

# Soliton solutions of the (2+1)-dimensional complex modified Korteweg-de Vries and Maxwell-Bloch equations

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## **Abstract.**

In this paper, we consider the (2+1)-dimensional complex modified Korteweg-de Vries and Maxwell-Bloch (cmKdVMB) equations. Lax pairs of cmKdVMB equations are presented. Using the Lax pair, we construct a Darboux transformation and namely one-fold transformations. The soliton solutions are obtained from the different "seeds" by using this Darboux transformation.

## **1. Introduction**

The theory of nonlinear partial differential equations has attracted a lot of attention among researcher and is fundamentally linked to several basic developments in this area of soliton theory. It is well known that nonlinear equations such as the Korteweg-de Vries (KdV) equation, modified Korteweg-de Vries (mKdV) equation and the nonlinear Schrodinger (NLS) equation are the most typical and well-studied integrable evolution equations which describe nonlinear wave phenomena for a range of dispersive physical systems. In the study of nonlinear waves their solutions play an important role [1-12]. One of the generalizations of the mKdV equation is complex mKdV (cmKdV) equation which is one of the well-known and completely integrable equations in soliton theory. It possesses all the basic characters of integrable models. From a physical point of view cmKdV equation has been derived for, e.g. the fundamental evolution of nonlinear lattice, fluid dynamics, plasma physics, ultra-short pulses in nonlinear optics, nonlinear transmission lines and so on [13-14].

At the present time, the cmKdV equation is used in pair with Maxwell-Bloch system of equations, and so it called as complex modified Korteweg-de Vries and Maxwell-Bloch (cmKdVMB) equations. Moreover, this equation can be received by the reduction of the Hirota-Maxwell-Bloch (HMB) system of equations. In (1+1)-dimensions cmKdVMB equations were studied in [15-17] by the reduction of HMB system of equations.

In this work, we consider (2+1)-dimensional cmKdVMB equations which were suggested in [18]. Here we use the method of Darboux transformation. It has been proved to be an efficient way to find the different solutions from "seed" solutions this transformation allows to construct non-trivial analytical solutions of nonlinear partial differential equations [19-23].



The paper is organized as follows. The (2+1)-dimensional cmKdVMB equations we present in Section 2. In Section 3, we construct the DT for the (2+1)-dimensional cmKdVMB equations. In Section 4, using the constructed one-fold DT, the one-soliton solutions of the (2+1)-dimensional cmKdVMB equations is given.

## 2. The (2+1)-dimensional complex modified Korteweg de Vries and Maxwell-Bloch equations

The complex modified Korteweg-de Vries and Maxwell-Bloch (cmKdVMB) equations reads as [18]

$$iq_t + iq_{xxy} - vq + i(wq)_x - 2ip = 0, \quad (1)$$

$$v_x - 2i\delta(q_{xy}^*q - q^*q_{xy}) = 0, \quad (2)$$

$$w_x - 2\delta(|q|^2)_y = 0, \quad (3)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (4)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (5)$$

where  $q, p$  are complex functions,  $v, w, \eta$  are real functions and  $\omega, \delta$  are real constants. This set of equations (1)-(5) is integrable by IST. The Lax pair corresponding to the cmKdVMB equations is given by [18], i.e.

$$\Psi_x = A\Psi, \quad (6)$$

$$\Psi_t = 4\lambda^2\Psi_y + B\Psi, \quad (7)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (8)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}, \quad (9)$$

with

$$B_1 = iw\sigma_3 + 2i\sigma_3 A_{0y}, \quad (10)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (11)$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -q_{xy} - wq \\ r_{xy} + wr & 0 \end{pmatrix}, \quad (12)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix} \quad (13)$$

and  $r = \delta q^*$ ,  $k = \delta p^*$ , where  $\delta = \pm 1$ . The compatibility condition of (6)-(7) is still

$$A_t - 4\lambda^2 A_y - B_x + [A, B] = 0. \quad (14)$$

## 3. DT for the (2+1)-dimensional cmKdVMB equations

In this section, we construct the DT for the (2+1)-dimensional cmKdVMB equations (1)-(5). In particular, we give in detail the one-fold DT and briefly the  $n$ -fold DT.

