

# Phase transitions for rotational states within an algebraic cluster model

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**Abstract.** The ground state and excited, rotational phase transitions are investigated within the *Semimicroscopic Algebraic Cluster Model* (SACM). The catastrophe theory is used to describe these phase transitions. Short introductions to the SACM and the catastrophe theory are given. We apply the formalism to the case of  $^{16}\text{O} + \alpha \rightarrow ^{20}\text{Ne}$ .

## 1. Introduction

The *Semimicroscopic Algebraic Cluster Model* (SACM) [1, 2] is a successful cluster model for nuclei. It takes into account the Pauli exclusion principle for any cluster model. It was successfully applied to nuclei in the sd shell, where one of the last applications is published in [3]. Cluster systems of astrophysical interest were considered and also spectroscopic factors [4] calculated. A detailed study of quantum phase transitions of the ground state were published in [5, 6]. However, this study was done in a pedestrian way, not using the latest *technology* available, which is the *catastrophe theory* [7]. Excited, rotational states were considered in [8], but also here the treatment was simple.

The objective of this contribution is to apply the *catastrophe theory* to nuclear cluster system, in order to study in a very effective and detailed manner the structure of phase transitions in nuclear cluster systems, in the ground state and for excited states. The *catastrophe theory* was applied with great success, also in nuclear systems [9, 10]. Of special interest, because of its relation to the case presented here, is [11].

## 2. The Semimicroscopic Algebraic Cluster Model

The basic degrees of freedom are the relative oscillations, described by the  $\pi_m^\dagger$  creation operators and by the corresponding annihilation operators  $\pi^m$  ( $m = \pm 1, 0$ ). The products of a creation with an annihilation operator form the  $U_R(3)$  group, where  $R$  refers to the relative motion. A cutoff is introduced by adding a scalar boson, whose creation and annihilation operator are  $\sigma^\dagger$  and  $\sigma$  respectively. These bosons do not have a physical meaning but are merely introduced to create a cut-off. With the help of these bosons, one can introduce a cut-off  $N = n_\pi + n_\sigma$ . The



products of all types of boson creation with boson annihilation operators form a  $U_R(4)$  group. For a detailed description of the SACM, please consult [1, 2].

The model space is constructed by first calculating the product  $(\lambda_1, \mu_1) \otimes (\lambda_2, \mu_2) \otimes (n_\pi, 0)$ , where the  $(\lambda_k, \mu_k)$  ( $k = 1, 2$ ) describe the structure of the clusters 1 and 2 and  $n_\pi$  is the relative oscillation number. This results in a sum of many  $SU(3)$  irreducible representations (irreps)  $(\lambda, \mu)$  with a multiplicity  $m_{(\lambda, \mu)}$ . This list has to be compared with the content of the shell model, which can be easily obtained by computer routines, which are available to us and can be obtained on request. When a  $SU(3)$  irrep in the former mentioned list does not appear in the shell model, this irrep is excluded, while when it appears it is included into the model space of the SACM. In this way, the model space of the SACM is microscopic.

The Hamiltonian, however, is phenomenological and is written as a sum of Casimir operators. When we restrict to spherical clusters, i.e.  $(\lambda_k, \mu_k) = (0, 0)$ , there is no contributions from the individual clusters. There are two dynamical symmetry group chains of importance, called the  $SU_R(3)$  and the  $SO_R(4)$  dynamical symmetries. For the case discussed here, the Hamiltonian has the form

$$\mathbf{H} = x\mathbf{H}_{SU(3)} + (1 - x)\mathbf{H}_{SO(4)} - \Omega\mathbf{H}_{\text{crank}} \quad (1)$$

with

$$\begin{aligned} \mathbf{H}_{SU(3)} &= \hbar\omega\mathbf{n}_\pi + (a - b\Delta\mathbf{n}_\pi)\mathbf{C}_2(\mathbf{n}_\pi, 0) + \xi\mathbf{L}^2 \\ \mathbf{H}_{SO(4)} &= \frac{c}{4}[(\boldsymbol{\pi}^\dagger \cdot \boldsymbol{\pi}^\dagger) - (\boldsymbol{\sigma}^\dagger)^2][(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) - (\boldsymbol{\sigma})^2] + \xi\mathbf{L}^2 \\ \mathbf{H}_{\text{crank}} &= \mathbf{L}_x \quad , \end{aligned} \quad (2)$$

where the  $\Delta\hat{\mathbf{n}}_\pi = \hat{\mathbf{n}}_\pi - n_0$  is the excitation number of quanta and  $n_0$  is the minimal number of relative oscillation quanta [12], a necessary condition to satisfy the Pauli exclusion principle. The  $\mathbf{L}_x$  is the x-component of the angular momentum operator and  $\Omega$  is the frequency parameter which fixes the amount of the angular momentum. The structure of the Hamiltonian is equivalent to introduce a Lagrange multiplier [13].

