

# Invariant surfaces of variable stability

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**Abstract.** This paper presents a brief description of the theory of invariant manifolds of variable stability in the context of their connection with the theory of solutions that are bounded on the whole axis. This approach allows various generalizations both to the case of increasing of the dimension of the invariant manifolds and to the case of multiple change of their stability. The sufficient conditions for the existence of an invariant manifold of variable stability are revealed. The continuity condition for the invariant manifold yields the analytic representation of the gluing function. The theoretical developments are illustrated by several examples.

## 1. Introduction

In this paper we outline an approach for constructing of invariant surfaces of variable stability. As a one-dimensional example of such objects it is possible to consider a *canard*. The term “canard” (or “duck-trajectory”) originally was introduced in the context of the van der Pol equation [1]–[4], but later was investigated for other types of singularly perturbed systems including the ones of higher dimensions, i.e. when the phase variables are vectors (see, for example, [5]–[10] and the references therein).

From the viewpoint of the geometrical theory of singular perturbations a canard may be considered as a result of gluing stable and unstable slow invariant manifolds at one point of the breakdown surface due to the availability of an additional scalar parameter in the differential system. This approach was first proposed in [11, 12] and was then applied in [7], [13]–[17].

Other example of invariant surfaces of variable stability may be a *canard cascade* [18, 19] which is a result of gluing stable (attractive) and unstable (repulsive) slow invariant manifolds at several points of the breakdown surface. For the possibility of this gluing an additional function of a slow variable and several additional parameters are required. If we take an additional function of a vector variable parameterizing the breakdown surface, we can glue the stable and unstable slow invariant manifolds at all points of the breakdown surface simultaneously. As a result we obtain the continuous stable/unstable invariant surface or *black swan* [8, 15, 19, 20, 21]. Such surfaces may be considered as a multidimensional analogue of the notion of a canard.

Thus, the applying of a gluing function instead of a single gluing parameter may lead to various types of the invariant surfaces of variable stability, some of which are discussed in this paper.

From the perspective of analytic representation an invariant surface of variable stability is closely connected to a bounded-on-the-whole-axis solution of the corresponding differential system. Furthermore, the continuity condition for the invariant surface yields the analytic representation of the gluing function.



In this paper we consider an approach for constructing of invariant surfaces of variable stability, which is based, on the one hand, on applying of a gluing function, and on the other hand on the analytic representation of a bounded-on-the-whole-axis solution of the corresponding differential system.

## 2. Theoretical Background

We briefly recall the basic definitions and notions concerning the invariant surfaces of variable stability of singularly perturbed systems. The main object of our consideration is the autonomous singularly perturbed system of the form

$$\frac{dx}{dt} = f(x, y, \alpha, \varepsilon), \quad (1)$$

$$\varepsilon \frac{dy}{dt} = g(x, y, \alpha, \varepsilon), \quad (2)$$

where  $x$  and  $y$  are vectors in Euclidean spaces  $R^n$  and  $R^m$ , respectively,  $\varepsilon$  is a small positive parameter, vector-functions  $f$  and  $g$  are sufficiently smooth and their values are comparable to unity. The slow and fast subsystems are represented by (1) and (2), respectively.

The degenerate equation  $g(x, y, \alpha, 0) = 0$  describes a *slow surface*  $S$  (or *slow curve* in the case of the scalar variable  $x$ ). The subset  $\Gamma$  of  $S$  given by

$$\det \left| \frac{\partial g}{\partial y}(x, y, \alpha, 0) \right| = 0 \quad (3)$$

determines a *breakdown surface* (*breakdown*, *turning* or *jump points* in the scalar case) [19]. The surface  $\Gamma$  divides  $S$  into foliations on which

$$\det \left| \frac{\partial g}{\partial y}(x, y, \alpha, 0) \right| \neq 0.$$

By the implicit function theorem, a leaf of the slow surface is determined by a well-defined vector-function:  $y = \varphi(x, \alpha)$ . A slow surface can consist of several foliations determined by different functions  $y = \varphi_i(x, \alpha)$ , domains of which can intersect, depending on the structure of the slow surface. It should be noted that the conditions of the implicit function theorem do not hold on the leaf boundary given by (3).

The leaf  $S^s$  of the slow surface is *stable* (or *attractive*) if the spectrum of the Jacobian matrix

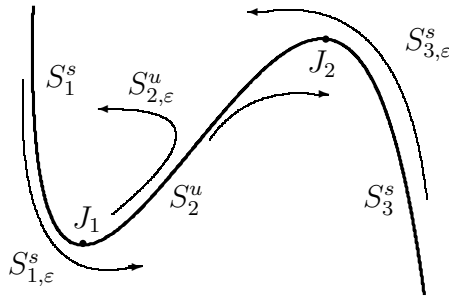
$$\frac{\partial g}{\partial y}(x, \varphi(x, \alpha), \alpha, 0)$$

is located in the left open complex half-plane. If there is at least one eigenvalue of the Jacobian matrix with a positive real part then the leaf of  $S$ , say  $S^u$ , is *unstable* (or *repulsive*). The case of complex eigenvalues is not considered in this paper.

In an  $\varepsilon$ -neighborhood of  $S^s$  ( $S^u$ ) there exists a stable (unstable) *slow invariant manifold*. The slow invariant manifold is defined as an invariant surface of slow motions.

**Definition 1** A smooth surface  $S_\varepsilon$  in  $R^m \times R^n$  is called an *invariant manifold* of the system (1), (2) if any trajectory of the system that has at least one point in common with  $S_\varepsilon$  lies entirely on  $S_\varepsilon$ .

