

Singular path-independent energy integrals for elastic bodies with thin elastic inclusions

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Abstract. An equilibrium problem for a two-dimensional homogeneous linear elastic body containing a thin elastic inclusion and an interfacial crack is considered. The thin inclusion is modeled within the framework of Euler–Bernoulli beam theory. An explicit formula for the first derivative of the energy functional with respect to the crack perturbation along the interface is presented. It is shown that the formulas for the derivative associated with translation and self-similar expansion of the crack are represented as path-independent integrals along smooth contour surrounding one or both crack tips. These path-independent integrals consist of regular and singular terms and are analogs of the well-known Eshelby–Cherepanov–Rice J -integral and Knowles–Sternberg M -integral.

1. Formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and let γ be the straight-line segment $(0, 2) \times \{0\}$, $\bar{\gamma} \subset \Omega$. Denote by γ_0 the set $(0, 1) \times \{0\}$, and consider the domain $\Omega_\gamma = \Omega \setminus \bar{\gamma}$. The geometrical setup is shown in Figure 1.

The domain Ω_γ is occupied by an elastic material that obeys the linear Hooke law, and γ corresponds to a thin elastic inclusion. We model a thin inclusion as a beam governed within the framework of Euler–Bernoulli beam theory. The inclusion is delaminated on γ_0^+ so that there is a crack on the interface between two media, while the part $\gamma \setminus \gamma_0$ is adhered perfectly to the elastic body.

The equilibrium problem reads as follows [5]. We want to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in Ω_γ , and thin inclusion displacements v, w defined on γ such that

$$-\operatorname{div} \sigma = f, \quad \sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \quad (1)$$

$$v_{,1111} = [\sigma_{22}] \quad \text{on } \gamma, \quad (2)$$

$$-w_{,11} = [\sigma_{12}] \quad \text{on } \gamma, \quad (3)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (4)$$

$$v_{,11} = v_{,111} = w_{,1} = 0 \quad \text{for } x_1 = 0, 2, \quad (5)$$

$$[u] = 0 \quad \text{on } \gamma \setminus \gamma_0, \quad (6)$$

$$v = u_2^-, \quad w = u_1^- \quad \text{on } \gamma, \quad (7)$$

$$\sigma_{22}^+ = 0, \quad \sigma_{12}^+ = 0 \quad \text{on } \gamma_0. \quad (8)$$



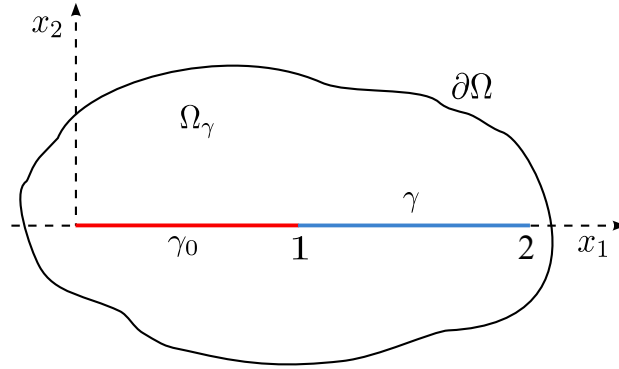


Figure 1. Geometrical setup of the problem.

Here, $\varepsilon = \{\varepsilon_{ij}(u)\}$ is the infinitesimal strain tensor, $\varepsilon_{ij}(u) = 1/2(u_{i,j} + u_{j,i})$. The elasticity tensor $A = \{a_{ijkl}\}$ is assumed to be symmetric and positive definite, i.e.

$$a_{ijkl} = a_{jikl} = a_{klij},$$

$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_A \xi_{ij}\xi_{ij}, \quad \xi_{ij} = \xi_{ji}, \quad c_A = \text{const} > 0.$$

For simplicity, we assume that a_{ijkl} are constants, and summation over repeated indices is employed. We identify functions defined on γ with the functions of variable x_1 . In accordance with the positive and negative directions of the x_2 axis, there are positive face γ^+ and negative γ^- . Then the quantity $[p] = p^+ - p^-$ denotes the jump of p across γ , and $p^\pm$ are the traces of p on $\gamma^\pm$. Finally, $f = (f_1, f_2) \in C^1(\bar{\Omega})^2$ is the density of the applied body forces. The first relations in (1) are the equilibrium equations for the elastic body Ω_γ , while the second relations are the linear Hooke law. Equations (2) and (3) are the Euler–Bernoulli governing equations for the inclusion γ . The right-hand sides of equations (2) and (3) represent the forces acting on γ from the surrounding elastic media. The inclusion displacements coincide with the vertical and the tangential displacements of the elastic body on γ^- in conformity with (7).

