Analysis of a QCD Hamiltonian in the low energy regime.

T. Yépez-Martínez[1]**, D. A. Amor Quiroz**[2]**, P. O. Hess**[2] **and O. Civitarese**[1]

[1] Departamento de Física, Universidad Nacional La Plata, C.C.67 (1900), La Plata, Argentina
[2] Instituto de Ciencias Nucleares, UNAM, Circuito Exterior, C.U., A.P. 70-543, 04510 México, D.F., México.

E-mail: yepez@fisica.unlp.edu.ar, arturo.amor@nucleares.unam.mx, hess@nucleares.unam.mx, osvaldo.civitarese@fisica.unlp.edu.ar

Abstract. We present a QCD motivated Hamiltonian for the light quark sector. Inspired from self-consistent analysis of the Coulomb interaction, we implement an interaction of the form $(-a/r + br)$ between color sources, which already consider gluonic dynamics by the linear potential contribution. A prediagonalization of the kinetic energy term followed by the implementation of the Tamm-Dancoff method are used to obtain the eigenvalues of the Hamiltonian. A variational analysis is implemented to obtain the optimized basis for the low energy meson spectrum. The potential parameter is compared to the reported lattice string tension with relatively good agreement. The obtained energies are located close to the experimental values and further improvements are discussed.

1. Introduction

The Coulomb gauge QCD Hamiltonian is derived in [1], where the underlying interactions are dominated by the instantaneous Coulomb potential acting between color charges. In the non-Abelian theory, the potential not only couples charges but it also depends on the gluon distribution of the state in which it is calculated [2, 3, 4]. An effective Hamiltonian with the aim of describing the low energy regime of the meson spectrum should consider the dynamics of quarks and their confinement on a finite volume. Following that direction, we take the kinetic plus the mass term and the Coulomb interaction for the quarks sector from the QCD Hamiltonian. The quark kinetic term at low energy has been analyzed using the three dimensional harmonic oscillator basis [5, 6] by analytic and semi-analytic methods as well as exact diagonalization [7]. On the other hand, in [5, 6, 8] the Coulomb QCD interaction between color charge densities had been approximated by a contact interaction resulting in a second order Casimir operator of the color group. The QCD Coulomb interaction has been formally analyzed using self-consistent methods like Dyson-Schwinger equations to obtain its behavior [2, 3, 4], which in coordinate representation seems to reproduce the confining potential reported from lattice calculations [9]. Motivated by this findings, in the present work we now consider a more realistic interaction of the type Coulomb $(-\frac{a}{r})$ plus linear (br) potential to analyze the low energy spectrum of mesons when the Tamm-Dancoff method is implemented to describe any possible combination of quark-antiquarks at low energy. Finally, we discus the procedure

to be taken in order to get better results compared to the reported [13] meson spectrum at low energies.

2. Coulomb gauge approach and the basic ingredients

The motivated QCD Hamiltonian, which we consider is built by the kinetic and mass terms together with a confining interaction between quark color charge densities. This Hamiltonian will resembles the gluon dynamics by introducing a linear potential.

$$H_{eff} = \int \left\{ \psi^\dagger(\mathbf{r})(-i\alpha \cdot \nabla + \beta m)\psi(\mathbf{r}) \right\} d\mathbf{r} - \frac{1}{2} \int \rho_a(\mathbf{r}) V(|\mathbf{r} - \mathbf{r}'|)\rho^a(\mathbf{r}') d\mathbf{r} d\mathbf{r}' \qquad (1)$$

where $\rho^a(\mathbf{r}) = \rho_q^a(\mathbf{r}) = \psi^\dagger(\mathbf{r})T^a\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r}) = (\psi_1^\dagger(\mathbf{r}, \sigma, c, f), \psi_2^\dagger(\mathbf{r}, \sigma, c, f))$ with

$$
\begin{aligned}
\psi_1^\dagger(\mathbf{r}, \sigma, c, f) &= \sum_{Nlm\sigma cf} \mathbf{b}_{\frac{1}{2},Nlm,\sigma cf}^\dagger R_{Nl}^*(r) Y_{lm}^*(\hat{\mathbf{r}}) \chi_\sigma^\dagger \\
\psi_2^\dagger(\mathbf{x}, \sigma, c, f) &= \sum_{Nlm\sigma cf} \mathbf{b}_{-\frac{1}{2},Nlm,\sigma cf}^\dagger R_{Nl}^*(r) Y_{lm}^*(\hat{\mathbf{r}}) \chi_\sigma^\dagger
\end{aligned}
\qquad (2)
$$

