

# Meshless local integral equation method for two-dimensional nonlocal elastodynamic problems

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**Abstract.** This paper presents the meshless local integral equation method (LIEM) for nonlocal analyses of two-dimensional dynamic problems based on the Eringen's model. A unit test function is used in the local weak-form of the governing equation and by applying the divergence theorem to the weak-form, local boundary-domain integral equations are derived. Radial Basis Function (RBF) approximations are utilized for implementation of displacements. The Newmark method is employed to carry out the time marching approximation. Two numerical examples are presented to demonstrate the application of time domain technique to deal with nonlocal elastodynamic mechanical problems.

## 1. Introduction

It is found that the distance between individual atoms or molecules cannot be ignored when it becomes obviously comparable to external scales and the discrete microstructure in materials can no longer be treated as homogeneous continuum at small size. Therefore, nonlocal elasticity theories have been initiated to model material properties exhibited at micro-, and even at nano-scale levels. Among others, Eringen [1] proposed an integral model for linear homogeneous isotropic media as

$$\sigma(\mathbf{x}) = \xi_1 \bar{\sigma}(\mathbf{x}) + \xi_2 \int_V \alpha(\mathbf{x}, \mathbf{x}', l) \bar{\sigma}(\mathbf{x}') dV(\mathbf{x}') \quad (1)$$

where  $\sigma$  is referred to as nonlocal stress,  $\bar{\sigma}$  is the classical stress,  $V$  represents the volume of domain,  $\mathbf{x}, \mathbf{x}'$  are collocation and domain integration points,  $l$  is the characteristic length.  $\xi_1$  and  $\xi_2$  are portion factors and  $\xi_1 + \xi_2 = 1$ .  $\alpha$  is the nonlocal kernel and usually taken as  $\alpha(\mathbf{x}, \mathbf{x}', l) = \lambda_0 \exp(-|\mathbf{x} - \mathbf{x}'|/l)$ ,  $\lambda_0 = 1/2\pi l^2$ .

## 2. LIEM for nonlocal elastodynamics

Consider a linear elastic body in two dimensional domain  $\Omega$  with boundary  $\Gamma$ . The weak form of governing equations of motion can be written as

$$\int_{\Omega_s} (\sigma_{ij,j} + f_i - \rho \ddot{u}_i) u_i^* d\Omega = 0 \quad (2)$$

where  $f$  denotes the body force,  $\rho$  is the mass density,  $\ddot{u}$  is the acceleration and  $u_i^*$  is the test function. By use of the divergence theorem, Eq.2 can be rewritten as

$$\int_{\Gamma_s} \sigma_{ij} n_j u_i^* d\Gamma - \int_{\Omega_s} (\sigma_{ij,j} u_i^* - f_i u_i^* + \rho \ddot{u}_i u_i^*) d\Omega = 0 \quad (3)$$



where  $n_j$  is the unit outward normal on the boundary. A unit test function  $\phi_i(\mathbf{x}) = 1$  is employed and thus, local integral equations in Eq.3 becomes

$$\int_{\Gamma_s} \sigma_{ij} n_j d\Gamma + \int_{\Omega_s} (f_i - \rho \ddot{u}_i) d\Omega = 0 \quad (4)$$

Assume the local domain is enclosed by several straight lines and recall the constitutive relation of Eringen's model in Eq.1, Eq.4 can be written as

$$\int_{\Gamma_s} \sigma_{ij}(\mathbf{x}) n_j(\mathbf{x}) d\Gamma(\mathbf{x}) = \xi_1 \sum_{l=1}^L n_j^l \int_{\Gamma_l} \sigma_{ij}(\mathbf{x}) d\Gamma(\mathbf{x}) + \xi_2 \sum_{l=1}^L n_j^l \Delta_l \int_V \alpha(\mathbf{x}_i, \mathbf{x}', l) \sigma_{ij}(\mathbf{x}') dV(\mathbf{x}') \quad (5)$$

where  $L$  is number of straight lines and  $\Delta_l$  is the length of the  $l^{th}$  straight line.

