .

Since the function  $I_1(x)$  increases exponentially largest value of  $x$  are not usual but the relative error is also small.

The maximum relative error is for values of  $x$  nearby 5. This is the usual behavior of the error using the multi-point quasi-rational approximant technique (MPQA) since the approximants are very precise for small and large values of the variable, and therefore the largest errors are for intermediate values. In any case the errors are always obtained using a small even for approximants low number of parameters. Including more parameters the largest relative errors decreases quickly, sometimes including one or two additional parameters the largest error decreases in magnitude order.

### 3. Conclusion

A precise approximant for the modified Bessel function of order one has been determined using the MPQA technique. The approximant is very simple and it is good for every positive value of the variable  $x$ . Using only a few parameters a good accuracy is obtained. This shows the goodness of this way to obtain approaches to spacial functions and in particular modified Bessel functions. Using this technique the largest error is usually for intermediate values, that is, no small and no large. Here we have determined an approximant with small number of parameters and the largest error was about 3 per cent nearby  $x \sim 5$ . Approximants with larger accuracy can be obtained using one or two parameters more, which are left for future works.

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