### 3.1. One-fold DT

Let  $\Psi$  and  $\Psi^{[1]}$  are two solutions of the system (6)-(7) so that

$$\Psi_x^{[1]} = A^{[1]}\Psi^{[1]}, \quad (15)$$

$$\Psi_t^{[1]} = 4\lambda^2\Psi_y^{[1]} + B^{[1]}\Psi^{[1]}. \quad (16)$$

We assume that these two solutions are related by the following transformation:

$$\Psi^{[1]} = T\Psi = (\lambda I - M)\Psi. \quad (17)$$

The matrix function  $T$  obeys the following equations

$$T_x + TA = A^{[1]}T, \quad (18)$$

$$T_t + TB = 4\lambda^2T_y + B^{[1]}T. \quad (19)$$

Then the relation between  $q, p, v, w, \eta$  and  $q^{[1]}, p^{[1]}, v^{[1]}, w^{[1]}, \eta^{[1]}$  can be reduced from (18)-(19), which is the Darboux transformation of cmKdVMB equations. Comparing the coefficient of  $\lambda^i$  ( $i = 0, 1, 2$ ) of the sides the equation (18), we have

$$\lambda^0 : M_x = A_0^{[1]}M - MA_0, \quad (20)$$

$$\lambda^1 : A_0^{[1]} = A_0 + i[M, \sigma_3], \quad (21)$$

$$\lambda^2 : iI\sigma_3 = i\sigma_3I. \quad (22)$$

The equation (21) gives

$$q^{[1]} = q - 2im_{12}, \quad (23)$$

$$q^{*[1]} = q^* - 2im_{21}, \quad (24)$$

where

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (25)$$

Hence we get  $m_{21} = -m_{12}^*$  in our attractive interaction case that is if  $\delta = +1$ . After comparing the coefficient of  $\lambda^i$  ( $i = 0, 1, 2$ ) the equation (19) gives us the following relations

$$\lambda^0 : -M_t = iB_{-1}^{[1]} - B_0^{[1]}M - iB_{-1} + MB_0, \quad (26)$$

$$\lambda^1 : B_0^{[1]} = B_0 - MB_1 + B_1^{[1]}M, \quad (27)$$

$$\lambda^2 : 4M_y = B_1^{[1]} - B_1, \quad (28)$$

$$(\lambda + \omega)^{-1} : 0 = -i\omega B_{-1}^{[1]} - iB_{-1}^{[1]}M + i\omega B_{-1} + iMB_{-1}. \quad (29)$$

Hence we get the DT

$$B_0^{[1]} = B_0 - MB_1 + (B_1 + 4M_y)M, \quad (30)$$

$$B_1^{[1]} = B_1 + 4M_y, \quad (31)$$

$$B_{-1}^{[1]} = (M + \omega I)B_{-1}(M + \omega I)^{-1}. \quad (32)$$

At the same, from the equation (30)-(31) we get

$$v^{[1]} = v + 4(m_{12}q_y^* + m_{12}^*q_y + 2im_{11}m_{11y} - 2im_{12}^*m_{12y}), \quad (33)$$

$$w^{[1]} = w - 4im_{11y} = w + 4im_{22y} \quad (34)$$

and we additionally have  $m_{22} = m_{11}^*$ . So the matrix  $M$  has the form

$$M = \begin{pmatrix} m_{11} & m_{12} \\ -m_{12}^* & m_{11}^* \end{pmatrix}, \quad M^{-1} = \frac{1}{|m_{11}|^2 + |m_{12}|^2} \begin{pmatrix} m_{11}^* & -m_{12} \\ m_{12}^* & m_{11} \end{pmatrix}, \quad (35)$$

$$M + \omega I = \begin{pmatrix} m_{11} + \omega & m_{12} \\ -m_{12}^* & \omega + m_{11}^* \end{pmatrix}, \quad (M + \omega I)^{-1} = \frac{1}{\square} \begin{pmatrix} m_{11}^* + \omega & -m_{12} \\ m_{12}^* & \omega + m_{11} \end{pmatrix}, \quad (36)$$

where

$$\square = \det(M + \omega I) = \omega^2 + \omega(m_{11} + m_{11}^*) + |m_{11}|^2 + |m_{12}|^2. \quad (37)$$

The equation (32) gives

$$\eta^{[1]} = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (38)$$

$$p^{[1]} = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}, \quad (39)$$

$$p^{*[1]} = \frac{p^*(\omega + m_{11}^*)^2 - pm_{12}^{*2} + 2\eta m_{12}^*(\omega + m_{11}^*)}{\square}. \quad (40)$$

We now assume that

$$M = H\Lambda H^{-1}, \quad (41)$$

where

$$H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & \psi_1(\lambda_2; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_2(\lambda_2; t, x, y) \end{pmatrix} := \begin{pmatrix} \psi_{1,1} & \psi_{1,2} \\ \psi_{2,1} & \psi_{2,2} \end{pmatrix}. \quad (42)$$