The index $\alpha = \pm\frac{1}{2}$ denotes the upper or lower Dirac level, N refers to the number of oscillation-quanta of the orbital level in the Dirac picture and l the angular momentum. The coupled representation of angular momentum and intrinsic spin-$\frac{1}{2}$ to total spin (j) is given by

$$\mathbf{b}_{\alpha(N,l,\frac{1}{2})j\lambda,cf}^\dagger = \sum_{m\sigma} \langle lm, \tfrac{1}{2}\sigma | j\lambda \rangle \mathbf{b}_{\alpha Nlm,\sigma cf}^\dagger \qquad (3)$$

Because of confinement, the domain of fields in Eq. (2) is expected to be restricted to a finite volume in space, where individual hadrons are located. Therefore, the eigenfunctions of the three dimensional harmonic oscillator are chosen as the single-particle orbitals.

$$R_{Nl}(r) = N_{Nl} \exp(-\frac{\gamma r^2}{2}) r^l L_{\frac{N-l}{2}}^{l+\frac{1}{2}}(\gamma r^2) \qquad (4)$$

Therefore, the particles are restricted to a finite volume. Below, we show that the variational analysis provides an optimized value for the oscillator length $(\frac{1}{\sqrt{\gamma}})$.

The use of the harmonic oscillator basis for the orbital part seems to restrict the validity of the approximations to a non-relativistic theory, but this is only true when individual levels are considered to describe quark states. When the complete basis is used, any relativistic state can be expanded into the non-relativistic basis. This requires to expand the relativistic states into, in general, an infinite sum. When sufficient basis states are included, this sum can be limited in such a way that adding new terms would not modify the results.

2.1. The Hamiltonian matrix elements.
Using the fermion field quatization given in Eqs. (2-4) the Hamiltonian is expressed as

$$
\begin{aligned}
H_{eff} &= \sum_{\{\alpha_i N_i l_i j_i\}} K_{\alpha_1 N_1 l_1, \alpha_2 N_2 l_2}^{j} \left(b_{\alpha_1(N_1 l_1 \frac{1}{2})j_1}^\dagger \cdot b^{\alpha_2(N_2 l_2 \frac{1}{2})j_2} \right) \\
&+ \sum_{\{\alpha_i N_i l_i j_i; L\}} V_{\{N_i l_i j_i\}}^L \left[\left[b_{\alpha_1(N_1 l_1 \frac{1}{2})j_1}^\dagger \otimes b_{\alpha_2(N_2 l_2 \frac{1}{2})j_2} \right]^\Gamma \otimes \left[b_{\alpha_3(N_3 l_3 \frac{1}{2})j_3}^\dagger \otimes b_{\alpha_4(N_1 l_1 \frac{1}{2})j_4} \right]^\Gamma \right]^{\hat{0}} (5)
\end{aligned}
$$

where $\Gamma = L, (1,1)_1, (0,0)_1$, indicating the intermediate couplings of spin, color and flavor quantum numbers. The kinetic matrix elements are given by

$$
K^j_{\alpha_1 N_1 l_1, \alpha_2 N_2 l_2} =
\begin{cases}
k^j_{N_1 N_2} & \text{if } l_1 = j + \frac{1}{2}, \ l_2 = j - \frac{1}{2}, \ \alpha_1 \neq \alpha_2 \\
k^j_{N_2 N_1} & \text{if } l_1 = j - \frac{1}{2}, \ l_2 = j + \frac{1}{2}, \ \alpha_1 \neq \alpha_2 \\
m_0 & \text{if } (N_1 l_1) = (N_2 l_2), \ \alpha_1 = \alpha_2 = +\frac{1}{2} \\
-m_0 & \text{if } (N_1 l_1) = (N_2 l_2), \ \alpha_1 = \alpha_2 = -\frac{1}{2} \\
0 & \text{in all other cases}
\end{cases}
$$

