Evaluation of Bi-Material Crack Using The Finite Block Method

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Abstract. The Finite Block Method has been employed in this paper to evaluate the stress intensity factor of a bi-material plate. The complex stress intensity factor components K_1 and K_2 determined by the Finite Block Method is compared with an equivalent Finite Element Method (ABAQUS) analysis. The paper demonstrates the accuracy of the Meshfree approach by the Finite Block Method without the arduous demand of meshing around the crack surface as seen on standard FEM crack analysis. This paper also describes the application of the polygonal singular core and the collocations points around the interface crack. A computational example for various E_1/E_2 material combinations is presented.

1. Introduction

The analytical solution of interface crack is difficult to obtain as work produced by Williams [1], England [2], Hutchinson [3] and many others have shown. However, the study of cracks in monolithic material has been in existence for decades and this has led to the development of Linear Elastic Fracture Mechanics, the J-Integral method, the strain energy release rate and many other analytical solutions [6]. In an attempt to tackle the challenges presented by the mixed mode coupling and oscillatory effect of interface cracks, many numerical solutions have been presented to determine the complex stress intensity factors.

The finite element method is a popular numerical method and it has been utilised in analysing two-dimensional cracks. This has been demonstrated in recent works by Lin [4]. Building on the success of the Finite Element Method, the Extended Finite Element Method as proposed by Belytschko utilize interpolation functions, the concept of partition of unity and the discontinuity of near crack tip displacement fields which is deemed independent from the finite element mesh to analyse interface cracks [7]. The Boundary Element method is another numerical method capable of evaluating stress intensity factor for bi-material. However, the finite element and the boundary element methods are heavily mesh based models and this presents its own challenges in modeling cracks. The Finite Block Method (FBM) based on the collocation method has been developed by Wen [5] for elasticity analysis. The FBM divide's the problem domain into blocks and the governing equations are satisfied in the strong form at certain collocation points. In this paper, the finite block method using the polygon single core as developed by the authors is used to evaluate the stress intensity factor for bi-material interfacial problem.

The paper is organized as follows. In section 2 an overview of FBM is presented, followed by the evaluations of SIF using the polygonal single core. In section 3, a numerical example of a bi-material plate with center and an edge crack is presented. The research is summarized in section 5.

2. Overview of the Finite Block Method For Interface Crack

The Finite Block Method formulation as reviewed in this paper makes use of a simple square to demonstrate the mathematical formulation. In addition, the reader is advised to continue reading the following papers [5, 8] for an in-depth understanding of FBM.

From the simple square plate shown in figure 1 a smooth function $u(\xi,\eta)$ can be approximated in the domain $|\xi| \le 1$ and $|\eta| \le 1$. Where the field function is given by

$$u(\xi,\eta) = \sum_{i=1}^{M}\sum_{j=1}^{N} F(\xi,\xi_i)G(\eta,\eta_j)u_l \tag{1}$$

In this case the number of collocation points along the two axes are represented by M and N, u_l indicates the nodal value, $l=(j-1)M+i$, with uniformly distributed nodes at $\xi_i = -1+2(i-1)/(M-1)$, $i=1,2,...,M$, $\eta_j = -1+2(j-1)/(N-1)$, $j=1,2,...,N$, and two polynomial functions;

$$F(\xi,\xi_i) = \prod_{\substack{m=1\\ m\neq i}}^{M} \frac{(\xi-\xi_m)}{(\xi_i-\xi_m)}, \qquad G(\eta,\eta_j) = \prod_{\substack{n=1\\ n\neq j}}^{N} \frac{(\eta-\eta_n)}{(\eta_j-\eta_n)}. \tag{2}$$