Here

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (43)$$

and  $\det H \neq 0$ , where  $\lambda_1$  and  $\lambda_2$  are complex constants. The matrix  $H$  obeys the system

$$H_x = -i\sigma_3 H\Lambda + A_0 H, \quad (44)$$

$$H_t = 4H_y^2 + B_1 H\Lambda + B_0 H + B_{-1} H\Sigma, \quad (45)$$

where

$$\Sigma = \begin{pmatrix} \frac{i}{\lambda_1 + \omega} & 0 \\ 0 & \frac{i}{\lambda_2 + \omega} \end{pmatrix}. \quad (46)$$

In order to satisfy the constraints of  $M$  and  $B_{-1}^{[1]}$  as mentioned above, we first notes that

$$\Psi^+ = \Psi^{-1}, \quad A_0^+ = -A_0, \quad (47)$$

$$\lambda_2 = \lambda_1^*, \quad H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & -\psi_2^*(\lambda_1; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_1^*(\lambda_1; t, x, y) \end{pmatrix}, \quad (48)$$

$$H^{-1} = \frac{1}{\Delta} \begin{pmatrix} \psi_1^*(\lambda_1; t, x, y) & \psi_2^*(\lambda_1; t, x, y) \\ -\psi_2(\lambda_1; t, x, y) & \psi_1(\lambda_1; t, x, y) \end{pmatrix}, \quad (49)$$

where

$$\Delta = |\psi_1|^2 + |\psi_2|^2. \quad (50)$$

So the matrix  $M$  has the form

$$M = \frac{1}{\Delta} \begin{pmatrix} \lambda_1 |\psi_1|^2 + \lambda_2 |\psi_2|^2 & (\lambda_1 - \lambda_2) \psi_1 \psi_2^* \\ (\lambda_1 - \lambda_2) \psi_1^* \psi_2 & \lambda_1 |\psi_2|^2 + \lambda_2 |\psi_1|^2 \end{pmatrix}. \quad (51)$$

Finally we can write the one-fold DT for the (2+1)-dimensional cmKdVMB equations as:

$$q^{[1]} = q - 2im_{12}, \quad (52)$$

$$v^{[1]} = v + 4(m_{12}q_y^* + m_{12}^*q_y + 2im_{11}m_{11y} - 2im_{12}^*m_{12y}), \quad (53)$$

$$w^{[1]} = w - 4im_{11y} = w + 4im_{22y}, \quad (54)$$

$$\eta^{[1]} = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (55)$$

$$p^{[1]} = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}. \quad (56)$$

At last, we note that the expressions of  $m_{ij}$  can be rewritten in the determinant form as

$$m_{11} = \frac{\lambda_1 |\psi_1|^2 + \lambda_2 |\psi_2|^2}{\Delta} = \frac{\Delta_{11}}{\Delta}, \quad m_{12} = \frac{(\lambda_1 - \lambda_2) \psi_1 \psi_2^*}{\Delta} = \frac{\Delta_{12}}{\Delta}, \quad (57)$$

where

$$\Delta_{11} = \det \begin{pmatrix} \psi_1 & -\lambda_2 \psi_2^* \\ \psi_2 & \lambda_1 \psi_1^* \end{pmatrix}, \quad \Delta_{12} = -\det \begin{pmatrix} \psi_1 & \lambda_1 \psi_1 \\ \psi_2^* & \lambda_2 \psi_2^* \end{pmatrix}. \quad (58)$$

#### 4. Soliton solutions

Having the explicit form of the DT, we are ready to construct exact solutions of the (2+1)-dimensional cmKdVMB equations. As an example, let us present the one-soliton solution. To get the one-soliton solution we take the seed solution as

$$q = v = w = p = 0, \quad \eta = 1. \quad (59)$$

Then the corresponding associated linear system takes the form

$$\psi_{1x} = -i\lambda\psi_1, \quad (60)$$

$$\psi_{2x} = i\lambda\psi_2, \quad (61)$$

$$\psi_{1t} = 4\lambda^2\psi_{1y} + \frac{i}{\lambda + \omega}\psi_1, \quad (62)$$

$$\psi_{2t} = 4\lambda^2\psi_{2y} - \frac{i}{\lambda + \omega}\psi_2. \quad (63)$$