The only invariant manifolds of system (1), (2) discussed here are those of dimension  $m$  (the dimension of the slow variable  $x$ ) that can be represented as graphs of vector-valued functions  $y = h(x, \alpha, \varepsilon) = \varphi(x, \alpha) + O(\varepsilon)$ ,  $\varepsilon \rightarrow 0$ . Here it is assumed that  $h(x, \alpha, \varepsilon)$  is a sufficiently smooth



**Figure 1.** The slow curve (thick line) and the trajectories (thin lines) of the van der Pol's type systems. The branches  $S_1^s$  and  $S_3^s$  of the slow curve are stable, the branch  $S_2^u$  is unstable,  $J_1$  and  $J_2$  are the jump points.

function of  $\varepsilon$ . Such integral manifolds are called manifolds of slow motions — the origin of this term lies in nonlinear mechanics.

The notions of (in)stability for slow invariant manifold become clear if we rescale the system (1), (2) via the change of variables  $\tau = t/\varepsilon$ . Hence we obtain the equivalent system

$$\frac{dx}{d\tau} = \varepsilon f(x, y, \alpha, \varepsilon), \quad (4)$$

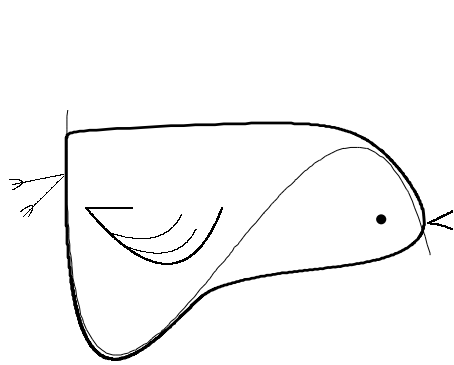
$$\frac{dy}{d\tau} = g(x, y, \alpha, \varepsilon). \quad (5)$$

For simplicity we consider the case when the variables  $x$  and  $y$  are scalar. The stable  $S^s$  and unstable  $S^u$  subsets of the slow curve  $S$  for which  $g_y < 0$  and  $g_y > 0$ , respectively, are separated by the jump point(s) at which  $g_y = 0$ . When  $\varepsilon = 0$   $S$  consists of equilibrium points of the system (4), (5) which are stable if  $g_y < 0$  and unstable if  $g_y > 0$ . When  $\varepsilon > 0$  and sufficiently small, the manifold of equilibria is replaced by a boundary layer close to  $S$  with slow dynamics. There is an attractive trajectory in the boundary layer close to the stable branch  $S^s$ , the *stable invariant manifold*  $S_\varepsilon^s$ , and there is a repulsive trajectory close to an unstable branch  $S^u$ , the *unstable invariant manifold*  $S_\varepsilon^u$  [22]. For example, the slow curve of the van der Pol system is S-shaped. It is divided by two jump points, say  $J_1$  and  $J_2$ , into two stable ( $S_1^s$  and  $S_3^s$ ) and one unstable ( $S_2^u$ ) branches, see Fig. 1. In the  $\varepsilon$ -neighborhood of the stable and the unstable branches of the slow curve there exist the stable ( $S_{1,\varepsilon}^s$  and  $S_{3,\varepsilon}^s$ ) and the unstable ( $S_{2,\varepsilon}^u$ ) slow invariant manifolds, respectively. A trajectory on  $S_{1,\varepsilon}^s$  must leave it when the jump point  $J_1$  is reached. After this moment two scenarios are possible depending on the value of the control parameter  $\alpha$ : the trajectory is either immediately repelled back to  $S_{1,\varepsilon}^s$  from  $S_{2,\varepsilon}^u$  (if  $S_{1,\varepsilon}^s$  is situated inside of  $S_{2,\varepsilon}^u$ ), or repelled to  $S_{3,\varepsilon}^s$  and makes a large excursion from the second jump point before it can return to  $S_{1,\varepsilon}^s$  (if  $S_{1,\varepsilon}^s$  is situated outside of  $S_{2,\varepsilon}^u$ ). While in the first case the limit cycles must be small, in the second case they are large and of relaxation type.

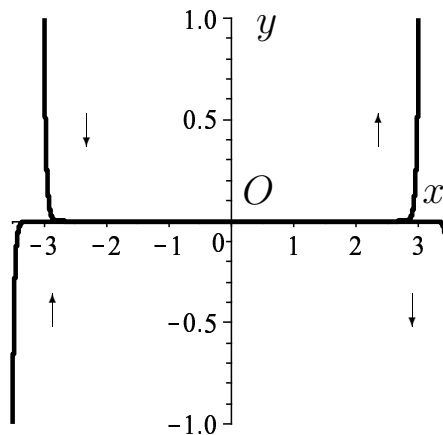
The presence of the additional scalar parameter  $\alpha$  provides the possibility of gluing the stable and unstable invariant manifolds at the jump point to form a single trajectory, the canard. The term “canard” (or duck-trajectory) had been originally given by French mathematicians to the intermediate periodic trajectories of the van der Pol equation between the small and the large orbits due to their special shapes, see Fig. 2. However, in our work a canard is a one-dimensional slow invariant manifold of variable stability.

**Definition 2** Trajectories which at first move along the stable slow invariant manifold and then continue for a while along the unstable slow invariant manifold are called canards or duck-trajectories.

**Definition 3** Trajectories which at first move along the unstable slow invariant manifold and then continue for a while along the stable slow invariant manifold are called false canard trajectories.



**Figure 2.** Canard (thick line) and the slow curve (thin line) of the van der Pol's type system.



**Figure 3.** Global (the longest) and local canards of (6). The arrows show the direction of increasing time,  $O$  is the turning point.

Therefore it is possible to consider a straight line as a basic model of a canard.

