

Statistical Research for Probabilistic Model of Distortions of Remote Sensing

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Abstract. In this work the new multivariate discrete probability model of distribution of processes distortion of radiation from remote sensing data is proposed and studied. Research was performed on a full cycle adopted in mathematical statistics, namely, the model was constructed and investigated, various methods for estimating the parameters was proposed and test the hypothesis that the model adequacy observations, was considered.

1. Introduction

On the Earth's near-Earth orbit for the purpose of remote sensing of the surface today launched a variety of satellites (Terra, Aqua, Suomi NPP, etc.) with the sensors, which transmit to the monitoring station huge data flows (see [1], p. 28). Problems with their storage and their processing in real time makes very urgent to study the probability of information about distortions.

In general, the processing of data from the satellite may be viewed as a conveyor. This technology can be shown as partitioning of data by the time, by the sensor type and by range. It is known that incoming files are passing sequential processing of.

International Satellite System based on geostationary spacecraft is designed to meet the challenges of the global meteorological support and for consumers in the European, Asian and African regions. The management and maintenance work on the program is implemented by Eumetsat.

The remote sensing (RS) is an observation of the surface of the Earth aviation and space vehicles equipped with different types of imaging equipment. Wavelength range, which is receiving apparatus, is constitutes from a fraction of micrometer (visible optical radiation) to m (radio waves). The methods can be passive, that is to use a natural reflection or secondary heat radiation of objects on the Earth's surface due to solar activity, and active, these are using stimulated emission objects initiated by artificial light source directed action. The datas of RS, which is received from the spacecraft, is characterized by a large degree of depending on the transparency of the atmosphere. So on spacecraft equipment the multichannel passive and active types are used, which have recording of electromagnetic radiation in different ranges.

RS can be represented as a process by which information is gathered about an object, area or phenomenon without direct contact with him. Part of RS are entering in the digital form that allows you to use them for processing modern computer technology. Pictures on photo can be



converted into a digital format for a bitmap using special scanning devices (scanners) . Digital image in the form of a raster is a matrix of numbers.

RS contain a series of random, systemic and systematic distortions due to the influence of the atmosphere, the Earth's curvature, motion shooting the vehicle with respect to its surface at the time of the shooting, the physical characteristics of sensors used and communication channels. To eliminate the mentioned quite numerous distortions, taking into account their specific uses several types of correction: radiation, radiometric, and geometric calibration.

Objectives of the study of: a) submission of a probabilistic model processes distortion RS; b) the construction of unbiased estimates of the probability distribution of the model under investigation. All major results are new and are as follows: a) proposed and studied by multivariate discrete probability model for the distribution of sums of unobservable independent identically distributed random matrices; b) it is shown that the best unbiased estimate of the probability of this distribution exist only if the uniqueness of the partition of the observed matrix of all possible components. c) if some element of the implementation of the sample has more than one partition on the possible components, then we construct a set of unbiased estimators studied for distribution; d) introduce the concept of the most appropriate estimate of the set of unbiased, which has good asymptotic properties.

The results and methods presented in the paper can be used in research specialists in Mathematical Statistics and in Space Monitoring .

Telemetry data transmitted from the Board of the satellite at a frequency of 1675.928 MHz. To send telemetry (see [2]) using directional antenna S-band high-gain, and at the stage of withdrawal satellite into orbit - directional antenna S-range frequencies. Command information is taken at a frequency of 2098.0 MHz using a directional antenna S-band high-gain.

On the energy characteristics of the radio transmission of data in HR with the IRS series of Meteosat satellites (see [3], p. 205) affect the EIIM, loss of distribution losses in the antenna pointing (± 1), gain receiving antenna with a mirror diameter of 4 m , losses in the feeder lines, return loss, received power $A + 2 + 3 + 4 + 5 + 6$, modulation loss, the effective received signal $G + 8$, the noise temperature of antenna elevation angle 30° , the noise temperature, system noise temperature $A0 + 11$, noise spectral density $198.6 + A2$, data transmission speed 166 kbit/second, the total noise power of the $A3 + A14$ and so on. In other words, the distortion of the influence factor of 23, ie, $d = 23$.

2. Multivariate discrete probability distribution processes remote sensing

Consider a probabilistic model of the processes of energy characteristics of radio satellite IRS Meteosat. In [4], a probabilistic-statistical model of weather prediction.

