Scattering of electromagnetic wave by dielectric cylinder in eikonal approximation

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Abstract. The scattering of the plane electromagnetic wave on a spatially extended, fiber-lake target is considered. The formula for the scattering cross section is obtained using the approximation analogous to eikonal one in quantum mechanics.

1. Introduction

The problem of multiple scattering of the electromagnetic wave under oblique incidence on the system of parallel dielectric fibers was considered in [1]. The scattering on the single fiber had been described there in the limit of infinitely thin fiber. The wave scattering cross section in that case possesses the axial symmetry for small angle of incidence $\psi \ll 1$.

The scattering cross section by the cylindric fiber of finite radius is calculated in the present article using eikonal approximation. The possibility of substantial axial anisotropy of the scattered radiation is demonstrated.

2. APPROXIMATIONS IN RADIATION SCATTERING THEORY

The equations for the electric field of a monochromatic wave $\mathbf{E}(\mathbf{r})e^{-i\omega t}$ in non-uniform medium

$$\left(\Delta + \frac{\omega^2}{c^2}\,\varepsilon\right)\mathbf{E} = \mathrm{grad}\,\mathrm{div}\,\mathbf{E}, \tag{1}$$

$$\mathrm{div}\,(\varepsilon\,\mathbf{E}) = 0 \tag{2}$$

(where Δ is the Laplasian operator and $\varepsilon(\mathbf{r})$ is the dielectric permittivity of the medium for the given frequency ω) could be easily derived from Maxwell equations (see, e.g. [2, §68]). The solution of equations (1), (2) for the case of a spatially localized target in vacuum is convenient to find as the superposition

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(0)}(\mathbf{r}) + \mathbf{E}^{(1)}(\mathbf{r}), \tag{3}$$

where $\mathbf{E}^{(0)}$ is the electric field of the incident wave that satisfies the equation

$$\left(\Delta + \frac{\omega^2}{c^2}\right)\mathbf{E}^{(0)} = 0. \tag{4}$$

Hence the field $\mathbf{E}^{(1)}$ in (3) has to be treated as the field of the scattered radiation.

Equation (1) can be written using (2), (3) and (4) in the form

$$\left(\Delta + \frac{\omega^2}{c^2}\right) \mathbf{E}^{(1)} = \frac{\omega^2}{c^2}(1-\varepsilon)\mathbf{E} + \mathrm{grad}\,\mathrm{div}\,((1-\varepsilon)\mathbf{E}). \tag{5}$$

The last equation could be presented in the integral form,

$$\mathbf{E}^{(1)}(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}') \left\{ \frac{\omega^2}{c^2}(1-\varepsilon(\mathbf{r}'))\mathbf{E}(\mathbf{r}') + \mathrm{grad}\,\mathrm{div}\,((1-\varepsilon(\mathbf{r}'))\mathbf{E}(\mathbf{r}')) \right\} d^3r', \tag{6}$$

where $G(\mathbf{r}-\mathbf{r}')$ is the Green function of equation (5),

$$G(\mathbf{r}-\mathbf{r}') = \int \frac{e^{i\boldsymbol{\kappa}(\mathbf{r}-\mathbf{r}')}}{(\omega/c)^2 - \kappa^2 + i0} \frac{d^3\kappa}{(2\pi)^3}. \tag{7}$$

The asymptotic of that function on large distances r from the domain where $\varepsilon(\mathbf{r}) \neq 1$,

$$G(\mathbf{r}-\mathbf{r}')\big|_{r\to\infty} \to -\frac{1}{4\pi}\frac{e^{ik_f r}}{r}\, e^{-i\mathbf{k}_f\mathbf{r}'}, \tag{8}$$

where $\mathbf{k}_f = (\omega/c)\mathbf{r}/r$ is the wave vector of the scattered wave, is needed to find the field of the scattered radiation. Substituting the last formula into (6) and integrating the second term by parts, we obtain

