# Periodic chain of resonators: gap control and geometry of the system

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**Abstract.** A periodic system of coupled resonators is considered. It has a form of zigzag line. A band structure of the spectrum is studied. Solvable model based on the theory of self-adjoint extensions of symmetric operators is used. The position of gaps depends on the zigzag line angle. It opens a way to control the gap position by geometrical parameters. Particularly, it is possible to construct a system with predetermined gap position

## 1. Introduction

Chain structures constructed from micro- or nano-resonators are often used in optical systems. Particularly, it allows one to modify optical waveguides system by adding such a chain playing the role of a SCISSOR device [1]. In other application it ensures a photonic crystal property to optical waveguide system [2] (an analogous effect can be reached by using of periodically coupled waveguides [3, 4]). This system is also used for constructing optical delay line [5]. The property of the operator spectrum for the system allows one to use it for creation of biosensors having many advantages in comparison with another ones [6].

One meets such systems in nanoelectronics too. An example of a chain of coupled conglobate quantum resonators (not unique but very interesting) is so-called nano peapod. It is a nanotube filled by a chain of fullerene molecules. There is a number of experimental works describing properties of such systems and the processes of its creation [7-12].

To describe the system, we use solvable model based on the operator extensions theory [13, 14]. It allows one to obtain the dispersion equation for the model operator in an explicit form. Particularly, our consideration is based on a variant of this approach known as a model of point-like window (zero-width slit) [15 - 18]. We consider two-dimensional zigzag-like chain of coupled resonator (see Fig. 1) and describe its spectrum. Bands and gaps positions depend on zigzag angle. It gives an opportunity to control the gap position by geometrical parameters.

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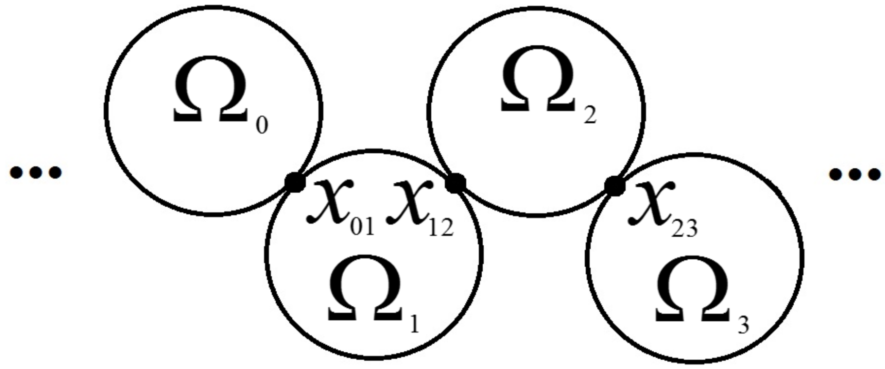


Fig. 1. Geometry of the system

## 2. Model construction and spectral analysis

First, consider two coupled two-dimensional identical resonators $\Omega_0, \Omega_1$ (disks) having common boundary point $x_{01}$. Let $-\Delta$ be the orthogonal sum of the Laplace operators with the Neumann boundary conditions in $L_2(\Omega_0)$ and $L_2(\Omega_1)$. Restrict the operator on the set of smooth functions vanishing at point $x_{01}$. We obtain symmetric non-self-adjoint operator with deficiency indices $(2,2)$. Its self-adjoint extension gives us the model of resonators coupled through point-like opening. The self-adjoint extension is a restriction of the adjoint operator. The domain of the adjoint operator consists of elements $(u_0, u_1)$ such that

$$u_{0,1}(x) = a_{0,1} G(x, x_{01}, k_0) + \breve{u}_{0,1}(x), \;\; x \in \Omega_{0,1},$$

where $G(x, x_{01}, k_0)$ is the Green function for the resonator, $k_0$ is some imaginary number, $\breve{u}_{0,1}(x)$ is an element from the domain of the Fridrichs extension, $\breve{u}_{0,1}(x_{01}) = \mathrm{b}_{0,1}$ . To construct the self-adjoint extension we should restrict the adjoint operator, i.e. to establish a relation between $a_o, a_1, \mathrm{b}_o, \mathrm{b}_1$. We choose the following (natural, see [17]) condition

