Logistic systems with linear feedback

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Abstract. A wide variety of systems may be described by specific dependence, which is known as logistic curve, or S-curve, between the internal characteristic and the external parameter. Linear feedback between these two values may be suggested for a wide set of systems also. In present paper, we suggest a bifurcation behavior for systems with both features, and discuss it for two cases, which are the Ising magnet in external field, and the development of manufacturing enterprise.

1. Introduction

Many systems, which are described in terms of external variable x and internal parameter y, may be characterized by specific dependence $y(x)$ known as logistic function, or S-function [1]:

$$y = S(x) = \frac{M\exp\{\alpha(x-C)\}+m}{\exp\{\alpha(x-C)\}+1}. \tag{1}$$

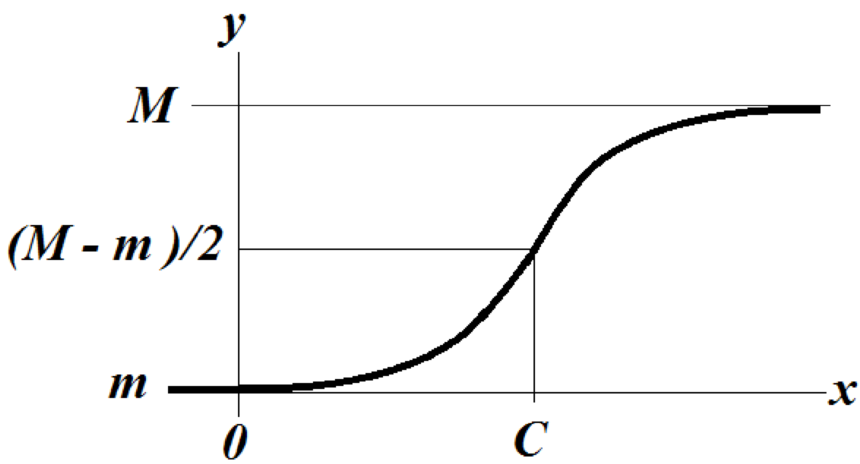

Fig.1. S-curve and its parameters.

This function arises for a system, which internal parameter y varies with x from its minimal value m at $x \to -\infty$ to the maximal M at $x \to +\infty$. The "velocity" of this varying is proportional to the deviation of y from its extreme values:

$$\frac{dy}{dx} = \alpha(y-m)(M-y), \tag{2}$$

where α is the coefficient of the proportionality. Function (1) is the solution of differential equation (2), and C is the integration constant, which corresponds to the choice of the x origin. The graph of the function (1) is presented in fig.1. S-curve describes population growth [2,3], technological transformations [4], dependence of the concentration on chemical potential [5], etc. In the present article, we discuss systems, in which linear feedback between the applied external variable and the internal parameter may be suggested. In the next section, we describe common features of such feedback, and in subsequent sections, we concern some cases from physics and economics.

2. Linear feedback and bifurcation.

Linear feedback implies that external effect described by the variable x is formed in accordance with the system behavior characterized by the parameter y:

$$x = ay + b. \tag{3}$$

Thus, the system has to correspond to those (x, y) that arise from the solution of equation

$$\frac{1}{a}x - \frac{b}{a} = \frac{M\exp\{\alpha(x-C)\}+m}{\exp\{\alpha(x-C)\}+1} \tag{4}$$

Depending on the a, b values, this equation may have 1,2 or 3 solutions, and that cases are presented in fig.2, where the graphical solution of eq.(4) is presented.

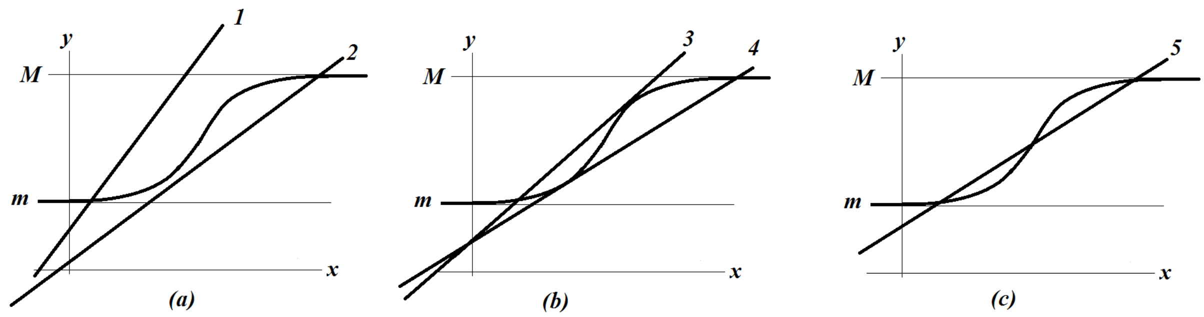

Fig.2. (a)-single solution of eq.(4), (b) – two solutions, (c) – three solutions.