with

$$
k^j_{N_1 N_2} = \sqrt{\gamma} \sqrt{\frac{N_1 - j + \frac{3}{2}}{2}} \delta_{N_2, N_1+1} + \sqrt{\gamma} \sqrt{\frac{N_1 + j + \frac{3}{2}}{2}} \delta_{N_2, N_1-1} \tag{6}
$$

and the potential matrix elements

$$
V^L_{\{N_i, l_i, j_i\}} = \sqrt{8(2L+1)} 2\mathrm{dimF} \frac{(-1)^L}{4(2L+1)} (-1)^{j_2 + \frac{1}{2} + j_4 + \frac{1}{2}} \prod_{i=1}^{4} \sqrt{(2l_i + 1)(2j_i + 1)}
$$

$$
\times \langle l_1 0, l_2 0 | L 0 \rangle \langle l_3 0, l_4 0 | L 0 \rangle
\begin{Bmatrix} j_1 & l_1 & \frac{1}{2} \\ l_2 & j_2 & L \end{Bmatrix}
\begin{Bmatrix} j_3 & l_3 & \frac{1}{2} \\ l_4 & j_4 & L \end{Bmatrix}
I^L_{n_i l_i} . \tag{7}
$$

with

$$
I^L_{n_i l_i} = \int r^2 dr \ r'^2 dr' R^*_{N_1 l_1}(r) R_{N_2 l_2}(r) R^*_{N_3 l_3}(r') R_{N_4 l_4}(r') \int_{-1}^{1} d(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') P_L(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') V(|\mathbf{r} - \mathbf{r}'|) . \tag{8}
$$

Using the confinning interaction

$$
V(|\mathbf{r} - \mathbf{r}'|) = -\frac{a}{|\mathbf{r} - \mathbf{r}'|} + b|\mathbf{r} - \mathbf{r}'| , \tag{9}
$$

we have found analytic expressions [15] for Eq. (8) which have been corroborated numerically.

3. The diagonalization of the kinetic energy term and the trial basis.
The kinetic term diagonalization is achieved by the method described in [7] which we resume here to consequently implement the trial basis to re-write the effective Hamiltonian.

The first step to diagonalize the kinetic energy is to consider a unitary transformation of the form

$$
\boldsymbol{b}^\dagger_{\alpha(N,l,\frac{1}{2})j\lambda, cf} = \sum_{\delta k} \left(\alpha^j_{\alpha Nl, \delta k} \right)^* \hat{\boldsymbol{b}}^\dagger_{\delta k j \lambda, cf} \tag{10}
$$

with the following identification

$$
\alpha^j_{\frac{1}{2} Nl, \frac{1}{2}k} = \gamma^j_{Nl,k} \quad , \quad \alpha^j_{\frac{1}{2} Nl, -\frac{1}{2}k} = -\beta^j_{Nl,k}
$$

$$
\alpha^j_{-\frac{1}{2} Nl, \frac{1}{2}k} = \beta^j_{Nl,k} \quad , \quad \alpha^j_{-\frac{1}{2} Nl, -\frac{1}{2}k} = \gamma^j_{Nl,k} \tag{11}
$$

leading to the transformation

$$
\boldsymbol{b}^\dagger_{\frac{1}{2}(N,l,\frac{1}{2})j\lambda, cf} = \sum_k \left(\gamma^j_{Nl,k} \boldsymbol{b}^\dagger_{kj\lambda, cf} - \beta^j_{Nl,k} \boldsymbol{d}_{kj\lambda, cf} \right) ,
$$