The problem (1)–(8) admits a variational formulation. Indeed, let us introduce the space

$$H_{\partial\Omega}^1(\Omega_{\gamma_0}) = \{u \in H^1(\Omega_{\gamma_0}) \mid u = 0 \text{ on } \partial\Omega\},$$

and the set of admissible displacements

$$K^0 = \{(u, v, w) \in H_{\partial\Omega}^1(\Omega_{\gamma_0})^2 \times H^2(\gamma) \times H^1(\gamma) \mid v = u_2^-, w = u_1^- \text{ on } \gamma\}.$$

Then problem (1)–(8) is equivalent to minimization of the energy functional

$$\Pi(u, v, w) = \frac{1}{2} \int_{\Omega_{\gamma_0}} \sigma_{ij}(u) \varepsilon_{ij}(u) dx - \int_{\Omega_{\gamma_0}} f_i u_i dx + \frac{1}{2} \int_{\gamma} v_{,11}^2 dx_1 + \frac{1}{2} \int_{\gamma} w_{,1}^2 dx_1$$

on the set K^0 and can be written in the form of the Euler equation

$$\begin{aligned} (u, v, w) \in K^0, \quad & \int_{\Omega_{\gamma_0}} \sigma_{ij}(u) \varepsilon_{ij}(\bar{u}) dx - \int_{\Omega_{\gamma_0}} f_i \bar{u}_i dx \\ & + \int_{\gamma} v_{,11} \bar{v}_{,11} dx_1 + \int_{\gamma} w_{,1} \bar{w}_{,1} dx_1 = 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K^0. \end{aligned} \quad (9)$$

Since the functional Π is coercive and weakly lower semicontinuous on the set K^0 , equation (9) has a (unique) solution.

2. Regularity of the solution

We discuss local higher regularity of the solution (u, v, w) to the Euler equation (9). First it should be noted that the H^2 -regularity of u inside Ω_γ and near $\partial\Omega$ follows from standard regularity theory for uniformly elliptic differential operators (see, e.g. [4]). We prove the following statement.

Theorem 1. Let $x^0 \in \gamma \setminus \{(1, 0)\}$ be any fixed point. Then there exists a neighborhood $S(x^0)$ of the point x^0 such that the solution of (9) satisfies

$$u \in H^2(S(x^0) \cap \Omega_\gamma)^2, \quad v \in H^4(S(x^0) \cap \gamma), \quad \text{and} \quad w \in H^2(S(x^0) \cap \gamma).$$

The proof of Theorem 1 relies upon difference quotient approximations for weak derivatives.

3. Derivative of the energy functional

In this section, we present a formula for the first derivative of the energy functional with respect to the perturbation of the crack γ_0 . The first goal of further considerations is to introduce a small perturbation parameter into the problem (9), i.e. to consider a family of perturbed problems depending on the parameter δ . Let the transformation of independent variables

$$y = T_\delta(x), \quad x \in \Omega_{\gamma_0}, \quad y \in \Omega_{\gamma_\delta}, \quad (10)$$

describe the perturbation of the domain Ω_{γ_0} , where $T_\delta(x) = x + \delta V(x)$, $V(x) = (V_1(x), V_2(x))$, and $V \in W^{1,\infty}(\Omega)^2 \cap W^{2,\infty}(\gamma)^2$. The perturbed domain $\Omega_{\gamma_\delta} = T_\delta(\Omega_{\gamma_0})$ containing the perturbed crack $\gamma_\delta = T_\delta(\gamma_0)$ for each fixed δ . The vector field V is supposed to be compactly supported in Ω and such that $\gamma = T_\delta(\gamma)$ and $\gamma_\delta \subset \gamma$. Consequently, we have $\Omega_{\gamma_\delta} = \Omega \setminus \bar{\gamma}_\delta$.