$$\mathbf{E}^{(scattered)}(\mathbf{r}) = \mathbf{E}^{(1)}(\mathbf{r})\Big|_{r\to\infty} = -\frac{1}{4\pi}\frac{e^{ik_f r}}{r}\left(\frac{\omega^2}{c^2}\mathbf{I} - \mathbf{k}_f(\mathbf{k}_f\cdot\mathbf{I})\right), \tag{9}$$

where

$$\mathbf{I} = \int (1-\varepsilon(\mathbf{r}))\,\mathbf{E}(\mathbf{r})\, e^{-i\mathbf{k}_f\mathbf{r}} d^3r. \tag{10}$$

We see that the integrand in (10) would be nonzero only in the domain where the dielectric permittivity is not equal to unit. The origin of the scattered radiation from the motion of the electrons in the medium excited by the incident wave becomes especially clear.

The integral form (6) of the field equation (5) that leads to (9), (10) is useful for construction of various approximate solutions. The simplest case is

$$|1-\varepsilon| \ll 1, \tag{11}$$

when the difference $(1-\varepsilon)$ in (10) could be treated as a small perturbation, hence the whole electric field in the target $\mathbf{E}(\mathbf{r})$ in (10) could be replaced by the field

$$\mathbf{E}^{(0)} = \mathbf{e}_i e^{i\mathbf{k}_i\mathbf{r}} \tag{12}$$

of the incident wave (where $\mathbf{k}_i$ is the incident wave vector, $|\mathbf{k}_i| = |\mathbf{k}_f|$, $\mathbf{e}_i$ is its polarization vector, and the wave amplitude has no matter for computation of the scattering cross section). This approximation is equivalent to Born approximation in the quantum theory of scattering (see, e.g., [3]). According to that analogy, the approximation is valid under

$$|1-\varepsilon|\left(\frac{a\omega}{c}\right)^2 \ll 1 \tag{13}$$

(where a is the characteristic size of the spatial domain with $\varepsilon \neq 1$) for $a\omega/c \ll 1$ and under

$$|1-\varepsilon|\frac{a\omega}{c} \ll 1 \tag{14}$$

for $a\omega/c \gg 1$.

The situation when (11) is valid, but (14) is not, needs another approach. The analogy to the quantum theory of scattering suggests the eikonal approximation in this case. The eikonal approximation means that the solution of equations (1), (2) has to be found in the form (see also [4])

$$\mathbf{E}(\mathbf{r}) = e^{i\mathbf{k}_i\mathbf{r}}\mathbf{\Phi}(\mathbf{r}) = e^{i(\omega/c)z}\mathbf{\Phi}(\mathbf{r}), \tag{15}$$

where the direction of the wave incidence on the target $\mathbf{k}_i$ is chosen as the z axis direction and the function $\mathbf{\Phi}(\mathbf{r})$ is supposed to change itself in space slow enough to neglect its second derivations in (1). Hence the solution of equation (1) that coincides with the non-disturbed field of the incident wave (12) under $z \to -\infty$ has the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_i \exp\left\{ i\frac{\omega}{c}\left[z - \frac{1}{2}\int_{-\infty}^{z} (1-\varepsilon(\mathbf{r}))\, dz' \right] \right\}. \tag{16}$$

The value $\mathbf{I}$ (10) with the field (16) is equal to

$$\mathbf{I} = \mathbf{e}_i \int e^{-i\mathbf{k}_{f\perp}\boldsymbol{\rho}}\, d^2\rho \int_{-\infty}^{\infty} (1-\varepsilon(\boldsymbol{\rho},z)) \exp\left\{ i\left[\left(\frac{\omega}{c} - k_{f_z}\right) z - \frac{\omega}{2c}\int_{-\infty}^{z} (1-\varepsilon(\boldsymbol{\rho},z'))\, dz' \right] \right\} dz. \tag{17}$$