$$
\boldsymbol{b}^\dagger_{-\frac{1}{2}(N,l,\frac{1}{2})j\lambda, cf} = \sum_k \left(\beta^j_{Nl,k} \boldsymbol{b}^\dagger_{kj\lambda, cf} + \gamma^j_{Nl,k} \boldsymbol{d}_{kj\lambda, cf} \right) \tag{12}
$$

and the matrix to be diagonalized given by

$$k^j_{k_1 k_2} = \sum_{\alpha_i \delta_i N_i l_i} K^j_{\alpha_1 N_1 l_1, \alpha_2 N_2 l_2} \left(\alpha^j_{\alpha_1 N_1 l_1, \delta_1 k_1} \right)^* \left(\alpha^j_{\alpha_2 N_2 l_2, \delta_2 k_2} \right) \qquad (13)$$

The corresponding eigenvalues (ϵ_{kj}) and eigenvectors allow us to re-write the Hamiltonian in the following way

$$
\begin{aligned}
H_{eff} &= \sum_{kj} \epsilon_{kj} \left[\left(b^\dagger_{kj} \cdot b^{kj} \right) - \left(d_{kj} \cdot d^{\dagger\, kj} \right) \right] \\
&\quad - \sum_{L k_i, j_i} \left\{ E^L_{k_i, j_i} \left[\left([b^\dagger_{k_1 j_1} b_{k_2 j_2}]^\Gamma - [d_{k_1, j_1} d^\dagger_{k_2 j_2}]^\Gamma \right) \left([b^\dagger_{k_3, j_3} b_{k_4, j_4}]^\Gamma - [d_{k_3, j_3} d^\dagger_{k_4 j_4}]^\Gamma \right) \right]^{\hat{0}} \right. \\
&\quad + F^L_{k_i, j_i} \left[\left([b^\dagger_{k_1 j_1} b_{k_2 j_2}]^\Gamma - [d_{k_1 j_1} d^\dagger_{k_2 j_2}]^\Gamma \right) \left([d_{k_3, j_3} b_{k_4 j_4}]^\Gamma + [b^\dagger_{k_3 j_3} d^\dagger_{k_4 j_4}]^\Gamma \right) \right]^{\hat{0}} \\
&\quad + F'^L_{k_i, j_i} \left[\left([d_{k_1 j_1} b_{k_2 j_2}]^\Gamma + [b^\dagger_{k_1 j_1} d^\dagger_{k_2 j_2}]^\Gamma \right) \left([b^\dagger_{k_3 j_3} b_{k_4 j_4}]^\Gamma - [d_{k_3 j_3} d^\dagger_{k_4 j_4}]^\Gamma \right) \right]^{\hat{0}} \\
&\quad \left. + G^L_{k_i, j_i} \left[\left([d_{k_1 j_1} b_{k_2 j_2}]^\Gamma + [b^\dagger_{k_1 j_1} d^\dagger_{k_2 j_2}]^\Gamma \right) \left(d_{k_3, j_3} b_{k_4, j_4} + b^\dagger_{k_3, j_3} d^\dagger_{k_4, j_4} \right) \right]^{\hat{0}} \right\} \qquad (14)
\end{aligned}
$$