The semi-classical potential is obtained by first defining a coherent state [14]

$$|\alpha\rangle = \frac{N!}{(N + n_0)!} \mathcal{N}_{N, n_0} \frac{d^{n_0}}{d\gamma^{n_0}} [\boldsymbol{\sigma}^\dagger + (\boldsymbol{\alpha}^* \cdot \boldsymbol{\pi}^\dagger)]^{N+n_0} |0\rangle \quad , \quad (3)$$

where

$$\mathcal{N}_{N, n_0}^{-2} = \frac{(N!)^2}{(N + n_0)!} \frac{d^{n_0}}{d\gamma_1^{n_0}} \frac{d^{n_0}}{d\gamma_2^{n_0}} [1 + \gamma_1\gamma_2(\boldsymbol{\alpha}^* \cdot \boldsymbol{\alpha})]^{N+n_0} \quad (4)$$

is the normalization constant. The  $\alpha_m$  ( $m = \pm 1, 0$ ) are the coherent state parameters. The  $\gamma_k$  are set to zero after having applied the derivatives.

Finally, calculating the expectation value of the total Hamiltonian, i.e.,  $\langle \alpha | \mathbf{H} | \alpha \rangle$ , the semi-classical potential is obtained. When  $\Omega = 0$ , then the only relevant parameter is  $\alpha = \alpha_0$ , describing the relative distance of the two clusters [14, 5, 6]. However, in general we need all three components when cranking is included. For reason of simplicity, we use polar coordinates, i.e.

$$\begin{aligned} \alpha_{\pm 1} &= \frac{\alpha}{\sqrt{2}} e^{\pm i\phi} \sin(\theta) \\ \alpha_0 &= \alpha \cos(\theta) \quad . \end{aligned} \quad (5)$$

The potential is obtained through the geometrical mapping. We add to the geometrical potential constant terms such that for  $\alpha \rightarrow 0$  the potential approaches zero. The final result is:

$$\begin{aligned} \tilde{V} = & A \left( \alpha^2 \frac{F_{11}(\alpha^2)}{F_{00}(\alpha^2)} - \frac{n_0}{N+n_0} \right) + \\ & (B + C \sin^2(2\theta)) \left( \alpha^4 \frac{F_{22}(\alpha^2)}{F_{00}(\alpha^2)} - \frac{n_0(n_0-1)}{(N+n_0)(N+n_0-1)} \right) + \\ & D \cos(2\theta) \alpha^2 \frac{F_{20}^{N-2}(\alpha^2)}{F_{00}(\alpha^2)} + \\ & \left( \alpha^6 \frac{F_{33}(\alpha^2)}{F_{00}(\alpha^2)} - \frac{n_0(n_0-1)(n_0-2)}{(N+n_0)(N+n_0-1)(N+n_0-2)} \right) - \\ & \Omega(\Omega') \sin(2\theta) \cos(\phi) \left( \alpha^2 \frac{F_{11}(\alpha^2)}{F_{00}(\alpha^2)} - \frac{n_0}{N+n_0} \right). \end{aligned} \quad (6)$$

The *Control Parameters*,  $c_i = \{A, B, C, D, \Omega\}$ , of the SE are functions of the parameters in the Hamiltonian.

As these control parameters varies, a complete family of analytic functions,  $\tilde{V}(\alpha, \theta, \phi : c_i)$ , is generated.

$$\begin{aligned} A(x) &= \frac{4x(\hbar\omega/4 + a - b + bn_0) + 2\xi - (1-x)\frac{\epsilon}{2}(N+n_0-1)}{(N+n_0-1)(N+n_0-2)(-xb)}, \\ B(x) &= \frac{x(a - b(6 - n_0)) + (1-x)\frac{\epsilon}{2}}{(N+n_0-2)(-xb)}, \\ C(x) &= \frac{\xi - (1-x)\frac{\epsilon}{4}}{(N+n_0-2)(-xb)}, \\ D(x) &= \frac{(1-x)\frac{\epsilon}{2}}{(N+n_0-2)(-xb)}, \\ \Omega(\Omega', x) &= \frac{\Omega'}{(N+n_0-1)(N+n_0-2)(-xb)}. \end{aligned} \quad (7)$$

This potential is the subject of investigation in this contribution. In the next sections the structure under phase transitions is investigated.

### 3. Catastrophe Theory and phase transitions

The semiclassical behavior of the system can be studied by the methods of catastrophe theory. It is useful to consider only the value  $\phi = 0$  (or  $\pi$ ), and the change of variables,  $v = \cos 2\theta$ . Then the potential, Eq.(6), can be written as

$$\tilde{V} = \frac{\alpha^2}{q_0(\alpha)} W_0(\alpha, v; c_i). \quad (8)$$

The  $c_i = \{A, B, C, D, \Omega\}$ , are the control parameters,  $q_0(\alpha)$  is a polynomial in even powers, positive integers,  $q_0(0) \neq 0$ , and

$$\begin{aligned} W_0 = & \left( A \mp \Omega \sqrt{1-v^2} \right) p_A(\alpha) + D v p_D(\alpha) \\ & (B + C(1-v^2)) p_B(\alpha) + p_0(\alpha). \end{aligned} \quad (9)$$

Here, for brevity, we omit the arguments into the adjustable control parameters in Eq.(7), and the variables  $\alpha$ , and  $v$ , are derived from those in Eq.(5). For any values of  $N$  and  $n_0$  chosen, the  $\{p_X(\alpha)\}$  in Eq.(9), results in a set of polynomials in even powers of  $\alpha$ , positive integer coefficients, and a non-zero constant, *i. e.*,  $p_X(0) \neq 0$ .

