

# Dispersive models describing mosquitoes' population dynamics

W M S Yamashita<sup>1</sup>, L T Takahashi<sup>1</sup> and G Chapiro<sup>1</sup>

<sup>1</sup>Universidade Federal de Juiz de Fora, 36036-900, Juiz de Fora-MG, Brazil

E-mail: grigori@ice.ufjf.br

**Abstract.** The global incidences of dengue and, more recently, zika virus have increased the interest in studying and understanding the mosquito population dynamics. Understanding this dynamics is important for public health in countries where climatic and environmental conditions are favorable for the propagation of these diseases. This work is based on the study of nonlinear mathematical models dealing with the life cycle of the dengue mosquito using partial differential equations. We investigate the existence of traveling wave solutions using semi-analytical method combining dynamical systems techniques and numerical integration. Obtained solutions are validated through numerical simulations using finite difference schemes.

## 1. Introduction

There is a renewed interest in controlling the proliferation of *Aedes aegypti* mosquito, which is the transmitter of Dengue virus, and more recently Chikungunya and Zika viruses, [1]. In order to minimize both social and economic costs, [2] analyze the Dengue vector control problem in a multi-objective optimization approach using a dynamic mathematical model representing the mosquitoes' population. There are several modeling approaches appearing in the study of biological invasions and spread of diseases. In [3, 4], the authors study spacial population dynamics of *Aedes aegypti* using partial differential equations (PDEs) describes the life cycle of the mosquito *Aedes aegypti*. In [5] was studied nonlinear modification of this model considering different diffusion (dispersion) and advection (wind) effects. We follow the same approach here using techniques presented in [6] applied to generalized non-linear problem.

The paper is organized as follows. Section 2 presents the mathematical model. Section 3 discusses the existence of the solution for this model in a form of traveling wave. In Section 4 we apply the algorithm and validate the results comparing them to the direct numerical simulation obtained by using Finite Difference Scheme. The final conclusions are presented in Section 5.

## 2. Mathematical model

We consider one-dimensional model from [5] considering  $p = 0$  and  $q = 1$ . To simplify the vital biological dynamics of mosquito, the model considers only two subpopulations: (1) eggs, larvae and pupae compose aquatic immobile phase ( $A(x, t)$ ) and (2) female mosquitoes compose winged mobile phase ( $M(x, t)$ ). The dimensional form of the model is

$$\begin{cases} \bar{M}_t &= \bar{D} \bar{M}_{xx} - 2\bar{\nu} \bar{M} / \bar{k}_1 \bar{M}_x + \bar{\gamma} \bar{A} (1 - \bar{M} / \bar{k}_1) - \bar{\mu}_1 \bar{M}, \\ \bar{A}_t &= \bar{r} (1 - \bar{A} / \bar{k}_2) \bar{M} - (\bar{\mu}_2 + \bar{\gamma}) \bar{A}, \end{cases} \quad (1)$$



where  $\bar{M}$  is the winged phase population,  $\bar{A}$  is the aquatic phase population,  $\bar{D}$  is the diffusion term,  $\bar{\nu}$  is the advection term,  $\bar{\mu}_i$  is the mortality rate ( $i = 1$  winged phase;  $i = 2$  aquatic phase),  $\bar{k}_i$  is the carrying capacity of each phase,  $\bar{\gamma}$  is the specific rate of maturation of the aquatic phase into winged phase and  $\bar{r}$  is the oviposition rate of the female mosquitoes. Following [3], the model can be written in dimensionless form:

$$\begin{cases} M_t &= M_{xx} - 2\nu M M_x + \gamma A(1 - M)/k - \mu_1 M, \\ A_t &= k(1 - A)M - (\mu_2 + \gamma)A, \end{cases} \quad (2)$$

where the partial derivatives in  $M$  are denoted as  $\partial M / \partial x_j = M_{x_j}$ , same for  $A$ .