**Example 1** As the simplest system with a canard we propose

$$\dot{x} = 1, \quad \varepsilon \dot{y} = xy. \quad (6)$$

Here  $g(x, y) = xy$ . It is clear that the trajectory  $y = 0$  is a canard. The left part ( $x < 0$ ) is attractive since  $\partial g / \partial y < 0$  and the right part ( $x > 0$ ) is repulsive. These two parts are divided by a turning point  $O$ , which separates stable and unstable parts of the slow curve, at  $x = 0$ .

Note that  $y = 0$  is global (the longest) canard and the unique bounded-on-the-whole-axis solution of system (6). Others solutions of system (6) which are local canards, see Fig. 3, are infinitely increasing (decreasing) as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ .

**Example 2** Some generalization of (6) is the system

$$\dot{x} = 1, \quad \varepsilon \dot{y} = xy + \phi(x, \varepsilon) + \alpha,$$

the unique bounded-on-the-whole-axis solution of which has the representation

$$y = \begin{cases} \frac{1}{\varepsilon} \int_{-\infty}^x \exp\left(\frac{x^2 - s^2}{2\varepsilon}\right) [\phi(s, \varepsilon) + \alpha] ds, & x \leq 0, \\ -\frac{1}{\varepsilon} \int_x^{\infty} \exp\left(\frac{x^2 - s^2}{2\varepsilon}\right) [\phi(s, \varepsilon) + \alpha] ds, & x > 0. \end{cases} \quad (7)$$

The “canard value” of the parameter  $\alpha$  can be obtained from the continuity condition for (7) at  $x = 0$ :

$$\varepsilon^{-1} \int_{-\infty}^{\infty} e^{-s^2/2\varepsilon} [\phi(s, \varepsilon) + \alpha] ds = 0$$

or

$$\alpha_\phi = -\frac{1}{\sqrt{2\pi\varepsilon}} \int_{-\infty}^{\infty} e^{-s^2/2\varepsilon} \phi(s, \varepsilon) ds.$$

The proof for the existence of a maximum canard is based on considering a solution to the corresponding differential system that is bounded on the whole axis. This technique canard constructing is also applicable to further generalizations.

### 3. Canard as a unique bounded-on-the-whole-axis solution

Will apply the technique of canard constructing described in the previous section for the planar system

$$\dot{x} = 1, \quad \varepsilon \dot{y} = 2xy + F(x, y, \alpha, \varepsilon), \quad (8)$$

where

$$F(x, y, \alpha, \varepsilon) = f(x, y) + \varepsilon g(x, y, \varepsilon) + \varepsilon \alpha (1 + h(x, y, \varepsilon)).$$

We assume that

- (i) The functions  $f$ ,  $g$  and  $h$  are sufficiently smooth in the domain

$$\Omega = \{x \in R, \quad y \in R, \quad \alpha \in R, \quad \varepsilon \in [0, \varepsilon_0]\}.$$

- (ii) The functions  $f$ ,  $g$  and  $h$  satisfy the following inequalities in the domain  $\Omega$

$$|f(x, y)| \leq C|y|^2, \quad (9)$$

$$|g(x, y, \varepsilon)| \leq C\mu, \quad (10)$$

$$|h(x, y, \varepsilon)| \leq C\mu, \quad (11)$$

$$|f(x, y) - f(x, \bar{y})| \leq \frac{L}{2} (|y| + |\bar{y}|) |y - \bar{y}|, \quad (12)$$

$$|g(x, y, \varepsilon) - g(x, \bar{y}, \varepsilon)| \leq L|y - \bar{y}|, \quad (13)$$

$$|h(x, y, \varepsilon) - h(x, \bar{y}, \varepsilon)| \leq L|y - \bar{y}|, \quad (14)$$

where  $C$  and  $L$  are the positive constants,  $\mu$  is a sufficiently small positive constant.

Let us reduce the system (8) to the form

$$\varepsilon \frac{dy}{dx} = 2xy + F(x, y, \alpha, \varepsilon). \quad (15)$$

Due to (9) the straight line  $y = 0$  is the slow curve of (15), the part of which with  $x < 0$  ( $x > 0$ ) is stable (unstable). Our goal is to prove that for some value of the parameter  $\alpha$  there exists a one-dimensional invariant manifold of variable stability (a canard) located in a  $\varepsilon$ -neighborhood of the slow curve  $y = 0$ .

We introduce the metric space  $\Phi$ . Its elements are the scalar functions  $\phi(x, \varepsilon)$ , bounded by the value  $\sqrt{\varepsilon}q\mu$ , i.e.

$$|\phi(x, \varepsilon)| \leq \sqrt{\varepsilon}q\mu, \quad (16)$$

and continuous in the domain

$$\Omega_1 = \{x \in R, \quad \mu \in (0, \mu_0), \quad \varepsilon \in (0, \varepsilon_0)\}.$$

$\Phi$  is a complete space with the metric

$$d(\phi, \bar{\phi}) = \sup_{\Omega_1} |\phi(x, \varepsilon) - \bar{\phi}(x, \varepsilon)|, \quad \forall \phi \in \Phi, \quad \forall \bar{\phi} \in \Phi.$$

On the space  $\Phi$  we define an integral operator  $T$  by the formula

$$T(\phi)(x) = \begin{cases} \frac{1}{\varepsilon} \int_x^x \exp\left(\frac{x^2 - s^2}{\varepsilon}\right) F(s, \phi(s, \varepsilon), \alpha_\phi, \varepsilon) ds, & x \leq 0, \\ -\frac{1}{\varepsilon} \int_{-\infty}^{\infty} \exp\left(\frac{x^2 - s^2}{\varepsilon}\right) F(s, \phi(s, \varepsilon), \alpha_\phi, \varepsilon) ds, & x > 0. \end{cases} \quad (17)$$

It should be noted that we use a modification of the usual technique of the integral manifold theory [23, 24], and the operator in the upper (lower) line in (17) is usually used for proving the existence of stable (unstable) invariant surfaces.