Assume that the true image can be represented as a matrix $\mathbf{l}_0 = \|l_{0,i,j}\|_{m \times q}$, which imposed distortion, consisting of four factors (matrices) of losses $\mathbf{u} = \|u_{i,j}\|_{m \times q}$, taking values from the set of $\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_d$.

Obviously, the factors (the matrix), the loss $\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_d$ are realizations of random matrices $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$, which appear relevant probabilities $\mathbf{p} = (p_1, \dots, p_d)$, where

$$\sum_{\alpha=1}^d p_{\alpha} = 1.$$

Assume that $V_{\mathbf{u}}$ is the number of possible combinations $r_{1,v_{\mathbf{u}}}\mathbf{L}_1, \dots, r_{d,v_{\mathbf{u}}}\mathbf{L}_d$, which together form a matrix of \mathbf{u} , where $r_{1,v_{\mathbf{u}}}, \dots, r_{d,v_{\mathbf{u}}}$ determine the possible number of balls taken out, which marked the relevant matrices $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$. In other words, from [5] it follows that $V_{\mathbf{u}}$ is the number of partitions on the part of the matrix \mathbf{u} on $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$.

Theorem 1. *The probability that the distortion takes the value u is determined by the formula*

$$P(\mathbf{U} = \mathbf{u}) = \sum_{v_{\mathbf{u}}=1}^{V_{\mathbf{u}}} n! \prod_{\alpha=1}^d \frac{r_{\alpha v_{\mathbf{u}}}}{r_{\alpha v_{\mathbf{u}}}!}. \quad (1)$$

3. Unbiased estimation of the probability distribution of the proposed model

In practice, as a rule, elements of the vector $\mathbf{p} = (p_1, \dots, p_d)$ are not known. It is also not known matrix $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$. Consequently, formula (1) does not find the actual application.

Assume that there are photos in the number of k particular locality with the distortions $\mathbf{x} = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$. In other words, a number of evidence- \mathbf{x} can be interpreted as a realization of a sample of k , whose elements are subject to distribution (1). We denote $\mathbf{r}_{v_{\beta}}$ vector $(r_{1v_{\beta}}, \dots, r_{dv_{\beta}})$, which defines v_{β} - th solution of equation

$$\begin{cases} \sum_{\alpha=1}^d L_{\alpha} r_{\alpha v_{\beta}} = \mathbf{u}, \\ \sum_{\alpha=1}^d r_{\alpha v_{\beta}} = n, \end{cases} \quad (2)$$

, where $v_{\beta} = 1, \dots, V_{\beta}$, V_{β} - the number of partitions of the matrix \mathbf{x}_{β} on the matrices $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$. Using the system of equations (3), the matrices $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$, and the actual data \mathbf{x} , we define for each $\beta = 1, \dots, k$ the number of partitions V_{β} matrix \mathbf{x}_{β} at $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$, and vectors $\mathbf{r}_{1\beta}, \dots, \mathbf{r}_{V_{\beta}}$.

Suppose that for each $j = 1, \dots, \mu$, where

$$\mu = \prod_{\beta=1}^k V_{\beta},$$

there is a vector $\mathbf{z}_j = (z_{1j}, \dots, z_{dj})$, defined as

$$\mathbf{z}_j = \sum_{\beta=1}^k \mathbf{r}_{v_{\beta}}, \quad (3)$$

and the indices on the right and left side are linked one-to-one correspondence, which is not unique.

Thus, from the above lemma that if some element of the implementation of the sample $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ of the distribution (1) has more than one partition on the submitted part, it is impossible, using the theorem Rao - Blackwell-Kolmogorov construct an unbiased estimate with minimum variance for the probability distribution (1).