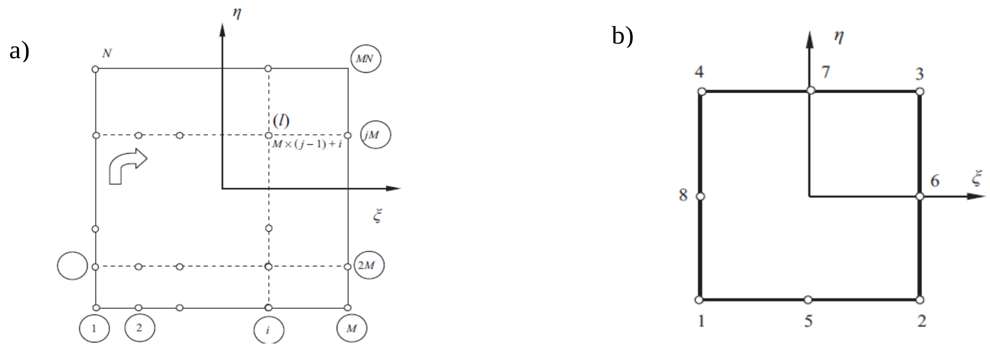

Figure 1. a) Simple 2D square domain with the local system nodes. b) The mapping geometry with the 8 seeds.

The total number of nodes is given by $Q = M \times N$. Then, the first order partial differential is determined with respect to ξ and η respectively

$$\frac{\partial u}{\partial \xi} = \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\partial F(\xi,\xi_i)}{\partial \xi} G(\eta,\eta_j)u_l\ , \quad \frac{\partial u}{\partial \eta} = \sum_{i=1}^{M}\sum_{j=1}^{N} F(\xi,\xi_i)\frac{\partial G(\eta,\eta_j)}{\partial \eta} u_l \tag{3}$$

Employing the same approach as used in the Finite Element Method and using a set of quadratic shape functions with 8 seeds, the 2D square problem in Cartesian coordinate system (x,y) is transformed into a square in the mapping domain (ξ,η) as shown in figure 1b. The quadratic shape functions are defined follows

$$N_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1), \qquad i=1,2,3,4, \tag{4}$$

$$N_i(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta_i\eta), \qquad i=5,7, \tag{5}$$

$$N_i(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1+\xi_i\xi), \qquad i=6,8. \tag{6}$$

From figure 1b, a quadratic block can be mapped into a square domain using the following expression

$$x = \sum_{k=1}^{8} N_k(\xi,\eta)x_k\ ,\ \ y = \sum_{k=1}^{8} N_k(\xi,\eta)y_k\ , \tag{7}$$

where (x_k, y_k) denotes the coordinate of seed k. The first order partial differentials of function $u(x,y)$ in the Cartesian coordinate system are obtained by

$$\frac{\partial u}{\partial x} = \frac{1}{J}\left(\beta_{11}\frac{\partial u}{\partial \xi} + \beta_{12}\frac{\partial u}{\partial \eta}\right), \quad \frac{\partial u}{\partial y} = \frac{1}{J}\left(\beta_{21}\frac{\partial u}{\partial \xi} + \beta_{22}\frac{\partial u}{\partial \eta}\right) \quad , \tag{8}$$

$$\beta_{11} = \frac{\partial y}{\partial \eta}, \beta_{12} = -\frac{\partial y}{\partial \xi}, \beta_{21} = -\frac{\partial x}{\partial \eta}, \beta_{22} = \frac{\partial x}{\partial \xi}, \ J = \beta_{22}\beta_{11} - \beta_{21}\beta_{12}. \tag{9}$$

Then, substituting (3) into (8) gives an expression interms of the nodal value of the first order partial differentials and the nodal value of displacement in the following manner

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{J}\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\beta_{11}\frac{\partial F(\xi,\xi_i)}{\partial \xi}G(\eta,\eta_j) + \beta_{12}F(\xi,\xi_i)\frac{\partial G(\eta,\eta_j)}{\partial \eta}\right]u_l = D_{xl}(\xi,\eta)u_l, \\ \frac{\partial u}{\partial y} &= \frac{1}{J}\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\beta_{21}\frac{\partial F(\xi,\xi_i)}{\partial \xi}G(\eta,\eta_j) + \beta_{22}F(\xi,\xi_i)\frac{\partial G(\eta,\eta_j)}{\partial \eta}\right]u_l = D_{yl}(\xi,\eta)u_l, \end{aligned} \tag{10}$$

Using equation 10, the general expression for a 2D constitutive equation and the equilibrium equation, the nodal value of the displacement (u_x,u_y) is determined for a given boundary condition.

3. Evaluation of SIF Using The Polygonal Singular Core

In this study a single polygonal core [8] centered at the crack tip is used in determining the near crack tip stress and displacement values. From figure 2, r_0 is the radius of the circle (dash line) originating from the crack tip. This indicates the size of the singular core and $P(r_k,\theta_k)$ $(k=1,...N_b)$ is the coordinate of the collocation point on the interface between the block and the core.