We define the set of admissible displacements associated with the perturbed problem as follows:

$$K^\delta = \{ (u, v, w) \in H_{\partial\Omega}^1(\Omega_{\gamma_\delta})^2 \times H^2(\gamma) \times H^1(\gamma) \mid v = u_2^-, w = u_1^- \text{ on } \gamma \}.$$

As before, a unique solution of the Euler equation exists

$$\begin{aligned} (u^\delta, v^\delta, w^\delta) \in K^\delta, \quad & \int_{\Omega_{\gamma_\delta}} \sigma_{ij}(u^\delta) \varepsilon_{ij}(\bar{u}) dy - \int_{\Omega_{\gamma_\delta}} f_i \bar{u}_i dy \\ & + \int_{\gamma} v_{,11}^\delta \bar{v}_{,11} dy_1 + \int_{\gamma} w_{,1}^\delta \bar{w}_{,1} dy_1 = 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K^\delta. \end{aligned} \quad (11)$$

For the solution $(u^\delta, v^\delta, w^\delta)$ to equation (11), we can find the potential deformation energy

$$\Pi^\delta(u^\delta, v^\delta, w^\delta) = \frac{1}{2} \int_{\Omega_{\gamma_\delta}} \sigma_{ij}(u^\delta) \varepsilon_{ij}(u^\delta) dy - \int_{\Omega_{\gamma_\delta}} f_i u_i^\delta dx + \frac{1}{2} \int_{\gamma} (v_{,11}^\delta)^2 dy_1 + \frac{1}{2} \int_{\gamma} (w_{,1}^\delta)^2 dy_1.$$

The main question in this section is as follows: does the first derivative of the energy functional with respect to the crack perturbation exist? More precisely, we refer to the existence of the limit

$$\mathcal{D}_V \Pi(u, v, w) = \lim_{\delta \rightarrow +0} \frac{\Pi^\delta(u^\delta, v^\delta, w^\delta) - \Pi(u, v, w)}{\delta}.$$

and an explicit representation for it. The following theorem gives a positive answer to this question.

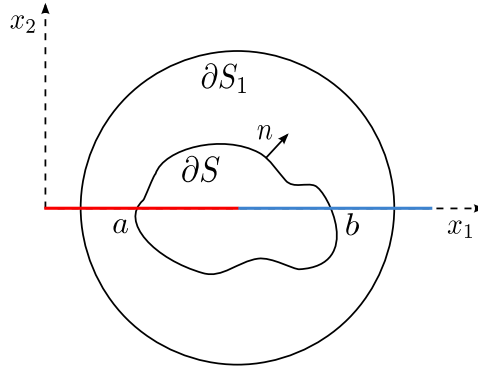


Figure 2. The integration path ∂S surrounding the crack tip.

Theorem 2. The first derivative of the energy functional with respect to the crack perturbation exists and equals

$$\begin{aligned} \mathcal{D}_V \Pi(u, v, w) = & \int_{\Omega_{\gamma_0}} \left(\frac{1}{2} \operatorname{div} V \sigma_{ij}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u) E_{ij} \left(\frac{\partial V}{\partial x}; u \right) \right) dx - \int_{\Omega_{\gamma_0}} \operatorname{div}(V f_i) u_i dx \\ & - \int_{\gamma} \left(\frac{3}{2} V_{1,1} v_{,11}^2 + V_{1,11} v_{,11} v_{,1} + \frac{1}{2} V_{1,1} w_{,1}^2 \right) dx_1, \quad (12) \end{aligned}$$

where the components of the transformed strain tensor E_{ij} have the form

$$E_{ij} \left(\frac{\partial V}{\partial x}; u \right) = \frac{1}{2} (u_{i,k} V_{k,j} + u_{j,k} V_{k,i}), \quad i, j = 1, 2.$$

4. Singular path-independent integrals

Let us now consider some particular cases of choosing the vector field V which will yield singular path-independent integrals by means of transformations of formula (12). In all cases, we will have to choose neighborhoods S and S_1 with smooth (Lipschitz) boundaries ∂S and ∂S_1 . In what follows, we assume that the boundaries of the domains $(S_1 \setminus S) \cap \Omega_{\gamma}$ also satisfy the Lipschitz condition and $f \equiv 0$ in $S \cap \Omega_{\gamma}$. The path-independent integrals are meaningful due to the regularity result from Section 2.