The continuity condition for  $T(\phi)(x)$  at  $x = 0$  defines the parameter value  $\alpha_\phi$  corresponding to the function  $\phi(x, \varepsilon)$  and implies the equation

$$\varepsilon^{-1} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon} F(s, \phi(s, \varepsilon), \alpha_\phi, \varepsilon) ds = 0$$

or

$$\alpha_\phi = -\frac{I_1}{I_0},$$

where

$$I_1 = \frac{1}{\varepsilon\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon} [f(s, \phi(s, \varepsilon)) + \varepsilon g(s, \phi(s, \varepsilon), \varepsilon)] ds,$$

$$I_0 = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon} [1 + h(s, \phi(s, \varepsilon), \varepsilon)] ds.$$

From (9), (10), (16) we obtain

$$|I_1| \leq \frac{1}{\varepsilon\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon} |C\varepsilon\mu^2q^2 + \varepsilon C\mu| ds = \sqrt{\pi}C(\mu^2q^2 + \mu),$$

and, taking into account (11), we get

$$|I_0| \geq \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon} (1 - C\mu) ds = \sqrt{\pi}(1 - C\mu).$$

The last two inequalities yield the estimate

$$|\alpha_\phi| \leq \frac{C\mu(\mu^2q^2 + 1)}{1 - C\mu}. \quad (18)$$

It is clear that the operator  $T(\phi)(x)$  satisfies the equation

$$\varepsilon \frac{dT(\phi)}{dx} = 2xT(\phi) + F(x, \phi(x, \varepsilon), \alpha_\phi, \varepsilon).$$

Hence, the existence problem of the canard reduces to the existence problem of the solution of the operator equation  $T(\phi) = \phi$ .

In accordance with (16) to check that the operator  $T(\phi)$  transforms the complete metric space  $\Phi$  into itself it is sufficiently to verify that

$$|T(\phi)(x)| \leq \sqrt{\varepsilon} q \mu \quad (19)$$

for all  $\phi \in \Phi$ . Applying (9)–(11) and (18) and taking into account the estimate

$$\int_x^\infty \exp\left(\frac{x^2 - s^2}{\varepsilon}\right) ds < \frac{\sqrt{\pi}}{2} \sqrt{\varepsilon}, \quad x > 0,$$

we get

$$|T(\phi)(x)| \leq \sqrt{\varepsilon} \frac{\sqrt{\pi}}{2} C \mu \left( \mu q^2 + 1 + \frac{1 + \mu q^2}{1 - C\mu} \right).$$

The similar inequality is obtained for  $x < 0$ .

The numbers  $\mu$  and  $q$  will be chosen to obtain the following inequality

$$\frac{\sqrt{\pi}}{2} C \left( \mu q^2 + 1 + \frac{1 + \mu q^2}{1 - C\mu} \right) \leq q, \quad (20)$$

which means that (19) is fulfilled.

For any  $\phi, \bar{\phi} \in \Phi$  from (9)–(14) and (18) we obtain

$$d\left(T(\phi), T(\bar{\phi})\right) \leq \frac{\sqrt{\pi}}{2} \left[ L\mu q + L\sqrt{\varepsilon} + \frac{C\mu L\sqrt{\varepsilon}(1 + \mu q^2)}{1 - C\mu} \right] d(\phi, \bar{\phi}).$$

The numbers  $\varepsilon_0$ ,  $\mu$  and  $q$  will be chosen to obtain both (20) and the inequality

$$\frac{\sqrt{\pi}}{2} \left[ L\mu q + L\sqrt{\varepsilon} + \frac{C\mu L\sqrt{\varepsilon}(1 + \mu q^2)}{1 - C\mu} \right] < 1$$

for sufficiently small  $\varepsilon$  ( $0 < \varepsilon \leq \varepsilon_0$ ). In this case the contracting operator  $T(p)$  transforms the complete metric space  $\Phi$  into itself. Hence, due the Banach Contraction Principle the operator  $T$  has a unique fixed point  $\phi(x, \varepsilon) \in \Phi$  corresponding to the canard of (15).

The above argument permits us to formulate the following statement.

**Theorem 1** *Let the conditions (i), (ii) hold. Then there are numbers  $\varepsilon_0 > 0$ ,  $q$  and  $\mu$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ , there exist  $\alpha = \alpha_\phi$  and  $\phi(x, \varepsilon) \in \Phi$  such that  $y = \phi(x, \varepsilon)$  is a canard of (8).*

**Remark 1** *Usually the conditions (9)–(14) are fulfilled for  $|x| \leq r_1$ ,  $\|y\| \leq r_2$  only. In this case we deal with a local canard.*

Note that the problem of the extendability of a local canard (or local invariant manifolds of variable stability) is important and complicated but requires a separate study which is beyond the scope of this paper.

**Example 3** Originally the existence of canards for the van der Pol system

$$\frac{dx}{dt} = y - \alpha, \quad (21)$$

$$\varepsilon \frac{dy}{dt} = y - y^3/3 - x \quad (22)$$

has been proven by the non-standard analysis methods [1, 2, 4]. We show that this statement also follows from the Theorem 1.