with

$$
\begin{aligned}
E^L_{k_i, j_i} &= \sum_{N_i, l_i} V^L_{\{N_i l_i j_i\}} \left[\left(\gamma^{j_1}_{N_1 l_1, k_1} \gamma^{j_2}_{N_2 l_2, k_2} + \beta^{j_1}_{N_1 l_1, k_1} \beta^{j_2}_{N_2 l_2, k_2} \right) \left(\gamma^{j_3}_{N_3 l_3, k_3} \gamma^{j_4}_{N_4 l_4, k_4} + \beta^{j_3}_{N_3 l_3, k_3} \beta^{j_4}_{N_4 l_4, k_4} \right) \right] \\
F^L_{k_i, j_i} &= \sum_{N_i, l_i} V^L_{\{N_i l_i j_i\}} \left[\left(\gamma^{j_1}_{N_1 l_1, k_1} \gamma^{j_2}_{N_2 l_2, k_2} + \beta^{j_1}_{N_1 l_1, k_1} \beta^{j_2}_{N_2 l_2, k_2} \right) \left(\gamma^{j_3}_{N_3 l_3, k_3} \beta^{j_4}_{N_4 l_4, k_4} - \beta^{j_3}_{N_3 l_3, k_3} \gamma^{j_4}_{N_4 l_4, k_4} \right) \right] \\
F'^L_{k_i, j_i} &= \sum_{N_i, l_i} V^L_{\{N_i l_i j_i\}} \left[\left(\gamma^{j_1}_{N_1 l_1, k_1} \beta^{j_2}_{N_2 l_2, k_2} - \beta^{j_1}_{N_1 l_1, k_1} \gamma^{j_2}_{N_2 l_2, k_2} \right) \left(\gamma^{j_3}_{N_3 l_3, k_3} \gamma^{j_4}_{N_4 l_4, k_4} + \beta^{j_3}_{N_3 l_3, k_3} \beta^{j_4}_{N_4 l_4, k_4} \right) \right] \\
G^L_{k_i, j_i} &= \sum_{N_i, l_i} V^L_{\{N_i l_i j_i\}} \left[\left(\gamma^{j_1}_{N_1 l_1, k_1} \beta^{j_2}_{N_2 l_2, k_2} - \beta^{j_1}_{N_1 l_1, k_1} \gamma^{j_2}_{N_2 l_2, k_2} \right) \left(\gamma^{j_3}_{N_3 l_3, k_3} \beta^{j_4}_{N_4 l_4, k_4} - \beta^{j_3}_{N_3 l_3, k_3} \gamma^{j_4}_{N_4 l_4, k_4} \right) \right]
\end{aligned}
$$

$$(15)$$

4. Tamm Dancoff equation and solutions.

We implement the Tamm Dancoff (TD) method [12], to define a new pair of quark anti-quark as a linear combination of the pairs $\gamma^\dagger_{12, \Gamma \mu} = \left[\boldsymbol{b}^\dagger_1 \otimes \boldsymbol{d}^\dagger_2 \right]^\Gamma_\mu$ where we used the short hand notation $\Gamma = (J, (0,0)_1, (\lambda_f, \lambda_f)_1)$ to denote the spin, color and flavor ($\lambda_f = 0, 1$) of the pair as well as $1 = (k_1, j_1)$ and $2 = (k_2, j_2)$. Therefore, the new pair is given by
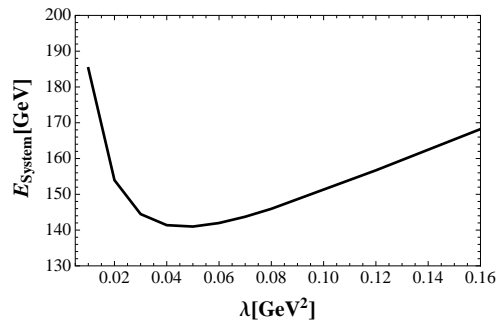
$$\tilde{\gamma}_{\kappa \Gamma \mu} = \sum_{12} C^{\kappa \Gamma}_{12} \gamma^\dagger_{12, \Gamma \mu} \qquad (16)$$

which represents a mixing between all pairs formed by all possible combinations of two orbitals. The index κ labels the new pairs and it runs from zero to the number of possibilities to form pairs. The old and new vacuum coincide, because it is a linear combination of the old annihilation operators. The resulting matrix equation is of the form

$$\sum_{1'2'} M^\Gamma_{12, 1'2'} C^{\kappa \Gamma}_{1'2'} = E^{\kappa \Gamma} C^{\kappa \Gamma}_{12} \qquad (17)$$

with

$$M^\Gamma_{12, 1'2'} = \langle \tilde{0} | \left[\gamma^{12, \Gamma \mu}, \left[\boldsymbol{H}, \gamma^\dagger_{1'2', \Gamma \mu} \right] \right] | \tilde{0} \rangle \ , \qquad (18)$$

Figure 1. Variation of the energy of the system as a function of the har. osc. basis parameter.