The time marching approximation of Newmark method [2] are given as:

$$\dot{u}_{n+1} = \frac{\gamma}{\beta \Delta t} (u_{n+1} - u_n) + \frac{\beta - \gamma}{\beta} \dot{u}_n - \frac{\gamma - 2\beta}{2\beta} \Delta t \ddot{u}_n \quad (6)$$

$$\ddot{u}_{n+1} = \frac{1}{\beta \Delta t^2} (u_{n+1} - u_n) - \frac{1}{\beta \Delta t} \dot{u}_n - \frac{1 - 2\beta}{2\beta} \ddot{u}_n \quad (7)$$

in which  $\dot{u}$ ,  $\ddot{u}$  and  $u$  represent the velocity, acceleration and displacement, respectively.  $u_n$  is the displacement at  $n\Delta t$ , where  $\Delta t$  is the selected time step. Parameters  $\gamma$  and  $\beta$  are selected as  $\gamma = 0.5$  and  $\beta = 0.25$  in this paper.

Consider the RBF approximation in [3], and the four-point standard integral scheme is adopted to carry out the domain integrals, the following discrete equations can be derived.

$$\begin{aligned} & \xi_1 C \sum_{k=1}^K u_{1(n+1)}^{(k)} \sum_{l=1}^L \left[ \sum_{i=1}^K (E' F_{1il} n_1^l + \mu F_{2il} n_2^l) \alpha_{ik} + \sum_{t=1}^T (E' G_{1tl} n_1^l + \mu G_{2tl} n_2^l) \beta_{tk} \right] + u_{1(n+1)}^{(k)} + \\ & \xi_2 C \sum_{k'=1}^{K'} u_{1(n+1)}^{(k')} \sum_{q=1}^V \sum_{p=1}^4 \sum_{l=1}^L (E' \phi_{k',1}(\mathbf{x}'_p) n_1^l(\mathbf{x}_l) + \mu \phi_{k',2}(\mathbf{x}'_p) n_2^l(\mathbf{x}_l)) \alpha(\mathbf{x}_l, \mathbf{x}'_p, l) \Delta_l w_p \Delta V_q + \\ & \xi_1 C \sum_{k=1}^K u_{2(n+1)}^{(k)} \sum_{l=1}^L \left[ \sum_{i=1}^K (\nu E' F_{2il} n_1^l + \mu F_{1il} n_2^l) \alpha_{ik} + \sum_{t=1}^T (\nu E' G_{2tl} n_1^l + \mu G_{1tl} n_2^l) \beta_{tk} \right] + \\ & \xi_2 C \sum_{k'=1}^{K'} u_{2(n+1)}^{(k')} \sum_{q=1}^V \sum_{p=1}^4 \sum_{l=1}^L (\nu E' \phi_{k',2}(\mathbf{x}'_p) n_1^l(\mathbf{x}_l) + \mu \phi_{k',1}(\mathbf{x}'_p) n_2^l(\mathbf{x}_l)) \alpha(\mathbf{x}_l, \mathbf{x}'_p, l) \Delta_l w_p \Delta V_q = I_1 \end{aligned} \quad (8a)$$

$$\begin{aligned} & \xi_1 C \sum_{k=1}^K u_{1(n+1)}^{(k)} \sum_{l=1}^L \left[ \sum_{i=1}^K (\nu E' F_{1il} n_2^l + \mu F_{2il} n_1^l) \alpha_{ik} + \sum_{t=1}^T (\nu E' G_{1tl} n_2^l + \mu G_{2tl} n_1^l) \beta_{tk} \right] + \\ & \xi_2 C \sum_{k'=1}^{K'} u_{1(n+1)}^{(k')} \sum_{q=1}^V \sum_{p=1}^4 \sum_{l=1}^L (\nu E' \phi_{k',1}(\mathbf{x}'_p) n_2^l(\mathbf{x}_l) + \mu \phi_{k',2}(\mathbf{x}'_p) n_1^l(\mathbf{x}_l)) \alpha(\mathbf{x}_l, \mathbf{x}'_p, l) \Delta_l w_p \Delta V_q + \\ & \xi_1 C \sum_{k=1}^K u_{2(n+1)}^{(k)} \sum_{l=1}^L \left[ \sum_{i=1}^K (E' F_{2il} n_2^l + \mu F_{1il} n_1^l) \alpha_{ik} + \sum_{t=1}^T (E' G_{2tl} n_2^l + \mu G_{1tl} n_1^l) \beta_{tk} \right] + u_{2(n+1)}^{(k)} + \\ & \xi_2 C \sum_{k'=1}^{K'} u_{2(n+1)}^{(k')} \sum_{q=1}^V \sum_{p=1}^4 \sum_{l=1}^L (E' \phi_{k',2}(\mathbf{x}'_p) n_2^l(\mathbf{x}_l) + \mu \phi_{k',1}(\mathbf{x}'_p) n_1^l(\mathbf{x}_l)) \alpha(\mathbf{x}_l, \mathbf{x}'_p, l) \Delta_l w_p \Delta V_q = I_2 \end{aligned} \quad (8b)$$