The change of variables

$$x = x_1 - \frac{2}{3}, \quad y = y_1 - 1, \quad \alpha = -1 + \varepsilon a,$$

transforms (21), (22) to the form

$$\begin{aligned} \dot{x}_1 &= y_1 - \varepsilon a, \\ \varepsilon \dot{y}_1 &= -x_1 + y_1^2 - \frac{1}{3}y_1^3. \end{aligned}$$

Applying a rectifying transformation

$$\varepsilon z = -x_1 + y_1^2 - \frac{1}{3}y_1^3 + \frac{\varepsilon}{y_1 - 2}$$

and eliminating the variable  $t$  we obtain

$$\varepsilon \frac{dz}{dy_1} = \frac{y_1(y_1 - 2)^2 z}{1 - z(y_1 - 2)} - \frac{\varepsilon}{(y_1 - 2)^2} - \frac{\varepsilon a(y_1 - 2)}{1 - z(y_1 - 2)}.$$

By the next change of the parameter  $a = 1/8 + a_1$  we get

$$\begin{aligned} \varepsilon \frac{dz}{dy_1} &= y_1(y_1 - 2)^2 z \left[ 1 + z(y_1 - 2) + \frac{z^2(y_1 - 2)^2}{1 - z(y_1 - 2)} \right] \\ &\quad - \varepsilon \left[ \frac{1}{(y_1 - 2)^2} + \frac{y_1 - 2}{8 - 8z(y_1 - 2)} \right] - \varepsilon a_1 \frac{y_1 - 2}{1 - z(y_1 - 2)}, \end{aligned}$$

or, finally,

$$\begin{aligned} \varepsilon \frac{dz}{dy_1} &= 4y_1 z + 4y_1^2 \left( \frac{y_1}{4} - 1 \right) z + y_1(y_1 - 2)^3 \frac{z^2}{1 - z(y_1 - 2)} \\ &\quad - \varepsilon \left[ \frac{1}{(y_1 - 2)^2} + \frac{y_1 - 2}{8 - 8z(y_1 - 2)} \right] - \varepsilon a_1 \frac{y_1 - 2}{1 - z(y_1 - 2)}. \end{aligned} \quad (23)$$

It is easy to see that (23) has the form of equation (8) and the conditions of the Theorem 1 are fulfilled in some neighborhood of  $y_1 = 0$ , outside of which the functions  $f$ ,  $g$ ,  $h$  should be extended with preservation of the properties (9)–(14).

The existence theorem for the canards can be extended to the case of vector variables  $x$  and  $y$  [7, 25] and can be obtained as a corollary of the existence theorem for the black swans, see Subsection 4.3.

#### 4. Further generalizations

The described method for constructing of the invariant surfaces of variable stability in the form of bounded-on-the-whole-axis solutions of corresponding differential systems assumes various generalizations, some of which are discussed in this section. A distinctive feature in these cases lies in replacing a single gluing parameter with a gluing function to ensure continuity of the invariant surface.



#### 4.1. Case of a multiple turning point

In this Subsection we deal with a multiple turning point.

**Example 4** Consider a modification of the Example 1 given by

$$\dot{x} = 1, \quad \varepsilon \dot{y} = x^k y,$$

or

$$\varepsilon \frac{dy}{dx} = x^k y. \quad (24)$$

For  $k = 3, 5, \dots$  the straight line  $y = 0$  is a canard.

**Example 5** Some generalization of (24) is the equation

$$\varepsilon \frac{dy}{dx} = x^k y + f(x, y, \alpha).$$

In this case the gluing function can be found in the form

$$\alpha = \alpha(x, \varepsilon) = \alpha_1(\varepsilon)x^{k-1} + \dots + \alpha_{k-1}(\varepsilon)x + \alpha_k(\varepsilon).$$

The other approach for investigation of canards in the case of a multiple turning point has been proposed in [26].

#### 4.2. Canard cascades

If it is necessary to glue stable and unstable slow invariant manifolds at several breakdown points, we need several additional parameters and as a result we obtain a slow invariant manifold with multiple change of stability (a canard cascade).

**Definition 4** The continuous one-dimensional slow invariant manifold of (1), (2) which contains at least two canards or false canards is called a canard cascade.

In the case of a planar system, if we take an additional function whose arguments are a vector parameter and a slow variable, we can glue the stable (attractive) and unstable (repulsive) slow invariant manifolds at all breakdown points at the same time. The outcome of this procedure is a canard cascade.

**Example 6** The differential system

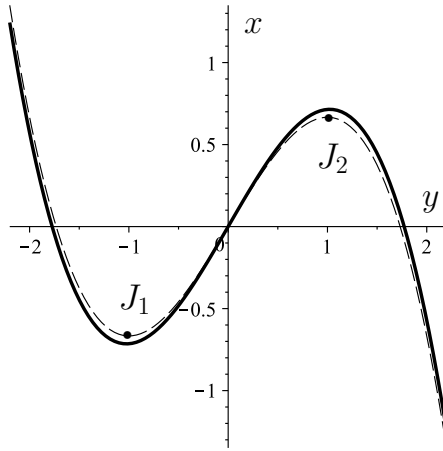
$$\dot{x} = 1, \quad \varepsilon \dot{y} = x(x-1)y$$

gives a simplest canard cascade  $y = 0$  which consists of two repulsive parts ( $x < 0$  and  $x > 1$  since  $\partial g / \partial y = x(x-1) > 0$ ) and one attractive part ( $0 < x < 1$  since  $\partial g / \partial y = x(x-1) < 0$ ) with two breakdown points  $x = 0$  and  $x = 1$ .