Calculating the average energy flux of the scattered wave (7) to the solid angle $d\Omega$ and dividing it by the average intensity of the incident wave we obtain the scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\omega^2}{(4\pi)^2 c^2} \left| \mathbf{k}_f \times \mathbf{I} \right|^2, \tag{18}$$

3. THE SCATTERING ON THE DIELECTRIC CYLINDER

The formulae (17) and (18) describe the radiation scattering by the target of arbitrary structure. Consider now the simplest case of uniform cylinder of the radius a and the length $L \to \infty$ as the target (figure 1), when the cylinder's axis makes the angle ψ with the direction of the wave incidence $\mathbf{k}_i$. The integrals in (17) could be calculated after rotation of the coordinate axes to coincidence of the new axis z' with the cylinder's axis; the result could be written in the form

$$\mathbf{I} = -i\mathbf{e}_i(1-\varepsilon)\frac{2\pi\sin\psi\ \delta(k_{i_\parallel} - k_{f_\parallel})}{\frac{\omega}{c} - k_{f_z} - \frac{\omega}{2c}(1-\varepsilon)} \times \tag{19}$$

$$\times \left\{ \int_{-a}^{a} e^{i\left[-k_{f_y}y + \left(\frac{\omega}{c}\varepsilon - k_{f_z}\right)\frac{\sqrt{a^2-y^2}}{\sin\psi} \right]} dy - \int_{-a}^{a} e^{i\left[-k_{f_y}y - \left(\frac{\omega}{c} - k_{f_z}\right)\frac{\sqrt{a^2-y^2}}{\sin\psi} \right]} dy \right\}.$$

The presence of the δ-function expresses the equality of the components of $\mathbf{k}_i$ and $\mathbf{k}_f$ that parallel to the cylinder's axis, that means the azimuthal character of the scattering: since the absolute values of the wave vectors $\mathbf{k}_i$ and $\mathbf{k}_f$ are also equal, the scattered radiation will be directed along the surface of the cone with the axis along the cylinder and the half-opening angle equal to the incidence angle ψ.

The azimuthal character of the scattering permits clear interpretation as a manifestation of Cherenkov mechanism. Indeed, the incident wave produces a perturbation in the medium that moves along the fiber with the phase velocity $v = \omega/(k_i)_\parallel = c/\cos\psi > c$. This superluminal motion generates the radiation analogous to Cherenkov one. The half-opening angle of the Cherenkov cone for this case $\theta = \arccos(c/v)$ is just equal to ψ (figure 2).

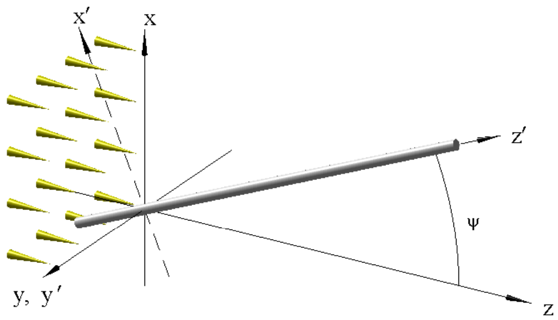

Figure 1. The radiation incidence along the z direction on the dielectric cylinder tilted by the angle ψ.

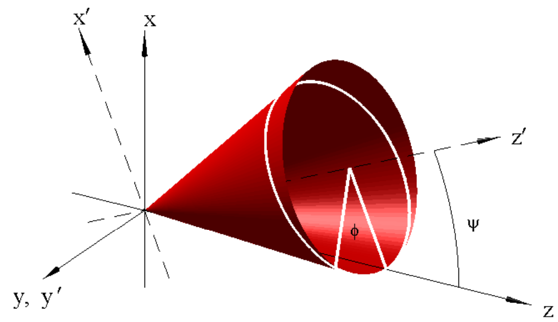

Figure 2. Direction diagram for the radiation scattered by the cylinder on figure 1. White curve represents the relative intensity of the scattered radiation in the limit of infinitely thin dielectric fiber (precise result [1]), the cone represents the azimuthally symmetric scattering. The angle of incidence is $\psi = 0.4$ radian.