in which  $C = -\frac{\beta \Delta t^2}{\rho \Omega_s}$ ,  $I_i = u_i + \Delta t \dot{u}_i + \frac{1 - 2\beta}{2} \Delta t^2 \ddot{u}_i$  (body force is ignored).  $E' = E/(1 - \nu^2)$ ,  $E$  is Young's modulus and  $\nu$  is the Poisson's ratio and shear modulus  $\mu = E/2(1 + \nu)$ .  $k = 1, 2, \dots, K$

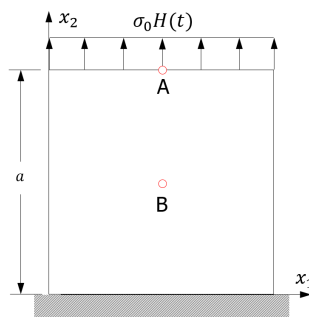
and  $k' = 1, 2, \dots, K'$  are the number of nodes in the support domain centred at  $\mathbf{x}$  and  $\mathbf{x}'$ , respectively.  $V$  is the number of sub integral domains and  $w_p = 0.25$ . Analytical solutions for boundary integrals of  $F_{jil}$  and  $G_{jtl}$  in closed form can be found in [4].

Traction boundary conditions are  $\int_{\Gamma-\Gamma_T} t_i d\Gamma - \int_{\Gamma_T} t_i^0 d\Gamma = I_i$  for  $\mathbf{x}_k$ ,  $k = 1, 2, \dots, M_T$ , where  $\Gamma_T$  is the traction boundary. While displacement boundary conditions can be introduced directly as  $u_i(\mathbf{x}_k) = u_i^0$ ,  $k = 1, 2, \dots, M_D$ .  $M_T$  and  $M_D$  are the number of nodes located on the traction and displacement boundaries, respectively.

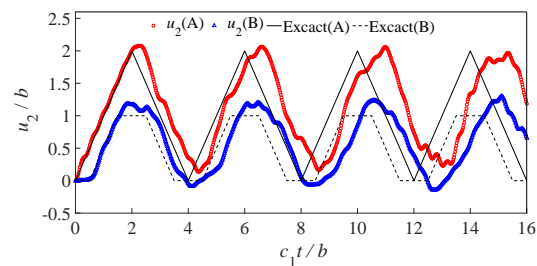
### 3. Numerical Examples

#### 3.1. A square plane subjected to dynamic load

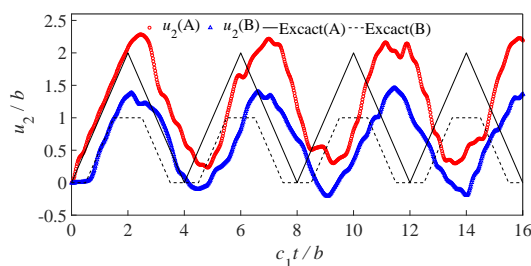
Consider a square plate of side  $a = 1.0$  subjected to a Heaviside load  $\sigma_0 H(t)$  on top and fixed along the edge at  $x_2 = 0$  as shown in Fig.1.  $11 \times 11$  nodes are distributed in the problem domain. Young's modulus is one unit and Poisson's ration is taken as zero. Normalized dynamic displacements on point A and B with the characteristic length  $l = 0.1$  and  $l = 0.2$  for different portion factors ( $\xi_1 = 0.8$  /  $\xi_1 = 0.5$ ) are plotted in Fig.2 to Fig.4, which show that the period of oscillation increases when the portion factor  $\xi_1$  decreases or the characteristic length  $l$  rises.



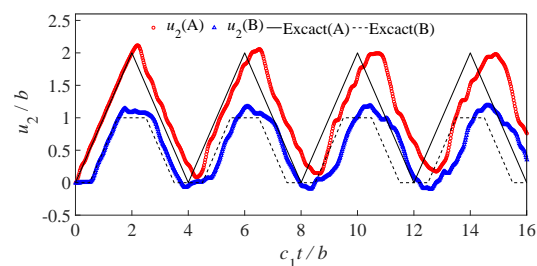
**Figure 1.** A square plate subjected to a Heaviside load.