In order to study the stability properties of the semiclassical system, we first evaluate the critical points of potential,  $\tilde{V}$ , by means of its first derivatives:  $U_\alpha \equiv \frac{\partial \tilde{V}}{\partial \alpha}$ , and  $U_v \equiv \frac{\partial \tilde{V}}{\partial v}$ . We find that these can be factored as:

$$U_\alpha = \frac{\alpha}{q_1(\alpha)} W_1(\alpha, v; c_i), \quad (10)$$

where,  $q_1(\alpha) = \kappa_1 \left( \frac{q_0(\alpha)}{(1+\alpha^2)} \right)^2$ ,  $\kappa_1$  a constant, and

$$U_v = \frac{\alpha^2}{q_0(\alpha)} \frac{\partial W_0}{\partial v}. \quad (11)$$

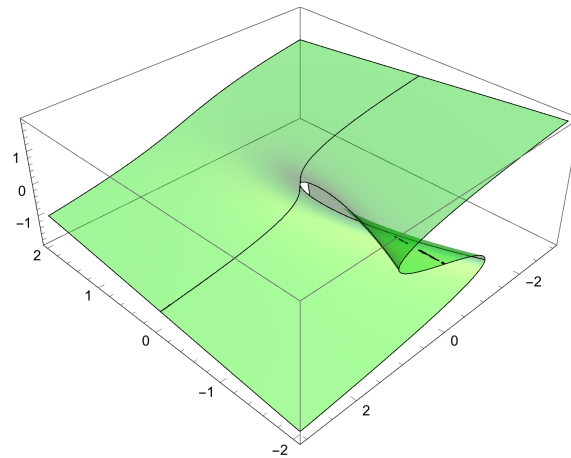
We can see from these last expressions that when the variable  $\alpha$ , takes the value,  $\alpha_0 = 0$ , it is a fundamental root; that is, it is a critical point for the potential  $\tilde{V}$ , for any choice of the control parameters. A degeneracy in any critical point means the coalescence of two or more of them due to a variation of control parameters. In this situation, the stability of the system can not be studied by means of the usual second derivatives criteria. The shapes and stability of the semiclassical system can be studied by means of the catastrophe theory program, as considered in the past [10], as follows:

(i) The fundamental root,  $\alpha_0 = 0$ , is selected in order to obtain the *germ* of the potential  $\tilde{V}$ , that is, to find the point within the control parameter space where the maximum degeneracy is presented. This *germ* is the first term of the Taylor series expansion of  $\tilde{V}$  which can not be cancelled by any set of control parameters. The first terms, in increasing order in the Taylor expansion, are succesively eliminated by means of a set of relations among the control parameters. This process permits us the determination of the essential parameters.

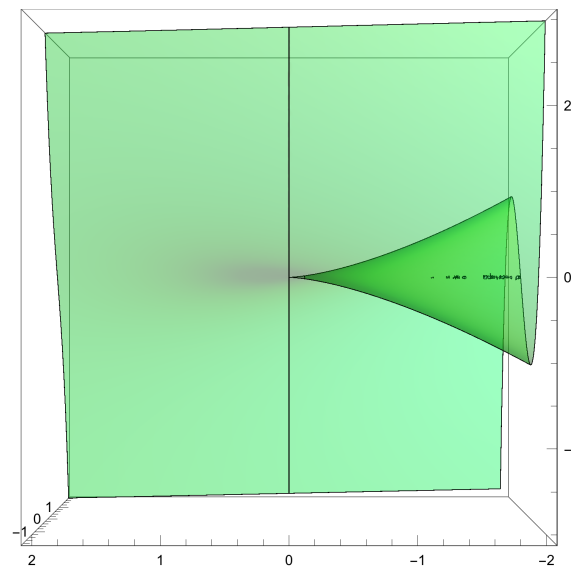
(ii) One constructs the Separatrix of the system. The first part of this separatrix consist of the bifurcation sets of the energy surface. These are the *loci* within the essential parameter space where a transition occurs from one minimum to another. The bifurcation set satisfies the condition that the matrix of second derivatives of the energy surface, when evaluated at the critical points, has a null determinant value. However, in this work, in order to evaluate the bifurcation set we shall employ an important topologic feature of catastrophe theory instead [7]: the mapping, from the critical manifold into the essential parameter space, becomes singular at the bifurcation set. The critical manifold is the surface  $\{(x_{cr}^{(j)}, c_i)\}$ , of those points obtained when all the  $c_i$ 's are continuously varied, for every  $j$ -critical point,  $(x_{cr}^{(j)}, c_i)$ . The mapping mentioned becomes singular if the Jacobian of the transformation is zero, and in general the mapping is invertible except for the set points at which the tangent plane to the manifold is vertical, meaning that they have associated the same control parameters, that is when the critical points present a coalescence.