Substitution of (19) into (18) gives (using the rule of δ-function squaring, $[\delta(k_{i_\parallel} - k_{f_\parallel})]^2 = \delta(k_{i_\parallel} - k_{f_\parallel}) \cdot L/2\pi$) the cross section (averaged over polarizations of the incident wave):

$$\frac{d\sigma}{d\Omega} = L\delta(k_{i_\parallel} - k_{f_\parallel})F(\phi), \tag{20}$$

where the function $F(\phi)$ describes the angular dependence of the cross section; this dependence is determined by the single parameter ϕ (the azimuthal angle in the plane perpendicular to the cylinder's axis) due to the azimuthal character of the scattering given by the delta-function in (20):

$$F(\phi) = \frac{1}{4\pi}\left(\frac{a\omega}{c}\right)^2 (1-\varepsilon)^2(1+\cos\theta)^2 \left(\frac{\sin\psi}{\frac{1+\varepsilon}{2} - \cos\theta}\right)^2 \times \tag{21}$$

$$\times \left\{ \left[\int_0^1 \cos\left(\frac{a\omega}{c} n_y \xi\right) \left[\cos\left(\frac{a\omega}{c}\frac{\varepsilon - \cos\theta}{\sin\psi}\sqrt{1-\xi^2}\right) - \cos\left(\frac{a\omega}{c}\frac{1-\cos\theta}{\sin\psi}\sqrt{1-\xi^2}\right)\right] dy\right]^2 + \right.$$

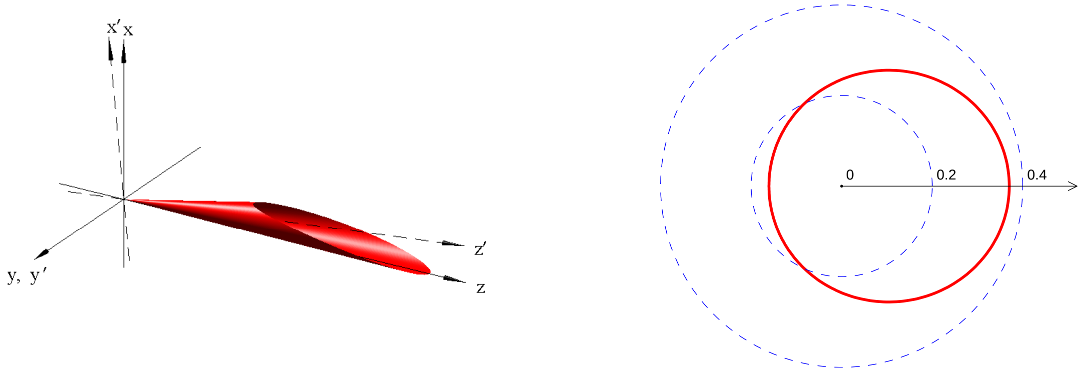

Figure 3. Plots of the function $F(\phi)$ (21) for $\varepsilon = 1.1$, $\psi = 0.1$, $a\omega/c = 50\sin\psi$ as the 3-dimensional direction diagram (left panel) and the polar plot (right panel; the polar axis corresponds to $\phi = 0$ that is the direction of the incident radiation).

$$+\frac{\pi^2}{4}\left[\frac{\varepsilon-\cos\theta}{\sqrt{n_y^2\sin^2\psi+(\varepsilon-\cos\theta)^2}}J_1\left(\frac{a\omega}{c\sin\psi}\sqrt{n_y^2\sin^2\psi+(\varepsilon-\cos\theta)^2}\right)-\right.$$

$$\left.\left.-\frac{1-\cos\theta}{\sqrt{n_y^2\sin^2\psi+(1-\cos\theta)^2}}J_1\left(\frac{a\omega}{c\sin\psi}\sqrt{n_y^2\sin^2\psi+(1-\cos\theta)^2}\right)\right]^2\right\}.$$