### 3. Traveling wave solution

In this section we investigate whether System (2) possesses a traveling wave solution, following the same steps of [6]. We change coordinates from  $(x, t)$  to  $(\xi, t)$ , where  $\xi = x - ct$  is called as traveling variable with constant propagation speed  $c$ . Following [7] we look for stationary solution in variable  $(\xi)$  as  $M(x, t) = m(\xi)$  and  $A(x, t) = a(\xi)$ , where  $m(\xi)$  and  $a(\xi)$  correspond to wave profiles of winged and aquatic mosquitoes population densities, respectively. Performing traveling coordinates substitution, we rewrite System (2) as a system of ODEs:

$$\begin{cases} m'(\xi) &= h(\xi) \\ h'(\xi) &= (2\nu m(\xi) - c)h(\xi) + (\mu_1 + \gamma a(\xi)/k)m(\xi) - \gamma a(\xi)/k \\ a'(\xi) &= k(a(\xi) - 1)m(\xi)/c + (\mu_2 + \gamma)a(\xi)/c, \end{cases} \quad (3)$$

where prime indicates derivative in  $\xi$ . Solving the system  $(m', h', a') = (0, 0, 0)$ , we find two stationary solutions (also called equilibria) of System (3)

$$A = (0, 0, 0), B = (m^*, 0, a^*), \text{ where } a^* = \frac{k(\gamma - \mu_1\mu_2 - \gamma\mu_1)}{\gamma(k + \mu_2 + \gamma)} \text{ and } m^* = \frac{\gamma - \mu_1(\mu_2 + \gamma)}{\gamma + k\mu_1}. \quad (4)$$

Here  $A$  represents the lack of both populations and  $B$  represents the maximum population in both phases.

The main goal consists in studying the existence of the traveling wave solution of System (3) in the form of heteroclinic orbit connecting equilibria  $A$  to  $B$  (or  $B$  to  $A$ ) in the following sense. The propagating profile corresponds to the solution connecting the region with no mosquitoes upstream the wave ( $\xi \rightarrow -\infty$ ) to the region with mosquitoes downstream the wave ( $\xi \rightarrow \infty$ ):

$$\lim_{\xi \rightarrow -\infty} (m(\xi), h(\xi), a(\xi)) = (0, 0, 0) \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} (m(\xi), h(\xi), a(\xi)) = (m^*, 0, a^*), \quad (5)$$

where  $h \rightarrow 0$  because  $m$  converges to constant state. The other case is analogous. In both cases constants  $m^*$  and  $a^*$  are given in Eq. (4). The matrices of derivatives of the flux in System (3) at equilibria  $A$  and  $B$  are

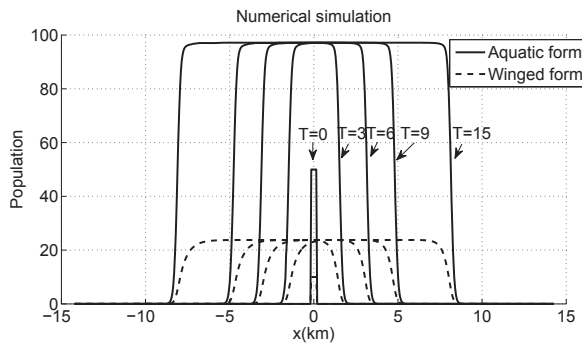
$$J(A) = \begin{bmatrix} 0 & 1 & 0 \\ \mu_1 & -c & -\frac{\gamma}{k} \\ -\frac{k}{c} & 0 & \frac{\mu_2 + \gamma}{c} \end{bmatrix}, \quad J(B) = \begin{bmatrix} 0 & 1 & 0 \\ \mu_1 + \frac{\gamma}{k}a^* & 2\nu(m^*) - c & \frac{\gamma}{k}(m^* - 1) \\ \frac{k}{c}(a^* - 1) & 0 & \frac{km^* + \mu_2 + \gamma}{c} \end{bmatrix}. \quad (6)$$