(iii) The Maxwell sets are determined. These sets constitute the locus of points in the essential parameter space for which the energy surface takes the same value in two or more critical points. They can be found through the Clausious-Clapeyron equations [10]; however, due to the big magnitude of the problem in the present case, we shall present an original approach when solving this problem below.

(iv) Finally, the separatrix shall be constructed by the union of the bifurcation and the maxwell sets. This procedure divides the essential parameter space into shape stability regions and identifies the loci where there are transitions according to its order.



**Figure 1.** The Cusp catastrophe. For the cusp potential  $V_{cusp}(x) = x^4/4 + ax^3 + bx + c$ , the set of all its critical points  $x_{cr}$ :  $dV_{cusp}(x; a, b)/dx = x^3 + ax + b = 0$ , when the essential parameters  $(a, b)$  change, spans the critical manifold of the Cusp:  $(a, b, x_{cr})$ .



**Figure 2.** Mapping the Cusp. A Cusp is a mapping singularity when the manifold in Fig.(1) is projected onto the essential parameters space. The Separatrix:  $(\frac{a}{3})^3 + (\frac{b}{2})^2 = 0$ .

#### 4. Application to $^{16}\text{O} + \alpha \rightarrow ^{20}\text{Ne}$

Next we follow the procedure indicated above, and evaluate the numerical results for the case of  $^{16}\text{O} + \alpha \rightarrow ^{20}\text{Ne}$ . We shall include,  $N = 12$ , and  $n_0 = 8$ , values into the previous expressions in what follows.

First, we evaluate the Taylor series expansion of the energy surface Eq.(8), around the fundamental root,  $\alpha_0 = 0$  ( $\phi = 0$ ):

$$\tilde{V} = T_2 \alpha^2 + T_3 v \alpha^2 + T_4 \alpha^4 + T_5 v \alpha^4 + T_6 \alpha^6 + \dots, \quad (12)$$

with the successive coefficients:

$$\begin{aligned}
 T_2 &= 18(28 + 57\rho_2 + 48\rho_1)/95, \\
 T_3 &= 594D/19, \\
 T_4 &= 18(28 + 57\rho_2 + 48\rho_1)/95, \\
 T_5 &= -508464D/19, \\
 T_6 &= 144(809893 + 1930077\rho_2 + 1519398\rho_1)/19, \\
 \rho_2 &\equiv (A - \Omega) \\
 \rho_1 &\equiv (B + C).
 \end{aligned} \tag{13}$$

The first order term in Eq.(13) is obviously eliminated because it is evaluated at a critical point.

The  $T_2$  term is eliminated at the points of the following hyper surface in the parameter space

$$\mathcal{C}_0 = \left\{ (\rho_2, \rho_{1c}) \left| \rho_{1c}(\rho_2) = -\frac{19}{16}\rho_2 - \frac{7}{12} \right. \right\}. \tag{14}$$

At  $\mathcal{C}_0$  the fundamental root becomes double degenerate because their second derivatives are cancelled.

The  $T_3$  term cancels, and thus the fundamental root will degenerate further, for those points in parameter space laying on the plane  $\mathcal{D}_c$ , defined by:

$$D = 0. \tag{15}$$

On this plane, variable- $v$  becomes double-degenerated, as we shall demonstrate later. It is important to observe that the plane  $\mathcal{D}_c$  corresponds to the limit  $x = 1$ , as it is clear from Eq.(7), that is to the limit  $SU(3)$  in Eq.(1).

The  $T_4$  term, corresponding to the fourth order derivatives evaluated at the fundamental root, is eliminated if Eqs.(14) and (15) are satisfied, and simultaneously the following point is chosen:

$$(\rho_2, \rho_{1c}(12/19)) = (12/19, -4/3), \tag{16}$$

increasing to the fourth order the  $\alpha$  degeneracy. Under these conditions, the restriction

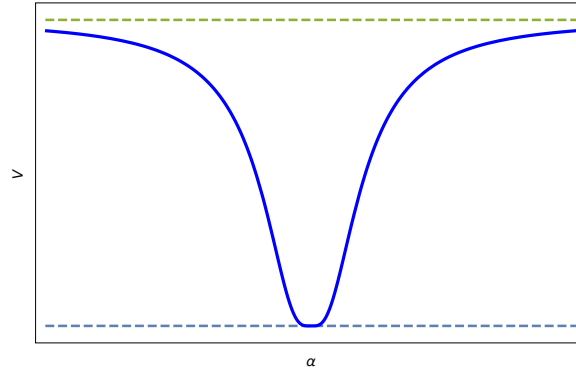
$$\Omega \rightarrow \Omega_c \equiv \frac{32}{19}C, \tag{17}$$

shall further increase the degeneracy of the  $v$ -variable to fourth order, being  $v_0 = 0$  the unique variable- $v$  critical point, as we shall show later.

The term  $T_5$  will be then automatically cancelled.