Indeed, both values in (21) that depend on the scattering direction, the angle θ between $\mathbf{k}_f$ and the z axis, and the projection n_y of the unit vector $(c/\omega)\mathbf{k}_f$ on the y axis, could be expressed via the azimuthal angle ϕ:

$$\cos\theta = (c/\omega)(k_f)_z = \cos\left[2\arcsin\left(\sin\psi\,\sin\frac{|\phi|}{2}\right)\right], \tag{22}$$

$$n_y = (c/\omega)(k_f)_y = \sin\psi\sin\phi, \tag{23}$$

where the angle ϕ is measured from the direction $(\mathbf{k}_i)_\perp$ (see figure 2).

The polar plots of the function $F(\phi)$ for some specific values of the parameters a, ω, ε and ψ are presented in figures 3 and 4. We see the substantial azimuthal asymmetry of the scattering cross section for large $a\omega/c$ (that is for large ratio of the cylinder's radius to the radiation wavelength). Similar behavior had been demonstrated earlier for the transition radiation of fast particles on the targets with cylindrical symmetry in Born [5] and eikonal [4] approximations.

4. Conclusion

The scattering of the electromagnetic wave under its oblique incidence on the linear extended target is considered. The formula for the scattering cross section is obtained using eikonal approximation. The azimuthal character of the scattering is demonstrated as well as the axial asymmetry around the target axis for the finite angle of incidence and the finite target thickness. The results could be used for improving the kinetic theory of propagation of the radiation through the system of parallel fibers [1].

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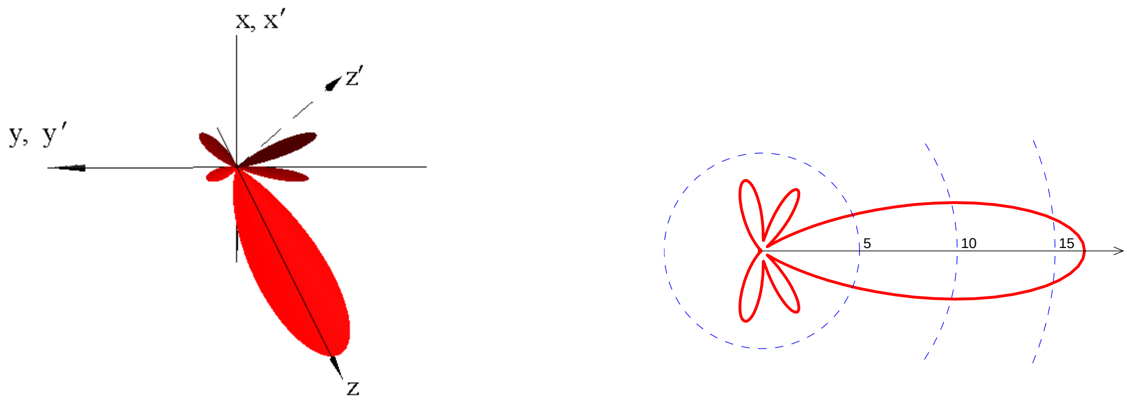

Figure 4. Plots of the function $F(\phi)$ (21) for $\varepsilon = 1.2$, $\psi = 0.1$, $a\omega/c = 500\sin\psi$ as the 3-dimensional direction diagram (left panel) and the polar plot (right panel; the polar axis corresponds to $\phi = 0$ that is the direction of the incident radiation).

References

[1] Shul'ga N F and Syshchenko V V 2014 *Problems of Atomic Science and Technjlogy* **93** (5) 115
[2] Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Media* (Oxford: Pergamon Press)
[3] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon Press)
[4] Shul'ga N F and Syshchenko V V 2006 *Advanced Radiation Sources and Applications* ed H Wiedemann (Springer Netherlands) pp 129–148
[5] Shul'ga N F and Syshchenko V V 2003 *Phys. Lett.* A **313** 307