

Mathematics of oscillations and waves

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Abstract. Analytical definition of the sine function and the number π which is related to sine function allows to understand, how harmonic oscillations and waves appear. The functional equation for the sine is received. The new fast algorithm of calculation of the π number is formulated. In the elementary case oscillations and waves are harmonious or sinusoidal. The sine function appears not accidentally. This function can be defined as the solution of the functional equation characterizing periodic properties of oscillations and waves.

1. The functional equation

We will consider the functional equation

$$f(x+p) = -f(x), \quad p > 0, \quad x \in \mathbf{R},$$

in which p is half-period of the f function. The half-period p unambiguously defines period $2p$ of a periodic function.

The general solution of the equation in the class of analytical functions bounded on the real straight line is a three-parameter family of sinusoids

$$\begin{cases} f(x) = a \sin(bx + c); \\ p = \frac{\pi}{b}, \end{cases}$$

where a, b, c are free parameters.

The particular solution of the equation

$$f(x) = r \sin \left[\frac{\pi}{p} (x - q) \right]$$

follows from the system

$$\begin{cases} f(x+p) = -f(x); \\ f(q) = 0; \\ f\left(q + \frac{p}{2}\right) = r \end{cases}$$

with given parameters p, q, r .

Finally, the system

$$\begin{cases} f(x+p) = -f(x); \\ f(0) = 0; \\ f'(0) = 1; \\ f\left(\frac{p}{2}\right) = 1 \end{cases}$$

gives the only solution

$$\begin{cases} f(x) = \sin x, \\ p = \pi. \end{cases}$$

Thereby, the functional equation allows to define the sine in a new way.

2. The number π

Important statements for the number π also follow from the functional equation.

We will consider the mapping

$$F = \{\sin, \arcsin\} = \sin \circ \arcsin + \arcsin \circ \sin = \text{id} + \arcsin \circ \sin$$

defined as anti-commutator of sine and arcsine. By means of the symbol id here is designated identical function.

Defenition. The number π is a stable fixed point of mapping F of a unit disc $|z-3| \leq 1$ of complex plane onto itself.

Contracted mapping F has the only one fixed point in the disc. Stability of the point means that $F(\pi + \varepsilon) = \pi$ for small absolute values of arbitrary complex number ε .

Theorem. $\pi \equiv F(z)$, where $|z-3| \leq 1$.

Corollary. $\pi = F(3)$ or $\pi = 3 + \arcsin \sin 3$.

The number π which is the $z = F(z)$ equation root we can find by the successive approximations method. The necessary condition of the method's convergence is defined by a double inequality $-1 < F(z) < 1$. As $F'(z) \equiv 0$ in a unit disc, so the method is reduced to one iteration $z \mapsto \pi$ that means infinitely high rate of convergence of successive approximations to a fixed point. In other words, mapping F is greatly contracted. Therefore, the theorem gives new fast algorithm for the number π calculation.

3. Rate of convergence

Rate of convergence of iterative process shows how quickly the sequence of approximations converge to the limit: the number of the correct decimal signs of one iteration approach increases in k times. For comparison, we will give values of convergence rate for the most important iterative methods of the solution of the algebraic and transcendental equations:

$k \geq 1$ Successive approximations method,

$k = \tau$ Secant method or chord method ($\tau = 1,618\dots$ – a golden section),

$k = 2$ Newton's method or method of tangents,

$k = 3$ Halley's method,

$k = \infty$ Chebyshev's method,

$k = \infty$ New fast algorithm of the number π calculation.

One iteration gives exact value of a limit when there is the infinitely large rate of convergence.

4. Calculations of the number

π by means of formulae

$$\begin{aligned}\pi &= \{ \sin, \arcsin \} (2) = \{ \sin, \arcsin \} (3) = \{ \sin, \arcsin \} (4) = \\ &= \{ \sin, \arcsin \} (3+i) = \{ \sin, \arcsin \} (\tau) = \{ \sin, \arcsin \} (e),\end{aligned}$$

subject to preservation N decimal digits for all received intermediate numbers, give N correct signs of the number π for each formula. Here $e = 2,718\dots$ is the natural logarithmic base, i is the imaginary unit. It speaks about the stability of computing process which is coming from stability of the fixed point of mapping F .

5. The law of similarity

Let the number π be calculated with a precision N decimal signs by means of our algorithm, let t be time of calculation of the number π , t_* be characteristic time of calculation.

Computational experiments give a formula $\frac{t}{t_*} = b N^a$ establishing a power dependence of the

calculation time of the received signs. Parameters a and b have constant values for this computer. We will say that two calculations (N_1, t_1) and (N_2, t_2) are similar to each other if they have the same a and b . Thus, the last formula can be characterized as an experimental law of similarity of calculations. The formula allows to estimate calculation time for a large number of signs of the number π after rather small number of such signs is found.

Today the number π is calculated with the astronomical precision: 13 000 000 000 000 decimal signs (2014). Now the hardly foreseeable set of formulae, mathematical and physical facts are connected with this number. Their quantity continues to grow promptly.

For comparison, we will consider the sequence of the famous mathematical constants, according to their complexity growth of their calculation: 1, τ , e , π , α & δ . Here α & δ are hardly computable constants of Feigenbaum.

6. The sine function

To be exact a sinusoid $y = a \sin(bx + c)$, is the simplest and most important function among all periodic functions. It was clear for Galileo, Newton, Fourier, Maxwell, Rayleigh and de Broglie.

Galileo Galilei (Dialogue Concerning the Two Chief World Systems – Ptolemaic and Copernican, The first day [1]) has formulated three simple curves: straight line, circle and spiral (helix). In fact, the simple curve is the same thing as a spiral, and the line and the circle – special cases. There is only one smooth connected regular curve in the three-dimensional space. This is the spiral. It is the regular curve in the sense that each its point is equally surrounded by the other points of the curve. According to differential geometry, curvature and torsion at each point of the spiral are constants. The projection of the spiral on a plane parallel to an axis of symmetry of the spiral gives a sinusoid.

Isaac Newton (Mathematical Principles of Natural Philosophy, Proposition XLVII, Theorem XXXVII [2]) asserted that: "If impulses are propagated through a fluid, the several particles of the fluid, going and returning with the shortest reciprocal motion, are always accelerated or

retarded according to the law of the oscillation pendulum". In the modern language atoms and molecules of liquid perform harmonic or sinusoidal oscillations. Another important facts relating to the sine function can be found in the literature [3-5].

7. The new formula

Which are given here for the sine and the number π will be useful in the theory of harmonic oscillations and waves.

References

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