It is important to note that when conditions, Eq.(14) to Eq.(17), are satisfied, then it is imposible to cancel the sixth order term in Taylor series expansion by any further parameters relation. In this way, we have obtained the *germ* of the energy potential as that point  $\mathcal{G}_0$ :  $(\rho_{2G}, \rho_{1G}) = (12/19, -4/3)$  within the parameter space satisfying all the above conditions. When evaluated at this *germ* in the parameter space, the potential becomes: a) six-fold degenerated at its minimum, b) parameter independent, and c) a ratio of two polynomials,  $\tilde{V}(\alpha, v = 0) = \mathcal{P}_n(\alpha)/\mathcal{P}_d(\alpha)$ ; with,  $\mathcal{P}_n(0) = 0$ ,  $\mathcal{P}_d(0) \neq 0$ , and  $\tilde{V}(\alpha \rightarrow \infty, v) = 11/57$  (see Fig.(3)). At its *germ* point, the potential  $\tilde{V}$  has no other minimal critical point, and its maximum is attained only at infinity. If any of the parameters vary, going out of this *germ*, then other shapes and phase-shapes-transitions can occur.

From the above discussion we found the *germ*,  $\mathcal{G}_0$ , and conclude that the separatrix for the fundamental root consist of the plane  $\mathcal{D}_c$ , Eq(15), and the straight line  $\mathcal{C}_0$ , Eq(14). On the one hand, out of separatrix  $\mathcal{D}_c$ , as it will be shown later, the doubly degenerate  $v_0 = 0$  critical point



**Figure 3.** Energy Surface  $\tilde{V}(\alpha, v)$ , evaluated at the critical *germ* in the essential parameter space. The minimum at the origin cannot be approximated by an harmonic oscillator. It has an asymptotic value,  $\tilde{V}(\alpha \rightarrow \infty, v = 0)$ , indicated as the upper horizontal line.

bifurcates, and  $v = 0$  is no longer an extreme value; this means a shape transition in variable  $v$ -direction for the fundamental root. On the other hand, at the separatrix line  $\mathcal{C}_0$  within the plane  $\mathcal{D}_c$  there is a shape transition in the variable  $\alpha$ . Crossing this line in one direction (from above) a minimum changes to a maximum. If one goes through a continuous change within the line  $\mathcal{C}_0$ , then there is a shape transition in the variable  $\alpha$ . Crossing the *germ* in one direction (from left): a minimum changes to a maximum.

In this contribution we shall be interested in points very near to the separatrix plane  $\mathcal{D}_c$ , so we consider parameter values  $D \sim 0$ ; however, we shall study the more general bifurcation and the Maxwell sets in the variable  $\alpha$ , for all  $\alpha$ -extreme values with  $v_{cr} = 0$ . At the last part of this work, a very brief numerical example of bifurcation in the  $v$ -critical point shows us that a spontaneous symmetry breaking phenomena is represented as a phase transition, and also we present a numerical example of the existence of Maxwell points not including the fundamental root.

*Bifurcation set within  $\mathcal{D}_c$  plane.-* In this section we take  $v_0 = 0$  critical point only. Then,  $W_0 \rightarrow w_0(\alpha; \rho_2, \rho_1)$ , and we can write Eqs.(9), and (10) as:

$$w_0(\alpha; \rho_2, \rho_1) = \rho_2 p_A(\alpha) + \rho_1 p_B(\alpha) + p_0(\alpha), \quad (18)$$

$$U_\alpha = \frac{\alpha}{q_1(\alpha)} w_1(\alpha; \rho_2, \rho_1). \quad (19)$$

We want to evaluate the bifurcation set of the  $\alpha_c \neq 0$  critical points. These critical points are the roots of the equation

$$w_1(\alpha; \rho_2, \rho_1) \Big|_{\alpha_c} = 0. \quad (20)$$

As we mention in Sect. III.(ii), we begin by considering the mapping,  $\mathcal{M}$ , from the critical manifold to the essential parameter space

$$\mathcal{M} : \quad \{(\alpha_c, \rho_2, \rho_1)\} \longrightarrow \{(\rho_2, \rho_1)\}. \quad (21)$$

In order to find the singular points at the parameter space of this mapping, instead of solving for  $\alpha_c$  in Eq.(20), we solve for  $\rho_1$ , and rename as follows:

$$\alpha_c \rightarrow \lambda_1, \quad \rho_2 \rightarrow \lambda_2, \quad \rho_1 \rightarrow S_1(\lambda_1, \lambda_2). \quad (22)$$

The mapping is singular if its Jacobian determinant is zero, or equivalently, if

$$\frac{\partial S_1(\lambda_1, \lambda_2)}{\partial \lambda_1} = 0. \quad (23)$$

Finally, we solve for  $\lambda_2$  in this last equation; this solution gives a critical  $\lambda_2$  as a function of  $\lambda_1$ :

$$\lambda_{2c} \equiv \mathcal{S}_2(\lambda_1). \quad (24)$$

The mapping  $\mathcal{M}$  in Eq.(21) is invertible, except for a smooth curve on the critical manifold in the tridimensional space, given by:

$$\{(\alpha_c, \rho_2, \rho_1)\} = (\lambda, \mathcal{S}_2(\lambda), \mathcal{S}_1(\lambda)) \quad (25)$$

where we have defined

$$\mathcal{S}_1(\lambda) \equiv S_1(\lambda, \mathcal{S}_2(\lambda)). \quad (26)$$