**Example 7** An example of a system with an infinite number of breakdown points is

$$\dot{x} = 1, \quad \varepsilon \dot{y} = y \sin x.$$

**Example 8** The slow curve of the van der Pol system (21), (22) has two jump points. To construct a canard cascade we need to have two independent parameters. We can consider  $\alpha$  as a gluing function  $\alpha = \alpha(x, \lambda)$  depending on the slow variable  $x$  and the vector  $\lambda = (\lambda_1, \lambda_2)$ . The vector  $\lambda$  is a function of  $\varepsilon$ :  $\lambda = \lambda(\varepsilon)$ . The consideration of a variant where the function  $\alpha$  is a polynomial in  $x$ , i.e.,  $\alpha = \lambda_1 x + \lambda_2$  seems quite natural. Of course, we can use a



**Figure 4.** Slow curve (thin dashed line) and canard cascade (thick full line) in the case of the van der Pol equation.  $J_1$  and  $J_2$  are jump points where stable and unstable slow invariant manifolds are glued.

trigonometric polynomial or the linear combination of any linearly independent functions in  $x$  with the coefficients  $\lambda_1, \lambda_2$ . We will search the canard cascade in the form of a polynomial

$$x = y - y^3/3 + \varepsilon q_m(y, \varepsilon), \quad (25)$$

where  $q_m(y, \varepsilon)$  is an  $m$ th-degree polynomial in  $y$ . From (21), (22) and (25) we have

$$\frac{\partial}{\partial y} \left( y - y^3/3 + \varepsilon q_m(y, \varepsilon) \right) (-q_m(y, \varepsilon)) = y - \alpha.$$

Balancing the degrees of polynomials on both sides of this equality implies that  $m = 1$  since the highest power of  $y$  in  $x$  is  $y^3$ , and therefore,  $q_m(y, \varepsilon) = ay + b$ . After some calculations we obtain [18, 19] the representation for gluing function

$$\alpha = \alpha(x, \varepsilon) = 3\mu(\varepsilon)x,$$

where

$$\mu(\varepsilon) = (\sqrt{1 + 2\varepsilon} - 1)/2\varepsilon,$$

and a canard cascade

$$x = y - y^3/3 + \varepsilon\mu(\varepsilon)y.$$

Figure 4 shows that the canard cascade (the one-dimensional slow invariant manifold) passes near the slow curve of the system. We glue the stable and the unstable slow invariant manifolds at two jump points  $J_1$  and  $J_2$ . In this case we have used only one additional parameter to obtain a canard cascade.

Note that the change of variable

$$x = -\varepsilon z + y - y^3/3 + \varepsilon\mu(\varepsilon)y$$

transforms (21), (22) to the system

$$\dot{y} = z - \mu y, \quad \varepsilon \dot{z} = (y - 1)(y + 1)z - 4\varepsilon\mu z$$

with the canard cascade  $z = 0$  which plays the role of a bounded-on-the-whole-axis solution.

In the general case, when the slow curve of a system has  $k$  jump points, it is necessary to use  $k$  additional parameters to construct a canard cascade.

### 4.3. Black swans

We return now to the more general case and consider the multidimensional invariant surfaces of variable stability or black swans [20, 21]. These surfaces are considered as natural generalizations of the notion of a canard.

We consider a non-autonomous system obtained from an original autonomous system of type (1), (2) with a scalar fast variable by eliminating the variable  $t$ . It is assumed that this system can be represented as:

$$\frac{dy}{dx} = Y(x, y, z, \varepsilon), \quad y \in R^n, \quad x \in R; \quad (26)$$

$$\varepsilon \frac{dz}{dx} = 2xz + a + Z(x, y, z, a, \varepsilon), \quad |z| \leq r, \quad (27)$$

where  $r$  and  $a_0$  are positive constants. It is supposed that the functions  $Y, Z$  are continuous and satisfy the following inequalities for  $x \in R$ ,  $y \in R^n$ ,  $|z| \leq r$ ,  $|a| \leq a_0$ ,  $\varepsilon \in [0, \varepsilon_0]$ :

$$\|Y(x, y, z, \varepsilon)\| \leq k, \quad |Z(x, y, z, a, \varepsilon)| \leq M(\varepsilon^2 + \varepsilon|z| + |z|^2), \quad (28)$$

$$\|Y(x, y, z, \varepsilon) - Y(x, \bar{y}, \bar{z}, \varepsilon)\| \leq M(\|y - \bar{y}\| + |z - \bar{z}|), \quad (29)$$

$$|Z(x, y, z, a, \varepsilon) - Z(x, \bar{y}, \bar{z}, \bar{a}, \varepsilon)| \leq M\{(\varepsilon + |\bar{z}|)|z - \bar{z}| + (\varepsilon^2 + \varepsilon|\bar{z}| + |\bar{z}|^2)\|y - \bar{y}\| + \varepsilon|a - \bar{a}|\}, \quad |\bar{z}| = \max\{|z|, |\bar{z}|\}, \quad (30)$$

where  $\|\cdot\|$  denotes the usual norm in  $R^n$  and  $|\cdot|$  denotes the absolute value of a scalar,  $k$  and  $M$  are positive constants.

We take  $a$  as a function:  $a = a(y, \varepsilon)$ .