Theorem 2. *The elements of $W(\mathbf{u}, \mathbf{z}) = W(\mathbf{u}, \mathbf{z}_1), \dots, W(\mathbf{u}, \mathbf{z}_{\mu})$ is an unbiased estimate for the probability $P(\mathbf{U} = \mathbf{u})$ distribution (1) that for $j = 1, \dots, \mu$ is defined as*

$$W(\mathbf{u}, \mathbf{z}_j) = \frac{\sum_{v_{\mathbf{u}}=1}^{V_{\mathbf{u}}} \prod_{\alpha=1}^d \binom{z_{\alpha j}}{r_{\alpha v_{\mathbf{u}}}}}{\binom{nk}{n}}, \quad (4)$$

where $V_{\mathbf{u}}$ - number of partitions on the part of the matrix \mathbf{u} $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$; for each partition $r_{1v_{\mathbf{u}}}, \dots, r_{dv_{\mathbf{u}}}$ determine the possible number of matrices $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_d$; $k \geq 1$ and $z_{\alpha j} \geq r_{\alpha v_{\mathbf{u}}}$, when $\alpha = 1, \dots, d$, $v_{\mathbf{u}} = 1, \dots, V_{\mathbf{u}}$.

4. The most suitable unbiased estimates for the probability distribution of the proposed model and their properties

Thus, we have a lot of unbiased estimates of the probability of distortion.

Definition 1. *Decision \mathbf{z}_g , based on observation, is the most appropriate set of $\mathbf{z} = \{\mathbf{z}_1, \dots, \mathbf{z}_m\}$, if*

$$\prod_{\beta=1}^k W(\mathbf{x}_\beta, \mathbf{z}_g) = \max_{j=1, \dots, \mu} \prod_{\beta=1}^k W(\mathbf{x}_\beta, \mathbf{z}_j), \quad (5)$$

where for $\beta = 1, \dots, k$ elements of $W(\mathbf{x}_\beta, \mathbf{z}) = \{W(\mathbf{x}_\beta, \mathbf{z}_1), \dots, W(\mathbf{x}_\beta, \mathbf{z}_\mu)\}$ is an unbiased estimate for the probability $P(\mathbf{U} = \mathbf{u})$ distribution (1) defined in (5).

Definition 2. *Unbiased estimate of $W(\mathbf{x}_\beta, \mathbf{z}_g)$ for the probability $P(\mathbf{U} = \mathbf{u})$ distribution (1) is the most suitable from the entire set of unbiased estimates of $W(\mathbf{x}_\beta, \mathbf{z}) = \{W(\mathbf{x}_\beta, \mathbf{z}_1), \dots, W(\mathbf{x}_\beta, \mathbf{z}_\mu)\}$ defined in (5), if \mathbf{z}_g - the most appropriate solution, based on observation.*

Theorem 3. *The most suitable unbiased estimate of $W(\mathbf{x}_\beta, \mathbf{z}_g)$ for the probability $P(\mathbf{U} = \mathbf{u})$ model (1) is consistent, asymptotically normal and asymptotically efficient.*

5. Statistical hypothesis testing

Consider construction of the chi-square test of the hypothesis

$$H_0 = \{\mathbf{U} \sim P(\mathbf{U} = \mathbf{u})\}.$$

Using the simple's realization

$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$$

of volume k for any

$$\mathbf{u} \in \Omega = \{\mathbf{u}\},$$

where Ω is the space of elementary events of model (1), define the following random variables for $k = 1, \dots, k$

$$\delta_{\mathbf{u}}(\mathbf{x}_\beta) = \begin{cases} 1, & \text{if } \mathbf{u} = \mathbf{x}_\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\delta_{\mathbf{u}}(\mathbf{x}) = \sum_{\beta=1}^k \delta_{\mathbf{u}}(\mathbf{x}_\beta)$$

is the frequency of the matrix

$$\mathbf{u} \in \Omega = \{bfu\}$$

from simple's realization

$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k).$$

Divide the set Ω on

$$\lambda \geq 2$$

sets by the following form

$$\Omega = \{\Omega_1, \dots, \Omega_\lambda\},$$

where

$$\bigcap_{i=1}^{\lambda} \Omega_i = \Omega$$

and for any $i_1 = 1, \dots, \lambda$, $i_2 = 1, \dots, \lambda$, if $i_1 \neq i_2$ then