The separatrix,  $\mathcal{C}_1$ , for the critical points  $\alpha_c \neq 0$ , is the set of points on the essential parameter space obtained parametrically as:

$$\mathcal{C}_1 : \quad \left\{ \left( \mathcal{S}_2(\lambda), \mathcal{S}_1(\lambda) \right) \mid \lambda \in \mathcal{R}e^+ \right\}. \quad (27)$$

This separatrix,  $\mathcal{C}_1$ , curve surges from the *germ* at the cusp point:

$$\left( \mathcal{S}_2(0), \mathcal{S}_1(0) \right) = (12/19, -4/3), \quad (28)$$

as a numerical evaluation can demonstrate.

*Maxwell set  $\mathcal{C}_M$  within  $\mathcal{D}_c$  plane.*- In the first place, we observe that for a given point on the essential parameter space, we obtain a particular graph of potential  $\tilde{V}$ . Further, suppose that we have chosen a point belonging to the Maxwell set. Then, at this particular point  $(\rho_2, \rho_1)$ , the form of the graph is such that both, the fundamental critical point  $\alpha_0 = 0$ , and another  $\alpha_c \neq 0$  critical point, satisfy the equality

$$\tilde{V}(\alpha_0; \rho_2, \rho_1) = \tilde{V}(\alpha_c; \rho_2, \rho_1) = 0. \quad (29)$$

Also, it is clear from Eqs.(8), and (18), that the following equation is satisfied

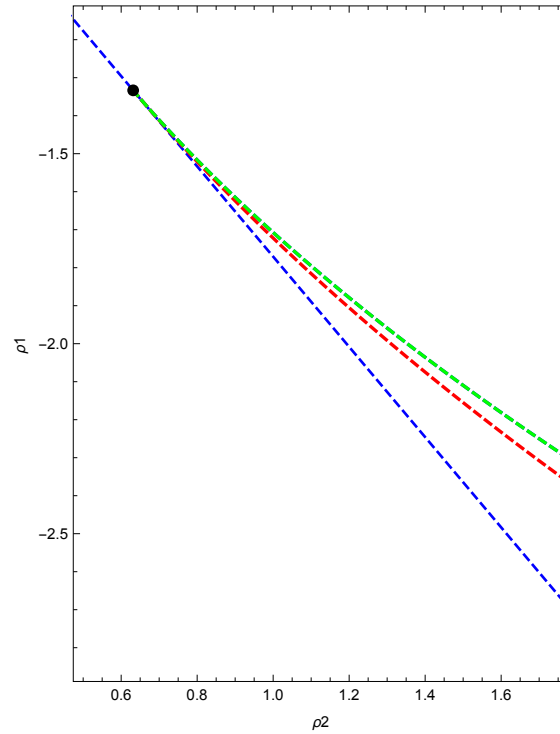
$$w_0(\alpha; \rho_2, \rho_1) \Big|_{\alpha_c} = 0. \quad (30)$$

At the second place, we observe that in a more general  $(\rho_2, \rho_1)$  point, a root  $\alpha^{(j)} \neq 0$  of the equation:

$$w_0(\alpha; \rho_2, \rho_1) = 0. \quad (31)$$

means that the graph of potential  $\tilde{V}$  is crossing the horizontal  $\alpha$ -axis, at a distance  $\alpha^{(j)}$ -away from the origin. As the  $(\rho_2, \rho_1)$  point changes towards a Maxwell point, two different roots,  $\alpha^{(j)}$  and  $\alpha^{(k)}$ , of this last equation coalesce. Obviously, this also means that at the coalescence  $\alpha^{(j)}$ -value the  $\alpha$ -horizontal axis becomes a tangent, and thus a critical Maxwell point will be found. From the facts pointed in this paragraph, we propose a method, inspired in catastrophe theory, that produces the Maxwell set as the singularities on the parameter space  $(\rho_2, \rho_1)$  of a mapping from the three-dimensional surface,  $(\alpha^{(j)}; \rho_2, \rho_1)$ , of all different  $\alpha^{(j)}$ -root points of Eq.(31), obtained by a continuous variation of the essential parameters, similar to the singularity illustrated in Fig.(2).





**Figure 4.** Separatrix. The straight line  $\mathcal{C}_0$  is a bifurcation set for the fundamental root,  $\alpha_0 = 0$ . The upper line,  $\mathcal{C}_1$ , is a bifurcation set for critical points,  $\alpha_c \neq 0$ . The intermediate line,  $\mathcal{C}_M$  is a Maxwell set. The cusp, signaled a big dot at the point  $(12/19, -4/3)$ , is the locus of the *germ*  $\mathcal{G}_0$ . This *germ* is in itself a separatrix within the subspace  $\mathcal{C}_0$ : it divides the behavior of the fundamental root, being a minima at any point from above and, going through a phase transition at this *germ*, it turns to maxima at the other side.