Thus, varying a and b, one can produce a transformation of the system from the position near $y \cong m$ (fig.2a, line 1) to the $y \cong M$ (fig.2a, line 2). The sequence of the left – side (straight line) positions of eq.(4) is 1 – 3 – 5 – 4 – 2. The stability of a certain solution may by estimated in terms of iteration procedure, which reproduces the system behavior: initial external impact x_0 results in system response $S(x_0)$, then the next – step value may be presented as $x_1 = aS(x_0) + b$, etc. Thus, one gets the iteration procedure

$$x_{n+1} = aS(x_n) + b \ . \tag{5}$$

From the other hand, procedure (5) reproduces Seidel method for eq.(4) numerical solution. With respect to (5), fig1(a) demonstrates two single stable points, fig.1(c) – two stable points, separated by the unstable one. Fig.1(b) shows the appearance of a couple of stable and unstable points, i.e. to the bifurcation that corresponds to the cusp catastrophe [6]. The cusp point in the (a,b) plane has the coordinates

$$a_c = \frac{4}{\alpha(M-m)}, \qquad b_c = C - \frac{2(M+m)}{\alpha(M-m)} . \tag{6}$$

3. Case 1: Ising model in the mean – field approximation.

The Ising model describes magnet in following terms (see, for example, [7]). Consider certain space lattice, in sites of which unit spins are placed. With respect to the applied external magnetic field h, each spin on site r may have only two projections $s(r) = \pm 1$ on its direction. Neighboring spins interact with each other, so that the energy of the system is

$$H = -\frac{1}{2}\sum_{r,r'} Js(r)s(r') - h\sum_{r} s(r) . \tag{7}$$

In the first term, summation goes over all pairs of nearest neighbors r, r'. Here, J is so called exchange integral, which is the gain in energy for neighboring spins to have equal orientations. For ferromagnets, J is positive. The second term is the energy of spins interaction with external field. To proceed with the thermodynamics of the system, one has to calculate the partition function

$$Z(T,h) = \sum_{\{s(r)\}} \exp\left\{-\frac{H}{T}\right\} , \tag{8}$$

where the summation goes over all possible configurations $\{s(r)\}$. Exact calculation in 3-dimensional space is still unsolved mathematical task. The most useful approximation is the mean – field approximation, which looks as follows. Consider single site surrounded by neighbors with mean spin projection value σ. For this site, the energy is

$$H(s) = -(\frac{\nu J}{2}\sigma + h)s , \qquad s = \pm 1, \tag{9}$$

where ν is the number of nearest neighbors. Instead of (8), one gets single site statistics with only two possible configurations. The partition, and the mean value $< s >$ can be easily calculated:

$$Z = \exp\left\{-(\frac{\nu J}{2T}\sigma + \frac{h}{T})\right\} + \exp\left\{+(\frac{\nu J}{2T}\sigma + \frac{h}{T})\right\} , \tag{10}$$

$$< s > = \frac{1}{Z}\left[\exp\left\{+(\frac{\nu J}{2T}\sigma + \frac{h}{T})\right\} - \exp\left\{-(\frac{\nu J}{2T}\sigma + \frac{h}{T})\right\}\right] \tag{11}$$

Obvious self-consistency condition $< s > = \sigma$ leads to the equation

$$\frac{T}{\nu J}x-\frac{2h}{\nu J}=\frac{\exp(x)-1}{\exp(x)+1}\ , \tag{12}$$

where

$$x=\frac{\nu J}{T}\sigma+\frac{2h}{T}\quad . \tag{13}$$

Equation (12) coincides with (4) at following parameters

$$M=+1,\ m=-1,\ a=\frac{\nu J}{T},\ b=\frac{2h}{T},\ \ C=0\,. \tag{14}$$

Thus, one can consider the Ising magnet as an example of logistic system with linear feedback. The cusp catastrophe corresponds to the hysteresis loop. Relations (6) give for the the cusp point

$$h_c=0\,,\ T_c=\frac{J\nu}{2}\ \ , \tag{15}$$

which is exactly the Curie point of second order phase transition.

4. Case 2: Manufacturing enterprise.

In this section, we discuss an enterprise in terms of its sales proceeds y and working capital (current assets) x. For an arbitrary case, the dependence $y(x)$ may be characterized by

$$\frac{dy}{dx}=c_0+c_1y-c_2y^2\,. \tag{16}$$

This dependence implies that at small y, the rate of its growth with respect to x is a linear function c_0+c_1y with positive c_0,c_1. At larger y, the quadratic term $-c_2y^2$, $c_2>0$ eliminates the rate of growth. Equation (16) suggests a general shape of dependence of the rate of y growth on its absolute value. From the other hand, (16) coincides with (2) after following notations

$$\alpha=c_2\,,\ M=\frac{c_1}{2c_2}+\sqrt{\left(\frac{c_1}{2c_2}\right)^2+\frac{c_0}{c_2}}\ \ ,\ \ m=\frac{c_1}{2c_2}-\sqrt{\left(\frac{c_1}{2c_2}\right)^2+\frac{c_0}{c_2}} \tag{17}$$