This system admits the following exact solutions

$$\psi_1 = e^{-i\lambda x + i\mu y + i(4\lambda^2\mu + \frac{1}{\lambda+\omega})t + \delta_1 + i\delta_2}, \quad (64)$$

$$\psi_2 = e^{i\lambda x - i\mu y - i(4\lambda^2\mu + \frac{1}{\lambda+\omega})t - \delta_1 - i\delta_2 + i\delta_0} \quad (65)$$

or

$$\psi_1 = e^{\theta_1 + i\chi_1}, \quad (66)$$

$$\psi_2 = e^{\theta_2 + i\chi_2}. \quad (67)$$

Here  $\mu = \eta_1 + i\nu_1$ ,  $\lambda = \alpha_1 + i\beta_1$ ,  $\delta_i$  are real constants then

$$\theta_1 = \beta_1 x - \nu_1 y - (8\eta_1\alpha_1\beta_1 + 4\nu_1(\alpha_1^2 - \beta_1^2) - \frac{\beta_1}{(\alpha_1 + \omega)^2 + \beta_1^2})t + \delta_1, \quad (68)$$

$$\chi_1 = -\alpha_1 x + \eta_1 y + (4\eta_1(\alpha_1^2 - \beta_1^2) - 8\nu_1\alpha_1\beta_1 + \frac{\alpha_1 + \omega}{(\alpha_1 + \omega)^2 + \beta_1^2})t + \delta_2 \quad (69)$$

and  $\theta_2 = -\theta_1$ ,  $\chi_2 = -\chi_1 + \delta_0$ . Then the one-soliton solution of the (2+1)-dimensional cmKdVMB equations (1)-(5) takes the form

$$q^{[1]} = 2\beta_1 e^{2i\chi_1 - i\delta_0} \operatorname{sech}[2\theta_1], \quad (70)$$

$$v^{[1]} = 16\beta_1(\alpha_1\nu_1 + \beta_1\eta_1) \operatorname{sech}^2[2\theta_1], \quad (71)$$

$$w^{[1]} = -8\beta_1\nu_1 \operatorname{sech}^2[2\theta_1], \quad (72)$$

$$\eta^{[1]} = \frac{(\omega + \alpha_1)^2 + \beta_1^2(1 - 2\operatorname{sech}^2[2\theta_1])}{(\omega + \alpha_1)^2 + \beta_1^2}, \quad (73)$$

$$p^{[1]} = \frac{2i\beta_1 e^{2i\chi_1 - i\delta_0}(\omega + \alpha_1 + i\beta_1 \tanh[2\theta_1])}{((\omega + \alpha_1)^2 + \beta_1^2) \cosh[2\theta_1]}. \quad (74)$$

Pictorial representations of one-soliton solutions (81)-(85) of the cmKdVMB equations is shown in figure 1 and figure 2

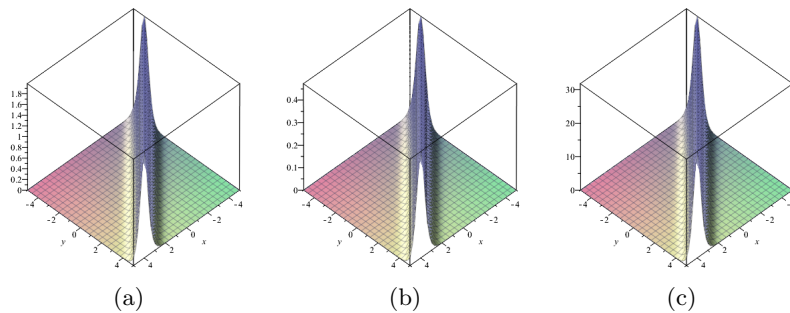


Figure 1: One-soliton solutions a)  $|q^{[1]}|$ , b)  $|p^{[1]}|$ , c)  $v^{[1]}$  of the cmKdVMB equations when  $\alpha_1 = 1, \beta_1 = 1, \omega = 1.5, \nu_1 = 1, \eta_1 = 1$

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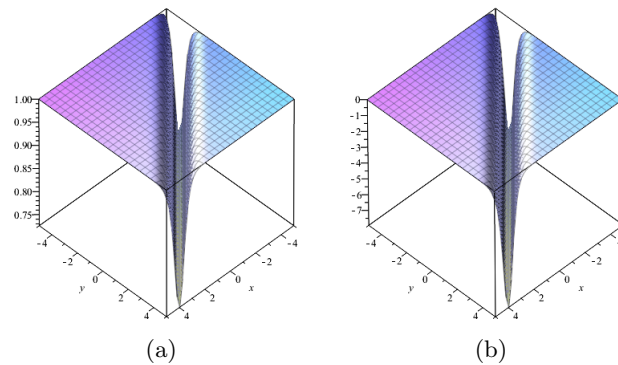


Figure 2: One-soliton solutions  $a)\eta^{[1]}, b)w^{[1]}$  of the cmKdVMB equations when  $\alpha_1 = 1, \beta_1 = 1, \omega = 1.5, \nu_1 = 1, \eta_1 = 1$

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