Let  $F$  be the complete metric space of functions  $a(y, \varepsilon)$  continuous with respect to  $y$  and satisfying

$$|a(y, \varepsilon)| \leq \varepsilon^2 K, \quad |a(y, \varepsilon) - a(\bar{y}, \varepsilon)| \leq \varepsilon^2 L \|y - \bar{y}\|,$$

for  $\varepsilon \in (0, \varepsilon_0]$ , where  $K$  and  $L$  are positive constants, with the metric defined by

$$\rho(a, \bar{a}) = \sup_{y \in R^n} |a(y, \varepsilon) - \bar{a}(y, \varepsilon)|.$$

Let  $H$  be the complete metric space of functions  $h(x, y, \varepsilon)$  mapping  $R \times R^n$  to  $R$  continuous with respect to  $x, y$  and satisfying

$$|h(x, y, \varepsilon)| \leq \varepsilon^{\frac{3}{2}} q,$$

$$|h(x, y, \varepsilon) - h(x, \bar{y}, \varepsilon)| \leq \varepsilon^{\frac{3}{2}} \delta \|y - \bar{y}\|,$$

for  $\varepsilon \in (0, \varepsilon_0]$ , where  $q$  and  $\delta$  are positive constants, with the metric

$$\rho(h, \bar{h}) = \sup_{x \in R, y \in R^n} |h(x, y, \varepsilon) - \bar{h}(x, y, \varepsilon)|.$$

On the space  $H$  we define an operator  $T$  by the formula

$$Th(x, y, \varepsilon) = \begin{cases} \frac{1}{\varepsilon} \int_x^{\infty} \exp\left(\frac{x^2 - s^2}{\varepsilon}\right) [Z(\cdot) + a(\varphi(s, x), \varepsilon)] ds, & x \leq 0, \\ -\frac{1}{\varepsilon} \int_{-\infty}^x \exp\left(\frac{x^2 - s^2}{\varepsilon}\right) [Z(\cdot) + a(\varphi(s, x), \varepsilon)] ds, & x > 0, \end{cases}$$

where  $Z(\cdot) = Z(s, \varphi(s, x), h(s, \varphi(s, x), \varepsilon), a(\varphi(s, x), \varepsilon), \varepsilon)$ , and  $\varphi(s, x)$  is defined as follows. For any element  $h \in H$  the initial value problem

$$\frac{d\varphi}{ds} = Y(s, \varphi, h(s, \varphi, \varepsilon), \varepsilon),$$

$$\varphi(x) = y,$$

is considered. This problem is obtained from (26) if we replace  $z$  by an element  $h \in H$ . The solution of this problem is denoted by  $\Phi(s, x, y, \varepsilon|h) = \varphi(s, x)$ . When the operator  $T$  possesses a fixed point  $h(x, y, \varepsilon)$  in  $H$  then the surface  $z = h(x, y, \varepsilon)$  is a slow stable/unstable invariant manifold (a black swan). It should be noted that the operator  $T$  defines a manifold of the solutions which are bounded with respect to  $z$ .

For the detailed proof, which is multidimensional modification of the proof of the Theorem 1, the readers are referred to [20, 21, 25]. The following theorem gives sufficient conditions for the system (26), (27) to have a black swan.

**Theorem 2** *Let the conditions (28)–(30) be satisfied. Then there are numbers  $\varepsilon_0 > 0$  and  $K, L, q, \delta$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ , there exist functions  $a(y, \varepsilon) \in F$  and  $h(x, y, \varepsilon) \in H$  such that  $z = h(x, y, \varepsilon)$  is a slow integral manifold (the black swan).*

**Remark 2** *A original singularly perturbed differential system can be reduced to the form (26), (27) by an multidimensional analogue of the rectifying transformation described in the Example 3 (see, for instance, [7, 25]).*

**Remark 3** *Usually the conditions (28)–(30) are fulfilled for  $|x| \leq r_1$ ,  $\|y\| \leq r_2$  only. Integral manifolds in this case are local.*

Note that Theorem 2 can be generalized to the case when  $z$  is a vector variable.

Let us consider now the system

$$\frac{dy}{dx} = Y(x, y, z_1, z_2, \varepsilon), \quad y \in R^n, \quad x \in R, \quad (31)$$

$$\varepsilon \frac{dz_1}{dx} = 2xz_1 + a(y, \varepsilon) + Z_1(x, y, z_1, z_2, \varepsilon), \quad z_1 \in R, \quad (32)$$

$$\varepsilon \frac{dz_2}{dx} = A(x)z_2 + a(y, \varepsilon)B + Z_2(x, y, z_1, z_2, \varepsilon), \quad z_2 \in R^m, \quad (33)$$

where  $A(x)$  is a bounded matrix, satisfying a Lipschitz condition, with eigenvalues  $\lambda_i(x)$ :

$$\operatorname{Re} \lambda_i(x) \leq -2\beta < 0 \quad (i = 1, 2, \dots, m),$$

$B$  is a constant vector, the continuous functions  $Y, Z_1, Z_2, a$  satisfy the inequalities

$$\|Y(x, y, z_1, z_2, \varepsilon)\| \leq k, \quad (34)$$

$$|Z_1(x, y, z_1, z_2, \varepsilon)| \leq M(\varepsilon^2 + \varepsilon\|z\| + \|z\|^2), \quad (35)$$

$$\|Z_2(x, y, z_1, z_2, \varepsilon)\| \leq M(\varepsilon^2 + \varepsilon\|z\| + \|z\|^2), \quad (36)$$

$$\|Y(x, y, z_1, z_2, \varepsilon) - Y(x, \bar{y}, \bar{z}_1, \bar{z}_2, \varepsilon)\| \leq M(\|y - \bar{y}\| + \|z - \bar{z}\|), \quad (37)$$

$$|Z_1(x, y, z_1, z_2, \varepsilon) - Z_1(x, \bar{y}, \bar{z}_1, \bar{z}_2, \varepsilon)| \leq M\{(\varepsilon + \|\bar{z}\|)\|z - \bar{z}\| + (\varepsilon^2 + \varepsilon\|\bar{z}\| + \|\bar{z}\|^2)\|y - \bar{y}\|\}, \quad (38)$$

$$\|Z_2(x, y, z_1, z_2, \varepsilon) - Z_2(x, \bar{y}, \bar{z}_1, \bar{z}_2, \varepsilon)\| \leq M \{(\varepsilon + \|\tilde{z}\|)\|z - \bar{z}\| + (\varepsilon^2 + \varepsilon\|\tilde{z}\| + \|\tilde{z}\|^2)\|y - \bar{y}\|\}, \quad (39)$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, \quad \|\tilde{z}\| = \max\{\|z\|, \|\bar{z}\|\}, \quad k > 0, M > 0.$$

We consider  $a(y, \varepsilon)$  to be a continuous function satisfying the inequalities

$$|a(y, \varepsilon)| \leq \varepsilon^2 K, \quad |a(y, \varepsilon) - a(\bar{y}, \varepsilon)| \leq \varepsilon^2 L \|y - \bar{y}\|, \quad (40)$$

for  $\varepsilon \in (0, \varepsilon_0]$  with some positive constants  $K$  and  $L$ .