$$\Omega_1 \Omega_2 = \emptyset.$$

Define the vector

$$\zeta = (\zeta_1, \dots, \zeta_\lambda)^T,$$

where for $i = 1, \dots, \lambda$

$$\zeta_i = \sum_{\mathbf{u} \in \Omega_i} \delta_{\mathbf{u}}.$$

Introduce the vector

$$\mathbf{B} = (B_1, \dots, B_\lambda)^T,$$

where for $i = 1, \dots, \lambda$

$$B_i = \sum_{\mathbf{u} \in \Omega_i} P(\mathbf{U} = \mathbf{u})$$

and

$$P(\mathbf{U} = \mathbf{u})$$

is the probability of distributuion (1). Define the vector

$$\mathbf{Y} = \frac{1}{\sqrt{k}}(\zeta_1 - A_1, \dots, \zeta_\lambda - A_\lambda),$$

where for $i = 1, \dots, \lambda$

$$A_i = k B_i.$$

From the construction in accordance with the central limit Theorem Lindeberg - Levy (see, for example, [13]) it follows that in the case of validity of the hypothesis

$$H_0 = \{\mathbf{u} \sim P(\mathbf{U} = \mathbf{u})\}$$

we have for $k \rightarrow \infty$ thr vector Υ has the asymptotically normal distribution

$$N(\mathbf{0}_\lambda, \Sigma),$$

where $\mathbf{0}_\lambda$ is the zero vector of dimension λ and the matrix Σ is defined as

$$\Sigma = \mathbf{D} - \mathbf{B}\mathbf{B}^T,$$

$$\mathbf{D} = \begin{vmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_\lambda \end{vmatrix}_{\lambda \times \lambda}$$

and

$$\mathbf{B}\mathbf{B}^T = \begin{vmatrix} B_1 B_1 & B_1 B_2 & \dots & B_1 B_\lambda \\ B_2 B_1 & B_2 B_2 & \dots & B_2 B_\lambda \\ \dots & \dots & \dots & \dots \\ B_\lambda B_1 & B_\lambda B_2 & \dots & B_\lambda B_\lambda \end{vmatrix}_{\lambda \times \lambda}.$$

Thus, we have following Theorem.

Theorem 4. *If $k \rightarrow \infty$, quadratic form*

$$\Phi_k^2 = \Upsilon^T \Upsilon$$

is seek to chi-square distribution with $\lambda - d + 1$ degrees of freedom, ie

$$\lim_{k \rightarrow \infty} P\{\Phi_k^2 \geq x | H_0\} = P\{\chi_{\lambda-d+1}^2 \geq x\}.$$

Let's

$$\Upsilon = \frac{1}{\sqrt{k}(\zeta_1 - A_1, \dots, \zeta_{\lambda-1} - A_{\lambda-1})}$$

and the matrix Σ_0 is obtained if the matrix Σ strike last row and column. Based on the work of the [14] we have the following Theorem.

Theorem 5. *The quadratic form*

$$\mathbf{I} = \Upsilon_0^T \Sigma_0^{-1} \Upsilon_0$$

for $k \rightarrow \infty$ has limits distribution Chi-square with $\lambda - d + 1$ degrees of freedom.

Let's

$$\mathbf{N} = \mathbf{K}(\mathbf{K}^T \mathbf{K})^{-} \mathbf{K}^T,$$

where

$$\mathbf{K} = \|a_{ij}\|_{\lambda \times d}|_{\tilde{\mathbf{p}}}$$

or

$$\mathbf{K} = \left\| \frac{\partial B_i}{\sqrt{B_i} \partial p_j} \right\|_{\lambda \times d} \Big|_{\tilde{\mathbf{p}}},$$

$(\mathbf{K}^T \mathbf{K})^{-}$ is the generalized inverse of the matrix for martix $\mathbf{K}^T \mathbf{K}$, that is

$$\mathbf{K}^T \mathbf{K} (\mathbf{K}^T \mathbf{K})^{-} \mathbf{K}^T \mathbf{K} = \mathbf{K}^T \mathbf{K},$$

the vector

$$\tilde{\mathbf{p}} = \{\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_\mu\}$$

is the vector of the unbiased estimations for parameters

$$\mathbf{p} = (p_1, p_2, \dots, p_d)$$

defined in (3).

Theorem 6. *If the vector of paramrters*

$$\mathbf{p} = (p_1, p_2, \dots, p_d)$$

isn't known and for $k \rightarrow \infty$, then quadratic form

$$\Theta_k^2 = \tilde{\Upsilon}^T \tilde{\Upsilon} - \tilde{\Upsilon}^T N \tilde{\Upsilon}$$

has limits distribution Chi-square with $\lambda - R(K) + 1$ degrees of freedom where

$$\tilde{\Upsilon}$$

is estimation for Υ , which obtained using a vector of estimates for \mathbf{p} and the most appropriate solutions \mathbf{z}_g , based on observation.