### 4. Numerical results

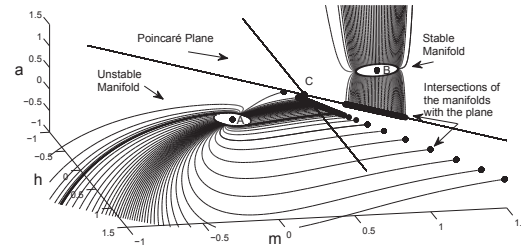
In this section we study numerically PDE System (1) and compare it with ODE System (3) using the algorithm of [6]. Both studies use same parameter values found in the literature. Dimensional parameter values are, [3],  $\bar{D} = 1.25 \times 10^{-2}$ ,  $\bar{\nu} = 5.0 \times 10^{-2}$ ,  $\bar{\gamma} = 0.2$ ,  $\bar{r} = 30$ ,  $\bar{k}_1 = 25$ ,  $\bar{k}_2 = 100$ ,  $\bar{\mu}_1 = 4.0 \times 10^{-2}$ ,  $\bar{\mu}_2 = 1.0 \times 10^{-2}$ , where space ( $\bar{x}$ ) is measured in  $km$  and time ( $\bar{t}$ ) in days. Dimensionless parameter values are, [3],  $\nu = 8.164 \times 10^{-2}$ ,  $\gamma = 6.66 \times 10^{-3}$ ,  $k = 2.5 \times 10^{-1}$ ,  $\mu_1 = 1.33 \times 10^{-3}$ ,  $\mu_2 = 3.33 \times 10^{-4}$ .

#### 4.1. Direct numerical simulation

System (1) is solved numerically using Crank-Nicolson finite difference scheme. We set initial data for the variables  $M$  and  $A$  at time  $t = 0$  as  $M_0 = 10$  and  $A_0 = 50$ . Dirichlet boundary conditions are considered for both left and right sides. Applying numerical scheme we obtain the solution of System (1) for times  $t = 0, 3, 6, 9, 15$  (days), see Fig. 1. This figure shows evidence that the solution of System (1) contains two traveling waves propagating forward and backward. From the biological point of view these results indicate the invasion phenomena in the population dynamics of *Aedes aegypti* with one profile propagating along the flow (from left to right with speed  $C_{al} = 0.2776 \text{ km/day}$  and dimensionless speed  $c_{al} = 0.4534$ ) and another profile propagating against the flow (from right to left with speed  $C_{ag} = -0.2775 \text{ km/day}$  and dimensionless speed  $c_{ag} = -0.4531$ ). This observation motivates the rigorous search for traveling wave solutions.



**Figure 1.** Numerical solution of System (1) for times  $t = 0, 3, 6, 9, 15$  days, using the dimensional parameter values.



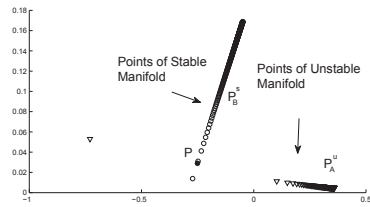
**Figure 2.** Numerical integration starts at small circles contained in  $E_A^u$  and  $E_B^s$  centered at equilibria  $A$  and  $B$ , respectively. Poincaré plane  $\pi$  is indicated by two straight lines; it intersects unstable manifold  $W_A^u$  and stable manifold  $W_B^s$  in points indicated by stars (\*).

#### 4.2. Semi-analytical method: invasion against the flow

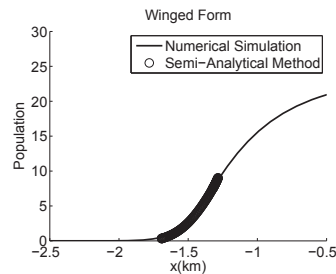
In this section we follow the steps of the semi-analytical method described in [6] in order to verify the existence of the traveling wave solution of System (3), i.e., the heteroclinic orbit connecting equilibrium  $A$  to equilibrium  $B$  (boundary conditions given by Eq. (5)). The algorithm for the orbit connecting  $B$  to  $A$  is analogous. We consider dimensionless parameter values given in Section 4 (introduction) with  $c_{ag} = -0.4531$  obtained in the numerical simulation.