Thus, S-function (1) describes general dependence of sales proceeds on current assets. The integration constant C should be chosen as to fulfill obvious condition $y(0)=0$. It always may be done, because in (17) $m<0$. From the other hand, any enterprise may be described in terms of the resources theory introduced by Barney [8,9]. According to the theory, every enterprise resource contributes into several features, which characterize the enterprise with respect to its Competitive Advantage. The features are V (valuable, i.e. can bring sales proceeds), R (rare, i.e. has low direct competition), I (imperfectly imitable, i.e. satisfies the need that cannot be satisfied by other products, or has low indirect competition), and O (organization, which characterizes technological complexity). These features form so called VRIO criteria for a manufacturing enterprise. The VRIO criteria correlate with the $y(x)$ S-dependence rather well (see fig.3). For the manufacturing cycle, current assets arise from the sales products of the previous cycle, and from the external funding. Thus, one gets the iteration procedure (5), where $0<a<1$ is the dimensionless coefficient of refinancing, and b is the external funding.

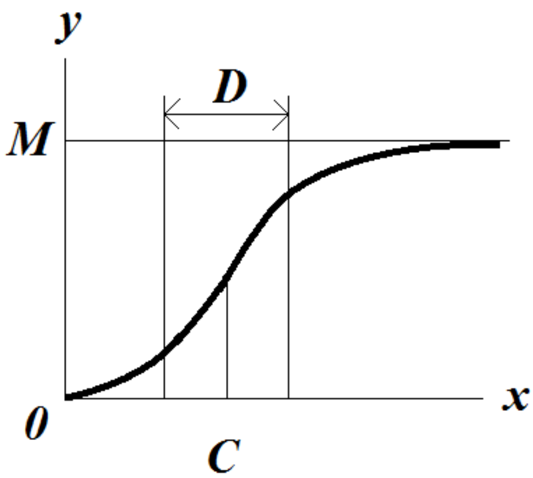

Fig.3. The dependence of sales proceeds on current assets according to (16,17). The maximal sales proceeds M corresponds to the V-feature, C shows the minimal assets that are necessary to get visible proceeds (O-feature), and the curve width D characterizes direct and indirect competition (R, I features). Thus, the VRIO criteria fit the $y(x)$ curve well.

Consider the case $b = 0$. Possible positions of the feedback line with respect to the S-curve are shown in fig.4. Line 1 corresponds to the small a coefficient, and there is only single stable zero point solution. Line 3 corresponds to the large coefficient of refinancing, but one has to remember that there is an upper limit $0 < a < 1$. The most probable situation corresponds to the line 2, which deals with the cusp catastrophe. If the initial cycle funding $x_0 < x_0^c$, then the iteration procedure (5) leads to the zero solution, i.e. the enterprise fails. When $x_0 > x_0^c$, the process gets into the attraction domain of the upper solution, and the enterprise becomes successful. Fig.4. allows one to formulate "startup rule": the current assets should rise after the very first production cycle.

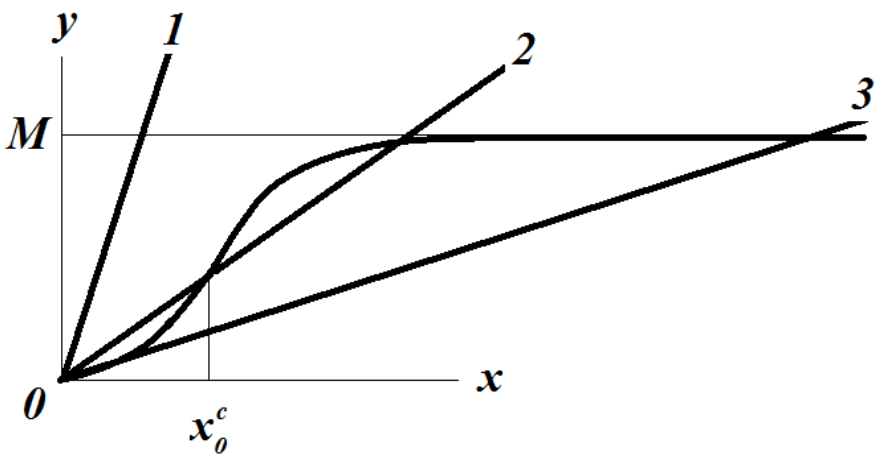

Fig.4. Manufacturing enterprise as S-system with linear feedback. Lines 1,2,3 correspond to the enlarging coefficient of refinancing.

5. Conclusion

We considered two cases from very different areas of science. The cases demonstrate wide universality of simple mathematical construction, which combines S-dependence of the internal variable on the external one, and the linear feedback. The combination results in the cusp catastrophe, and may be suggested for a wide variety of nonlinear systems.

6. Acknowledgements

The work was supported by Russian Ministry of Science and Education (Government job 2014/392 projects 2391 and 1177), and by Goverment of the Russian Federation, act 211 contract № 02.A03.21.0006.

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