The validity of this Theorem follows from [8].

Let's for $\beta = 1, \dots, \lambda$

$$S_\beta = \sum_{\mathbf{u} \in \Omega_\beta} W(\mathbf{u}, \mathbf{z}_g),$$

where $W(\mathbf{u}, \mathbf{z}_g)$ is most suitable unbiased estimation for distribution (1), and for $\alpha = 1, \dots, d$

$$Q_{\alpha\beta} = \sum_{\mathbf{u} \in \Omega_\beta} r_{\alpha_{v_b f u}} W(\mathbf{u}, \mathbf{z}_g),$$

where $r_{\alpha_{v_b f u}}$ is defined from z_α by using Lemma 1. Define

$$\begin{aligned} \Sigma_1 &= \mathbf{D}_1 - \mathbf{S}\mathbf{S}^T, \\ \mathbf{D}_1 &= \begin{vmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_{\lambda-1} \end{vmatrix}_{(\lambda-1) \times (\lambda-1)}, \\ \mathbf{S}\mathbf{S}^T &= \begin{vmatrix} S_1 S_1 & S_1 S_2 & \dots & S_1 S_{\lambda-1} \\ S_2 S_1 & S_2 S_2 & \dots & S_2 S_{\lambda-1} \\ \dots & \dots & \dots & \dots \\ S_{\lambda} S_1 & S_{\lambda} S_2 & \dots & S_{\lambda-1} S_{\lambda-1} \end{vmatrix}_{(\lambda-1) \times (\lambda-1)}, \\ \Sigma_2 &= \begin{vmatrix} np_1(1-p_1) & -np_1 p_2 & \dots & -np_1 p_d \\ -np_1 p_2 & np_2(1-p_1) & \dots & -np_2 p_d \\ \dots & \dots & \dots & \dots \\ -np_1 p_d & -np_2 p_d & \dots & np_d(1-p_d) \end{vmatrix}_{d \times d} \end{aligned}$$

and

$$\Sigma_3 = \begin{vmatrix} Q_{11} - S_1 np_1 & Q_{12} - S_2 np_1 & \dots & Q_{1\lambda-1} - S_{\lambda-1} np_1 \\ Q_{21} - S_1 np_2 & Q_{22} - S_2 np_2 & \dots & Q_{2\lambda-1} - S_{\lambda-1} np_2 \\ \dots & \dots & \dots & \dots \\ Q_{d1} - S_1 np_d & Q_{d2} - S_2 np_d & \dots & Q_{d\lambda-1} - S_{\lambda-1} np_d \end{vmatrix}_{d \times (\lambda-1)},$$

the elements of vector

$$\mathbf{p} = (p_1, p_2, \dots, p_d)$$

are parameters of distribution (3). Define the matrix

$$\mathbf{C} = \Sigma_1 - \Sigma_3 \Sigma_2^{-1} \Sigma_3$$

and vector

$$\mathbf{M} = \frac{1}{\sqrt{k}(\zeta_1 - kS_1, \dots, \zeta_\lambda - S_{\lambda-d+1})}.$$

From results [15] - [16] we have following Theorem.

Theorem 7. *If $k \rightarrow \infty$ then the quadratic form*

$$\Phi_k^2 = \mathbf{M}^T \mathbf{C}^{-1} \mathbf{M}$$

has limits distribution Chi-square with $\lambda - d + 1$ degrees of freedom.

Conclusion

Analysis conducted in this paper, allows us to formulate the following main results.

- Proposed and studied a new probability distribution of distortions of radiation processes from remote sensing data.
- Define the generating function for the distribution of the proposed model.
- The set of unbiased estimates for the probability distribution of the proposed model and the variance of these estimates.
- Introduced a new concept of the most appropriate evaluation of the set of unbiased estimates, with asymptotic properties.
- From the results of a method comparison of the estimates for the probability distribution of this model implies that the most suitable unbiased estimation is the best.
- Chi-square test for the adequacy of the model to experimental data in the case, where the distributions of the parameters of this model are known and unknown, is built.

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