$$a_o = -a_1, \; b_o = b_1 \tag{1}$$

As for the periodic system in question, we use the transfer-matrix approach (see, e.g., [19]) To find the transfer-matrix, let us consider one of the resonators (they are identical), say, $\Omega_1$. A non-trivial solution corresponding to a value of the spectral parameter $k^2$ has the form

$$\alpha_{01}^{+} G(x, x_{01}, k) + \alpha_{11}^{-} G(x, x_{11}, k) \; .$$

Here plus and minus mark the right and left coefficients for the common boundary point. Correspondingly,

$$\begin{cases} a_{01}^{+} = \alpha_{01}^{+}, \\ b_{01}^{+} = \alpha_{01}^{+} \left( G(x, x_{01}, k) - G(x, x_{01}, k_0) \right)\Big|_{x = x_{01}} + \alpha_{11}^{-} G(x_{01} x_{11}, k). \end{cases}$$

For the next point, one has (we take into account conditions (1)):

$$\begin{cases} a_{11}^{+} = \alpha_{11}^{+} = -\alpha_{11}^{-} = \left( \alpha_{01}^{+} g(k_1 k_0) - b_{01}^{+} \right) \dfrac{g(k_1 k_0)}{G(x_{01}, x_{11}, k)}, \\ b_{11}^{+} = b_{11}^{-} = \alpha_{01}^{+} G(x_{01}, x_{11}, k) - \alpha_{11}^{-} \left( G(x, x_{12}, k) - G(x_1, x_{12}, k_0) \right)\Big|_{x = x_{12}}, \end{cases}$$

where

$$g(k, k_0) = \left( G(x, x_{01}, k) - G(x, x_{01}, k_0) \right)\Big|_{x = x_{01}} .$$

Hence,

$$b_{11}^{+}=\alpha_{01}^{+}G(x_{01},x_{11},k)-\left(b_{01}^{+}-\alpha_{01}^{+}g(k,k_0)\right)\frac{g(k,k_0)}{G(x_{01},x_{12},k)}.$$

Thus, one gets

$$\begin{pmatrix}\alpha_{11}^{+}\\ b_{11}^{+}\end{pmatrix}=\begin{pmatrix}\dfrac{g^2(k,k_0)}{G(x_{01},x_{12},k)} & -\dfrac{g(k,k_0)}{G(x_{01},x_{12},k)}\\ G(x_{01},x_{12},k)+\dfrac{g(k,k_0)}{G(x_{01},x_{12},k)} & -\dfrac{g(k,k_0)}{G(x_{01},x_{12},k)}\end{pmatrix}\begin{pmatrix}\alpha_{01}^{+}\\ b_{01}^{+}\end{pmatrix}=M\begin{pmatrix}\alpha_{01}^{+}\\ b_{01}^{+}\end{pmatrix}.$$

Here $M$ is the transfer-matrix. Let us find its eigenvalues by solving the equation

$$\det(M-\lambda I)=\lambda^2-\lambda\left(g(k_1k_0)-1\right)\frac{g(k,k_0)}{G(x_{01},x_{12},k)}-\frac{g^3(k,k_0)}{G^2(x_{01},x_{12},k)}+g(k,k_0)=0$$

The eigenvalues of the transfer-matrix are as follows.

$$\lambda_{1,2}=\frac{\dfrac{(g-1)g}{G}\pm\left(\dfrac{(g-1)^2g^2}{G^2}+\dfrac{4g^3}{G^2}-4g\right)^{1/2}}{2},\quad G=G(x_{01},x_{12},k)$$

It is known (see, e.g., [20], [21]) that $k^2$ belongs to the continuous spectrum (band) of the model operator if

$$|\lambda_{1,2}|=1.$$

In our case, it means

$$(g-1)^2+4g-\frac{4G^2}{g}<0,$$

or

$$(g+1)^2g<4G^2. \qquad (2)$$

One can see that the left hand side of (2) does not depend on the form of the zigzag line. In contrast, the right hand side depends on the position of the disks touching points, i.e. on the form of the zigzag line (zigzag line angle).

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