**Figure 2.** Normalized displacement for point A and B against normalized time  $c_1 t/b$  when  $\xi_1 = 0.5$ ,  $l = 0.1$ .



**Figure 3.** Normalized displacement for point A and B against normalized time  $c_1 t/b$  when  $\xi_1 = 0.5$ ,  $l = 0.2$ .

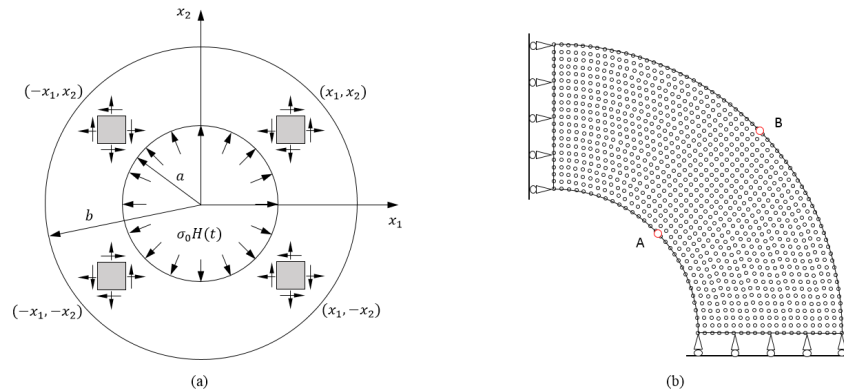


**Figure 4.** Normalized displacement for point A and B against normalized time  $c_1 t/b$  when  $\xi_1 = 0.8$ ,  $l = 0.2$ .

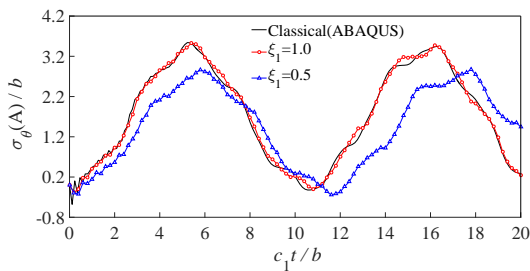
#### 3.2. A ring under internal dynamic pressure

A ring under internal dynamic pressure is analysed as shown in Fig.5(a). Dimensions  $a = 1\text{cm}$ ,  $b = 2\text{cm}$ . Young's modulus  $E = 1.0$  and Poisson's ratio  $\nu = 0.3$ . Due to symmetry, only one quarter of the ring is considered. However, the contributions of strains from the whole ring

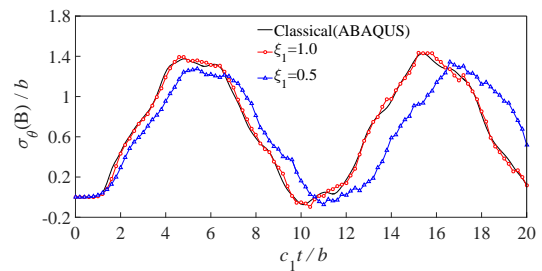
must be taken into account in the domain integral. Nodes distributions can be seen in Fig.5(b). The normalized dynamic tangential stress at point A and B are plotted in Fig.6 and Fig.7 with  $l = 0.2$  for different portion factors ( $\xi_1 = 1.0$  and  $\xi_1 = 0.5$ ). ABAQUS solutions are presented as well for comparisons when  $\xi_1 = 1.0$  and good agreement have been obtained, which validates the accuracy of time domain technique to general dynamic problems.



**Figure 5.** A ring under internal dynamic pressure :(a) geometry; (b) node distributions.



**Figure 6.** Normalized tangential stress against normalized time  $c_1 t/b$  for point A when  $l = 0.2$  for  $\xi_1 = 1.0$  and  $\xi_1 = 0.5$ .



**Figure 7.** Normalized tangential stress against normalized time  $c_1 t/b$  for point B when  $l = 0.2$  for  $\xi_1 = 1.0$  and  $\xi_1 = 0.5$ .

#### 4. Conclusions

In this paper, LIEM formulations are presented for two-dimensional nonlocal elastodynamics. The Newmark method is selected as the approximation scheme to deal with time-dependent cases. Two numerical examples demonstrated that LIEM embedded with time domain technique is stable, convergent and accurate for solving nonlocal elastodynamic problems.

#### References

- [1] Eringen A C and Kim B S 1974 *Mechanics Research Communications* **1** 233–237
- [2] Newmark N M 1959 *Journal of the engineering mechanics division* **85** 67–94
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