**Theorem 3** *Let the conditions (34)–(39) be satisfied. Then there are numbers  $\varepsilon_0 > 0$  and  $K, L, q, \delta$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ , there exist functions  $a(y, \varepsilon)$  satisfying (40) and a continuous function  $h(x, y, \varepsilon)$  satisfying*

$$|h_1(x, y, \varepsilon)| \leq \varepsilon^{3/2} q, \quad |h_1(x, y, \varepsilon) - h_1(x, \bar{y}, \varepsilon)| \leq \varepsilon^{3/2} \delta \|y - \bar{y}\|, \quad (41)$$

$$\|h_2(x, y, \varepsilon)\| \leq \varepsilon^2 q, \quad \|h_2(x, y, \varepsilon) - h_2(x, \bar{y}, \varepsilon)\| \leq \varepsilon^2 \delta \|y - \bar{y}\|, \quad (42)$$

$$h(x, y, \varepsilon) = \begin{pmatrix} h_1(x, y, \varepsilon) \\ h_2(x, y, \varepsilon) \end{pmatrix}, \quad h_1 \in R, \quad h_2 \in R^m,$$

such that  $z = h(x, y, \varepsilon)$  is a black swan of the system (31)–(33).

The proof of Theorem 3 follows the lines of the proof of Theorem 2, but now  $H$  is the complete metric space of functions  $h(x, y, \varepsilon)$  mapping  $R \times R^n$  to  $R^{m+1}$ , continuous with respect to  $x, y$  and satisfying (41), (42) for  $\varepsilon \in (0, \varepsilon_0]$ . The operator  $T$  is defined by the formula  $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ :

$$T_1 h(x, y) = \begin{cases} -\varepsilon^{-1} \int_{-\infty}^{\infty} e^{(x^2-s^2)/\varepsilon} [Z_1(\cdot) + a(\varphi(s, x), \varepsilon)] ds & , \quad x \geq 0, \\ \varepsilon^{-1} \int_{-\infty}^x e^{(x^2-s^2)/\varepsilon} [Z_1(\cdot) + a(\varphi(s, x), \varepsilon)] ds & , \quad x < 0, \end{cases}$$

$$T_2 h(x, y) = \varepsilon^{-1} \int_{-\infty}^x W(x, s, \varepsilon) [Z_2(\cdot) + Ba(\varphi(s, x), \varepsilon)] ds,$$

where

$$Z_{1,2}(\cdot) = Z_{1,2}(s, \varphi(s, x), h_1(s, \varphi(s, x), \varepsilon), h_2(s, \varphi(s, x), \varepsilon), \varepsilon)$$

and  $W(x, s, \varepsilon)$ ,  $W(s, s, \varepsilon) = I$ , is the fundamental matrix of the equation

$$\varepsilon \frac{dz_2}{dx} = A(x)z_2.$$

**Remark 4** *The proof of the Theorem 3 can be extended to the case when the system (31)–(33) has the form*

$$\frac{dy}{dx} = Y(x, y, z_1, z_2, \varepsilon), \quad y \in R^n, \quad x \in R, \quad (43)$$

$$\varepsilon \frac{dz_1}{dx} = 2xb(x, y)z_1 + a(y, \varepsilon) + Z_1(x, y, z_1, z_2, \varepsilon), \quad z_1 \in R, \quad (44)$$

$$\varepsilon \frac{dz_2}{dx} = A(x, y)z_2 + a(y, \varepsilon)B(x, y, \varepsilon) + Z_2(x, y, z_1, z_2, \varepsilon), \quad z_2 \in R^m. \quad (45)$$

Here the matrix  $A(x, y)$  and the functions  $Y$ ,  $Z_1$ ,  $Z_2$ , and  $a$  satisfy the conditions of the Theorem 3. Moreover, the matrix  $A(x, y)$ , the scalar function  $b(x, y)$  ( $0 < c_1 \leq b(x, y) \leq c_2$ ) and the vector function  $B(x, y, \varepsilon)$  are continuous and bounded, and satisfy a Lipschitz condition.

**Remark 5** *The existence of canards for the systems (31)–(33) and (43)–(45) follows from the Theorem 3. Indeed, the black swan  $z = h(x, y, \varepsilon)$  can be considered as a surface consisting entirely of the canards, each of which passes through a certain point of the breakdown surface. If we take the gluing function has to be a constant,  $a(y, \varepsilon) = a(y_0, \varepsilon)$ , the stable and unstable invariant manifolds can be glued at one point  $y = y_0$  of the breakdown surface only. The canard passes only through this point.*

## 5. Summary

In this paper we have looked from unified positions at the invariant surfaces of variable stability both from a geometric viewpoint, when dealing with gluing the stable and the unstable slow invariant manifolds, and in terms of their analytic representations as the unique bounded-on-the-whole-axis solutions. The invariant surfaces of variable stability are interesting not only from a mathematical viewpoint, they are successfully used for modeling of critical phenomena of diverse nature. It should be noted that the considered cases do not exhaust the possible generalizations.

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