- (i) Equilibria solutions of System (3) are  $A = (0, 0, 0)$  and  $B = (0.951, 0, 0.971)$  with corresponding matrices of derivatives given by  $J(A)$  and  $J(B)$ , Eq. (6).
- (ii)
  - $J(A)$  possesses two eigenvalues with positive real part:  $\lambda_1 = 0.1999$  and  $\lambda_2 = 0.4743$ , with corresponding eigenvectors  $w_1$  and  $w_2$ . These eigenvectors define the unstable subspace  $E_A^u$  which approaches locally the unstable manifold  $W_A^u$ ;
  - $J(B)$  possesses two eigenvalues with negative real part:  $\lambda_3 = -0.0467$  and  $\lambda_4 = -0.5403$ , with corresponding eigenvectors  $w_3$  and  $w_4$ . These eigenvectors define the stable subspace  $E_B^s$  which approaches locally the stable manifold  $W_B^s$ .
- (iii) Using point  $C = (A + B)/2$  and the vector  $N = \overrightarrow{CB}$ , we defined the Poincaré plane  $\pi$ . The plane  $\pi$  intersects both stable and unstable manifolds transversally, see Fig. 2, because  $\zeta(C) \cdot N \neq 0$ , where  $\zeta(m, h, a) = (m', h', a')$  is given by Eq. (3).
- (iv) Numerical integration starts at circles contained in  $E_A^u$  and  $E_B^s$  centered at equilibria  $A$  and  $B$ , respectively. The intersection sets between integration curves and  $\pi$  are denoted by

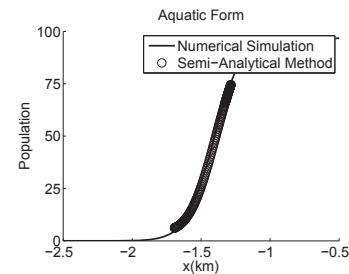
$P_A^u$  and  $P_B^s$ . The lines  $P_A^u$  and  $P_B^s$  intersect on the Poincaré plane at point  $P_A^u \cap P_B^s = P$ , see Fig. 3. This ensures the existence of the heteroclinic orbit connecting equilibrium  $A$  to equilibrium  $B$ .



**Figure 3.** Poincaré plane  $\pi$  from Fig. 2 indicating the intersection sets  $P_A^u$  ( $P_A^u \approx W_A^u \cap \pi$ ) and  $P_B^s$  ( $P_B^s \approx W_B^s \cap \pi$ ). Point  $P$  represent the intersection between both manifolds.



**Figure 4.** Against the flow invasion profiles for the winged phase obtained by using semi-analytical method and direct numerical simulation.



**Figure 5.** Against the flow invasion profiles for the aquatic phase obtained by using semi-analytical method and direct numerical simulation.

- (v) In order to validate the proposed semi-analytical method we compare the solution profile obtained by the proposed method with the one from direct numerical simulation for the winged phase and aquatic phase, see Figs. 4 and 5. The small numerical difference between profiles observed on Figs. 4 and 5 are due to diffusion effect of the numerical scheme and the radius of the small circles used for starting the integration.

## 5. Conclusions

In this work we complete the results presented in [5] showing that both waves appearing in the numerical simulation are traveling waves. Our semi-analytical method verified the existence of traveling wave profiles appearing in the system describing population dynamics of *Aedes aegypti* mosquitoes. The results were validated through numerical simulations.

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## References

- [1] Carneiro L and Travassos L 2016 *Microbes and Infect.*
- [2] Dias W O, Wanner E F and Cardoso R T 2015 *Math. Biosci.* **269** 37–47
- [3] Takahashi L, Maidana N, Ferreira Jr W, Pulino P and Yang H 2005 *B. Math. Biol.* **67** 509–528
- [4] Maidana N and Yang H 2008 *Math. Biosci.* **215** 64–77
- [5] Freire I and Torrisi M 2013 *Nonlinear Anal.-Real* **14** 1300 – 1307
- [6] Yamashita W, Takahashi L and Chapiro G 2016 *Manuscript submitted*
- [7] Volpert A, Volpert V and Volpert V 2000 *Traveling Wave Solutions of Parabolic Systems (Translations of mathematical monographs vol 140)* (American Mathematical Society) ISBN 9780821811436