Let the support of a smooth function θ lie in a small neighborhood S_1 of the point $(1, 0)$ and $\theta = 1$ in a neighborhood S of the point $(1, 0)$, $S \subset S_1$. The path ∂S intersects the axis x_1 at the points with the abscissae a and b as shown in Figure 2. Denote by $n = (n_1, n_2)$ the outward normal vector to ∂S . We choose the coordinate transformation (10) in the form

$$y_1 = x_1 + \delta \theta(x_1, x_2), \quad y_2 = x_2, \quad (13)$$

where $(x_1, x_2) \in \Omega_{\gamma_0}$ and $(y_1, y_2) \in \Omega_{\gamma_{\delta}}$. The vector field V is determined by the formula $V(x) = (\theta(x), 0)$, and formula (12) can be rewritten as

$$\begin{aligned} \mathcal{D}_{(\theta,0)} \Pi(u, v, w) = & \int_{\Omega_{\gamma_0}} \left(\frac{1}{2} \theta_{,1} \sigma_{ij}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u) u_{i,1} \theta_{,j} \right) dx - \int_{\Omega_{\gamma_0}} (\theta f_i)_{,1} u_i dx \\ & - \frac{1}{2} \int_{\gamma} (3 \theta_{,1} v_{,11}^2 + 2 \theta_{,11} v_{,11} v_{,1} + \theta_{,1} w_{,1}^2) dx_1. \quad (14) \end{aligned}$$

The formula (14) is an analog of the well-known Griffith formula for the energy release rate associated with crack extension.

Let the map $J_B : (0, 2) \rightarrow \mathbb{R}$ be given by

$$J_B(x_1) = \frac{1}{2}v_{,11}^2(x_1) - v_{,11}v_{,11}(x_1) + \frac{1}{2}w_{,1}^2(x_1).$$

Integrating by parts in (14) and taking into account (1)–(8), we obtain the singular J -integral

$$\mathcal{J}_{\text{tip}} = \int_{\partial S} \left(-\frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)n_1 + \sigma_{ij}(u)u_{i,1}n_j \right) ds + J_B(b) - J_B(a). \quad (15)$$

It is immediate from the construction that $\mathcal{J}_{\text{tip}}$ does not depend on the integration path ∂S (and hence on the points a and b). In other words, the singular J -integral (15) has the same value for all closed curve surrounding the crack tip. The J -integral obtained consists of two parts. The first is a regular part that coincide with the classical Eshelby–Cherepanov–Rice J -integral, and the second is a singular part that depends on the displacement field (v, w) of the inclusion at the points where the path ∂S intersects the segment γ .

Let the coordinate transformation (10) have the form

$$y_1 = x_1(1 + \delta\theta(x_1, x_2)), \quad y_2 = x_2(1 + \delta\theta(x_1, x_2)), \quad (16)$$

where $(x_1, x_2) \in \Omega_{\gamma_0}$ and $(y_1, y_2) \in \Omega_{\gamma_\delta}$. In this case, the vector field V is determined by the formula $V(x) = (x_1\theta(x), x_2\theta(x))$, and formula (12) can be rewritten as

$$\begin{aligned} \mathcal{D}_{x\theta}\Pi(u, v, w) = & \int_{\Omega_{\gamma_0}} \left(\frac{1}{2}(x_l\theta)_{,l}\sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)u_{i,l}(x_l\theta)_{,j} \right) dx - \int_{\Omega_{\gamma_0}} (x_l\theta f_i)_{,l}u_i dx \\ & - \frac{1}{2} \int_{\gamma} (3(x_1\theta)_{,1}v_{,11}^2 + 2(x_1\theta)_{,11}v_{,11}v_{,1} + (x_1\theta)_{,1}w_{,1}^2) dx_1. \end{aligned} \quad (17)$$