Table 1. Tamm-Dancoff solutions for the lowest six $J = 0$ and 1 states.

| E_{TD}[GeV]	J=0	J=1
E_1	0.466	0.920
E_2	0.683	0.977
E_3	0.807	1.037
E_4	0.914	1.066
E_5	0.917	1.087
E_6	1.013	1.114

where the matrix M does not depend on the magnetic numbers and it is well defined by using Eqs. (7) and (15) and numerical techniques [10].

One feature we noted is that the $-\frac{a}{r}$ interaction does not have a big influence on the results, which is due to the low energy, when long-range effects are dominant. This leaves only one parameter free (b). In order to obtain the spectrum, we have performed a variational approach on the energy of the system to optimize the harmonic oscillator parameter (γ) as it is shown on Fig. 1. This indicates a minimum on the energy of the system at about $\gamma = 0.05\text{GeV}^2$ corresponding to radius $R \approx 0.9$fm. The energy scales for the meson energy spectrum of spin $J = 0, 1$ are shown on Tab. 1. In the example presented here, we took for $b = 0.4\text{GeV}^2 \approx 1.5 \times \sigma_{lattice}$, where the lattice string tensions is typically 0.26GeV^2 [9].

In this sample calculation, we restricted to massless quarks, total single particle spin $j = \frac{1}{2}$, and take the maximal number of oscillation quanta $N = 12$. Eventhough the calculation is not yet in position to be compared with the experimental values [13], it certainly shows some improvement at the TD-level calculation compared to other TD-approach [14], Fig. 2. Showing that the finite range imposed on the domain of the fermion fields points in the right direction.

5. Conclusions and Discussion.
We have briefly presented the exact diagonalization of the kinetic energy of the QCD Hamiltonian at low energy which provides the trial basis to be used for the full diagonalization of the motivated QCD Hamiltonian. For the case of no interaction, there is a two-fold degeneracy of the states at low energy. The calculation including a confining interaction of the type $(-\frac{a}{r} + br)$ with massless quarks restricted to only $j = \frac{1}{2}$ column and a maximal number of oscillation quanta equal 12, shows no more such degeneracies. The optimization of the basis indicates that the calculation also consider a finite volume corresponding to a oscillator length of about 0.9fm.
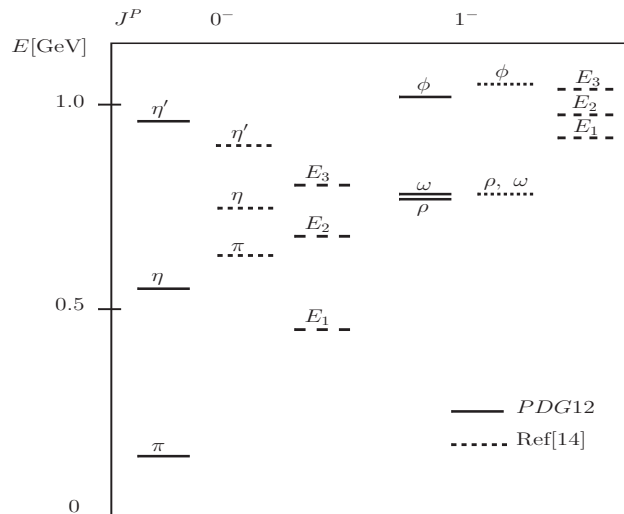
Figure 2. Tamm Dancoff energies compared to the reported in Refs. [13] and [14].

Noticing that the $-\frac{a}{r}$ contribution does not modify sensibly the results at low energy, we use only the linear interaction in order to investigate the spectrum of mesons when the Tamm-Dancoff method is implemented for each flavor-spin combination. The results indicate a patter which resembles some features of the low energy meson spectrum.

Work is in progress to include flavor symmetry breaking by using $m_{u,d} \neq m_s$ as well as the implementation of the Random Phase Approximation (RPA) in order to take into account particle-hole correlations in the vacuum of QCD, the results will be published somewhere else [15].

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