The coalescence points of the roots  $\alpha^{(j)} \neq 0$  of Eq.(31) can be found by considering the singularities of the mapping:

$$\mathcal{M}_M : \{(\mu_1, \mu_2, T_1(\mu_1, \mu_2))\} \longrightarrow \{(\rho_2, \rho_1)\} . \quad (32)$$

$T_1(\mu_1, \mu_2)$  obtained solving for  $\rho_1$  in Eq.(31), and being renamed as follows:

$$\alpha^{(j)} \rightarrow \mu_1, \quad \rho_2 \rightarrow \mu_2, \quad \rho_1 \rightarrow T_1(\mu_1, \mu_2) . \quad (33)$$

The mapping is singular if its Jacobian determinant is zero, or equivalently, if

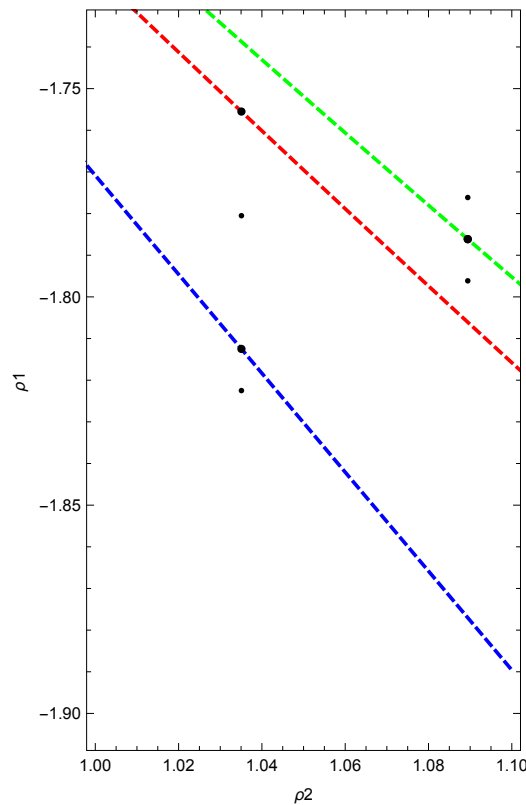
$$\frac{\partial T_1(\mu_1, \mu_2)}{\partial \mu_1} = 0 . \quad (34)$$

We obtain the critical  $\mu_2$ , by solving it from this last equation as a function of  $\mu_1$ :

$$\mu_{2c} \equiv \mathcal{T}_2(\mu_1) . \quad (35)$$

The separatrix,  $\mathcal{C}_M$ , for the roots  $\alpha^{(j)} \neq 0$ , is the loci on the essential parameter space obtained as:

$$\mathcal{C}_M : \left\{ \left( \mathcal{T}_2(\mu), \mathcal{T}_1(\mu) \right) \middle| \mu \in \mathcal{R}e^+ \right\} ; \quad (36)$$



**Figure 5.** Separatrix. Selected points at the separatrix within the subspace  $\mathcal{D}_c$ : it divides the behavior of the fundamental root, and other  $\alpha_c \neq 0$  critical points, as it is shown in the next figures.

where we have defined

$$\mathcal{T}_1(\mu) \equiv T_1(\mu, \mathcal{T}_2(\mu)). \quad (37)$$

This Maxwell curve surges from the *germ* at the cusp point:

$$(\mathcal{T}_2(0), \mathcal{T}_1(0)) = (12/19, -4/3), \quad (38)$$

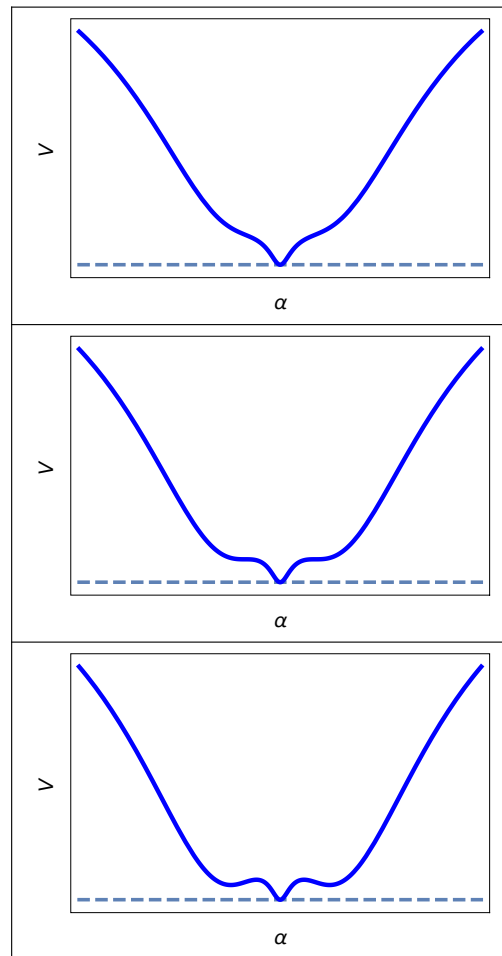
as a numerical evaluation can demonstrate (see Fig. 4).