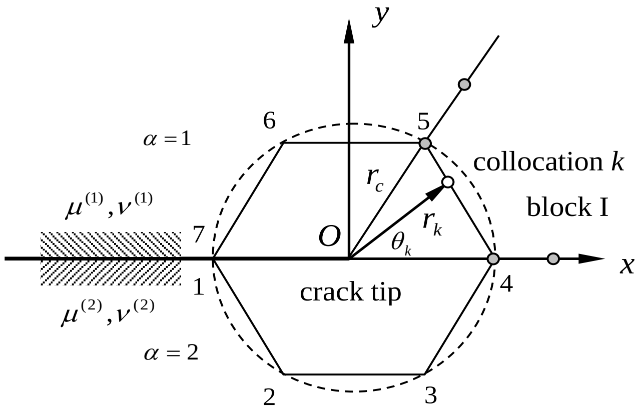

Figure 2. A single polygonal core around the crack tip. In this case a six sided core is used.

With the aid of a singular core around a straight interfacial crack, the singular stress field around the crack tip is determined by the following expression [8]

$$\sigma_y^{(\alpha)} + i\tau_{xy}^{(\alpha)} = A_{C0}^{(1)}\left(\frac{1}{2} + i\varepsilon\right)(1+\hat{\kappa})r^{-\frac{1}{2}+i\frac{\varepsilon}{2}} = \frac{K}{\sqrt{2\pi}}r^{-\frac{1}{2}+i\frac{\varepsilon}{2}} \tag{11}$$

where K, the complex stress intensity factor is defined by

$$\frac{K}{\sqrt{\pi a}} = \frac{K_1 + iK_2}{\sqrt{\pi a}} = A_{C0}^{(1)}\left(\frac{1}{2} + i\varepsilon\right)(1+\hat{\kappa})\sqrt{2/a} \tag{12}$$

where $\kappa^{(\alpha)} = 3 - 4\nu^{(\alpha)}$ for plane strain and $\kappa^{(\alpha)} = \dfrac{3 - \nu^{(\alpha)}}{1+\nu^{(\alpha)}}$ for plane stress

The superscript $\alpha = 1, 2$ represents material 1 above and material 2 below the plate.

$$\hat{\kappa} = \frac{\kappa^{(1)}\mu^{(2)} + \mu^{(1)}}{\kappa^{(2)}\mu^{(1)} + \mu^{(2)}}, \qquad \varepsilon = \ln(\hat{\kappa})/2\pi \tag{13}$$

Using the Williams' series [8] the coefficient $A_{Cn}^{(1)}$ $(n = 0,1,.., N_b - 1)$ is determined for each node.

4. Numerical Example

A central crack of length *2a* in a bi-material plate is studied in this example. The plate has width w = 1 and height H = 4. The uniformly distributed load σ_0 is applied on the top and bottom. Owing to the symmetry with respect to *y* axis, it is equivalent to solve the boundary value problem for a half plate with a reasonable mesh. The FEM (ABAQUS) analysis was conducted under plane strain condition with a Poison ratios $\nu_1 = \nu_2 = 0.3$ and the crack length a = 0.5. The results of SIFs versus the ratio of Young's modulus E_1/E_2 are shown in table 1 for different cases. The solutions given by ABAQUS are provided in the table for comparison.

a/w	E_1/E_2	$K_1/\sigma_0\sqrt{\pi a}$			$K_2/\sigma_0\sqrt{\pi a}$		
		Present	ABAQUS	[9]	Present	ABAQUS	[9]
0.5	1	1.190	1.188	1.1893	0.000	0.000	0
	2	1.182	1.179	1.1798	-0.056	-0.051	-0.0566
	5	1.154	1.148	1.1483	-0.110	-0.098	-0.1053
	10	1.131	1.123	1.1237	-0.132	-0.116	-0.124

Table 1 Stress intensity factors of central crack bi-material plate

5. Summary

The Finite Block Method has been utilised in this paper to study interface crack of bi-material. It is easily observed that the SIF as deduced by the FBM are in good agreement with ABAQUS and the reference.

6. References

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