Let the map $M_B : (0, 2) \rightarrow \mathbb{R}$ be given by

$$M_B(x_1) = \frac{3}{2}(x_1 - 1)v_{,11}^2(x_1) + v_{,1}(v_{,11} - v_{,111})(x_1) + \frac{1}{2}x_1w_{,1}^2(x_1).$$

Integrating by parts in (17), we obtain the singular path-independent M -integral in the equilibrium problem (1)–(8):

$$\mathcal{M}_{\text{tip}} = \int_{\partial S} \left(-\frac{1}{2}(n_l x_l)\sigma_{ij}(u)\varepsilon_{ij}(u) + \sigma_{ij}(u)u_{i,l}x_l n_j \right) ds + M_B(b) - M_B(a). \quad (18)$$

The M -integral (18) consists of two parts. The first is a regular part coinciding with the classical Knowles–Sternberg M -integral, please refer to [7], and the second is a singular part depending on the displacement field (v, w) of the inclusion γ .

In spite of the fact that transformations (13) and (16) are different, the perturbed domains Ω_{γ_δ} coincide for each fixed δ . Since the solution $(u^\delta, v^\delta, w^\delta) \in K^\delta$ is unique, it follows that the values of the potential deformation energy for the perturbed problems $\Pi^\delta(u^\delta, v^\delta, w^\delta)$ are both equal to each other. Consequently, the corresponding derivatives $\mathcal{D}_{(\theta,0)}\Pi(u, v, w)$ and $\mathcal{D}_{x\theta}\Pi(u, v, w)$ and therefore two singular path-independent integrals $\mathcal{J}_{\text{tip}}$ and $\mathcal{M}_{\text{tip}}$ are related through the relationships

$$\mathcal{D}_{(\theta,0)}\Pi(u, v, w) = \mathcal{D}_{x\theta}\Pi(u, v, w), \quad \mathcal{J}_{\text{tip}} = \mathcal{M}_{\text{tip}}.$$

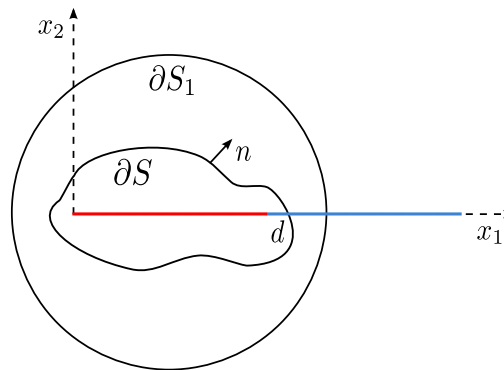


Figure 3. The integration path ∂S surrounding the whole crack.

The next step is to discuss a situation when the integration path surrounds the whole crack. Indeed, let a η be a smooth function with support located in a small neighborhood S_1 of the segment γ_0 . Moreover, $\eta = 1$ in a neighborhood S of the segment γ_0 , $S \subset S_1$. We assume that the path ∂S intersects the straight-line segment $\gamma \setminus \gamma_0$ at the point with the abscissa d as depicted in Figure 3. As previously, if $f \equiv 0$ in $S \cap \Omega_\gamma$, then we can obtain the path-independent M -integral in the form

$$\mathcal{M}_{\text{crack}} = \mathcal{D}_{x\eta} \Pi(u, v, w) = \int_{\partial S} \left(-\frac{1}{2} (n_l x_l) \sigma_{ij}(u) \varepsilon_{ij}(u) + \sigma_{ij}(u) u_{i,l} x_l n_j \right) ds + M_B(d). \quad (19)$$

In this case, the path ∂S is taken around the whole crack, and n is the outward normal to ∂S . The following interesting observation can be made in relation to (15) and (19):

$$\mathcal{J}_{\text{tip}} = \mathcal{M}_{\text{crack}}.$$

In other words, one quantity $\mathcal{M}_{\text{crack}}$ connected with the whole crack is related to another one, namely $\mathcal{J}_{\text{tip}}$, associated with the crack tip only. A similar relation between the classical M - and J -integrals was firstly found in certain special cases by Freund in [3].

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