The parametric plotting on the essential parameter space of the bifurcation sets:  $\mathcal{C}_0$  (Eq. (14)),  $\mathcal{C}_1$  (Eq. (27)), and the Maxwell set  $\mathcal{C}_M$  (Eq. (36)), defines the Separatrix of the system. The character of each equilibrium points remain the same at each distinct region, and a sudden qualitative changes take place when this separatrix is crossed as a result of a general variation of the Hamiltonian parameters. These sudden changes are treated as a mathematical catastrophes, and constitute the quantum phase transitions of the ground and the excited states for the cluster model Hamiltonian in Eq. (1).

In Fig.(5) we choose a point at every region of the Separatrix, and at every component of this Separatrix. In the following figures (Fig.(6) to Fig.(8)), we plot the corresponding potentials.

## 5. Conclusions

In this work, by means of coherent states and the catastrophe theory, the separatrix of an algebraic cluster model was constructed. We have discussed the salient features that allow to discriminate between a region in which phase transitions appear and a region that exhibits

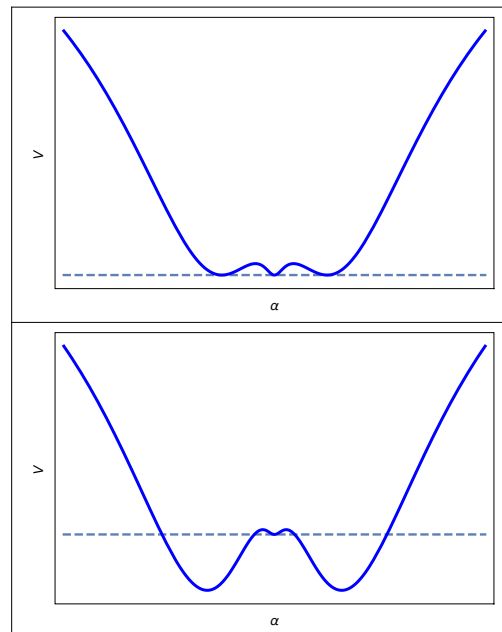


**Figure 6.** Shape transition at bifurcation set  $\mathcal{C}_1$ , at the parameter points indicated in Fig.5. The top figure shows a minimum at the  $\alpha = 0$  coordinates origin. This is the general behavior at the upper region, above the  $\mathcal{C}_1$  set. The middle figure corresponds to a point on this set; there, a critical degenerated saddle point appears. The bottom figure shows the behavior of the potential at the region in between the  $\mathcal{C}_1$ , and the  $\mathcal{C}_M$  sets: a metastable  $\alpha \neq 0$  appears.

coexisting shapes of this model. We established the germ point on the essential parameter space. When evaluated at this *germ* in the parameter space, the potential becomes: a) six-fold degenerated at its minimum, b) parameter independent, and c) a ratio of two polynomials. At its *germ* point, the potential  $\tilde{V}$  has no other minimum critical point, and its maximum is attained only at infinity. If any of the parameters vary, going out of this *germ*, then other shapes and phase-shapes-transitions can occur. The *germ* is the locus of the separatrix cusp point, and by crossing this point we observe a sixth order phase transition, the highest one in this model. Here we limit our attention to regions very near the  $SU(3)$  limit on the model, and then the fundamental root of the potential play a central role. In a future work, it remains to consider equilibria minimal points far from the origin, in order to extend the Maxwell set of shape coexistence forms.

### Acknowledgments

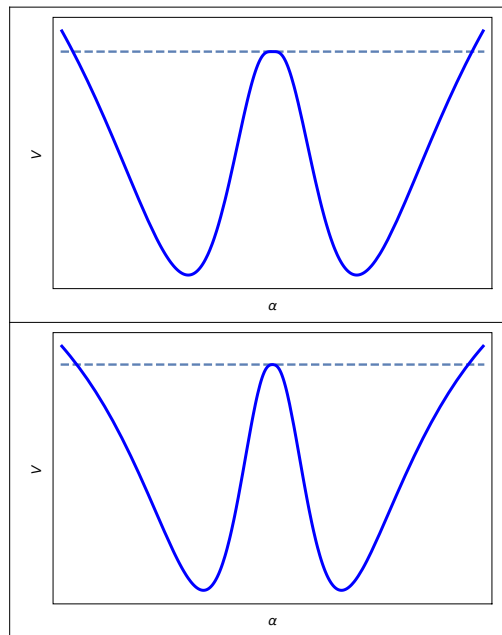
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**Figure 7.** Shape transition at Maxwell set  $\mathcal{C}_M$ , at the parameter points indicated in Fig.5. The top figure exhibits the coexistence of two minima at the Maxwell set. At the bottom figure, the potential is evaluated at the point between the  $\mathcal{C}_M$  and the  $\mathcal{C}_0$  sets. There, the fundamental root corresponds to an excited metastable state, and the ground state is an  $\alpha \neq 0$  deformed state.

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**Figure 8.** Shape transition at the  $\mathcal{C}_0$  set in Fig.5. The top figure exhibits degenerated maximum at the origin and a deformed minimum at  $\alpha_c \neq 0$ . The bottom figure shows the behavior for all those points below the  $\mathcal{C}_0$  set: The potential has only one  $\alpha_c \neq 0$  minimum and a regular maximum at the origin.