

Evolution of localized asymptotic solutions for linearized Navier — Stokes and MHD equations

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Abstract. In the paper, the asymptotic behavior of solutions of the Cauchy problem is described for the linearized Navier–Stokes and MHD equations with the initial condition localized in a neighborhood of a two-dimensional surface in three-dimensional space. In particular, conditions for the growth of the perturbation in plane-parallel, two-dimensional, and helical external flows are obtained.

1. Introduction

The evolution of small localized perturbations of the flow of an incompressible fluid was studied in many papers devoted to the theory of hydrodynamic stability. From the mathematical point of view, we speak of properties of solutions of the linearized Euler equations or of the Navier–Stokes equations. In particular, in the works [1] – [5], the asymptotic behavior of solutions of the Cauchy problem was described for the initial condition localized in a neighborhood of a single point (solitary vortex), and the possibility of growth of such a solution in the course of time in diverse external flows was discussed. It turned out that, in plane-parallel flows, only a power-law growth of perturbations is possible, while the growth can be exponential in flows with closed trajectories (a similar result for the spectral problem was obtained earlier by Bailey [6]; the growth conditions generalize the well-known Rayleigh instability conditions [7]).

In the present paper, the asymptotic behavior of a solution of the Cauchy problem for the linearized Navier–Stokes equations was described for the initial condition localized in a neighborhood of a two-dimensional surface in three-dimensional space. The asymptotic solution is also localized near a surface; it is obtained from the initial object by a shift along the trajectories of the external flow. The evolution of the profile of the perturbation is described by a system of linear ordinary differential equations along the trajectories; for diverse classes of external flows, we study in detail the behavior of the leading term of the asymptotic behavior for large times. In particular, conditions for the growth of the perturbation in plane-parallel and two-dimensional external flows are obtained.

We study also analogous localized asymptotic solutions for the linearized MHD equations. The main effect which differs the situation from the previous one, is the effect of interaction of Alfvén modes. From the mathematical point of view, we deal here with characteristics of variable multiplicity; for WKB-type solutions this situation was studied in [8]. The solution is



now localized near a number of different surfaces — they correspond to different modes and to the mode interaction. We discuss the structure of the localized solutions as well as their asymptotics as $t \rightarrow \infty$. Here we only formulate the results — the proofs are postponed to further publications.

2. Localized solutions for the linearized Navier — Stokes Equations

A small perturbation u of the velocity field of an incompressible viscous fluid (a time-dependent vector field in \mathbf{R}^3) satisfies the linearized Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (V, \nabla)u + (u, \nabla)V + \nabla P = \nu \varepsilon^2 \Delta u, \quad (\nabla, u) = 0. \quad (1)$$

Here $V(x)$ is a given smooth divergence-free vector field bounded together with all its derivatives (stationary external flow), P is a perturbation of the pressure, and $\varepsilon^2 \nu > 0$ is the viscosity coefficient (below we describe the asymptotic behavior of solutions of this system as $\varepsilon \rightarrow 0$ in different situations; it is convenient to single out the parameter ν to clarify the influence of viscosity on the leading part of the asymptotics). We will suppose that V satisfies Euler equations

$$(V, \nabla)V + \nabla P_0 = 0, \quad (\nabla, V) = 0.$$

Let us pose the Cauchy problem for system (1) with a localized initial condition

$$u|_{t=0} = \hat{u}_0 (\Phi^0(x)/\varepsilon, x) \quad (2)$$

where \hat{u}_0 is a smooth compactly supported divergence-free vector field and $\Phi^0(x)$ is a smooth scalar function bounded together with all its derivatives, and the function $\hat{u}_0(y, x)$ decreases as $|y| \rightarrow \infty$ more rapidly than any power of $|y|$ (here y stands for the “fast” variables $y = \Phi^0(x)/\varepsilon$). We suppose that the equation $\Phi^0 = 0$ defines a regular smooth compact two-dimensional surface M_0 with $\nabla \Phi^0 \neq 0$; the initial condition as $\varepsilon \rightarrow 0$ is localized near this surface.

We study the asymptotic behavior of the Cauchy problem (1) — (2) as $\varepsilon \rightarrow 0$.

Remark 1. *The case of initial condition localized near a single point or near a curve was discussed in [1, 4, 5, 9].*

3. Leading term of the asymptotic solution

Introduce the following notation. Let $X(\xi, t)$ be the trajectory of the field V issuing from a point ξ , i.e., the solution of the Cauchy problem

$$\dot{X} = V(X, t), \quad X|_{t=0} = \xi,$$

and let $x_0(x, t)$ be the initial point of the trajectory which came at time t to the point x , i.e., the solution of the equations $X(x_0, t) = x$ (under the conditions formulated above, $X(\xi, t)$ and $x_0(x, t)$ are smooth functions on $\mathbb{R}^3 \times [0, T]$). The shift of the initial surface M_0 along the trajectories is defined by the equation $\Phi(x, t) = 0$, where $\Phi(x, t) = \Phi^0(x_0(x, t))$. Denote by $k(\xi, t)$ the cotangent vector moving along a trajectory X and annihilating the tangent plane to M_t ,

$$k(\xi, t) = ((\partial X / \partial \xi)^*)^{-1} k^0, \quad k^0 = \nabla \Phi^0(\xi).$$

Finally, denote by $Q(\xi, t)$ the fundamental matrix of the system of equations

$$\dot{A} + Q(k^0, \xi, t)A = 0, \quad Q = \left(E - 2 \frac{k \otimes k}{k^2} V_x(\xi, t) \right).$$

Theorem 1. *Let*

$$U(x, t, y) = A(\xi, t) \int_{-\infty}^{\infty} e^{iyp} e^{-\nu p^2 \int_0^t k^2 dt} \tilde{u}_0(p) dp|_{\xi=x_0(x,t)}$$

where \tilde{u}_0 is the Fourier transform of the function \hat{u}_0 with respect to the variable y .

The solution $u(x, t, \varepsilon)$ of the Cauchy problem (2.1), (2.2) on every chosen time interval $t \in [0, T]$ can be represented in the form

$$u(x, t, \varepsilon) = U(x, t, \Phi(x, t)/\varepsilon) + w, \quad \text{where } |w| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

4. Time evolution of localized perturbations

From the point of view of the theory of hydrodynamic stability, the behavior of a localized solution for large times t is of interest, namely, if the perturbation grows, then the external flow V is unstable with respect to these localized perturbations. Below we consider the asymptotic behavior as $t \rightarrow \infty$ of the leading term of the asymptotics, i.e., of the vector field U . Recall that an asymptotic solution approximates the exact one on a finite time interval $t \in [0, T]$. Thus, we first proceed to the limit of the exact solution as $\varepsilon \rightarrow 0$, and then as $t \rightarrow \infty$; from the physical point of view, the problem under consideration concerns the behavior of a solution at times which are large but much less than ε^{-r} , $r > 0$.

The time evolution of a localized perturbation depends substantially on the properties of the external flow $V(x)$; below we consider classes of fields which are well known in hydrodynamics, namely, the plane-parallel flows and two-dimensional flows with closed trajectories.

4.1. Flows with constant direction

Let $V(x) = (0, 0, V_3(x_1, x_2))$. The trajectories of this vector field are straight lines parallel to the x_3 axis, and the velocity of the motion of liquid particles varies from line to line.

Assertion 1. *Let V be a flow with constant direction and $k_3^0 = \partial\Phi^0/\partial x_3 = 0$ at some point ξ in a neighborhood of the initial surface M . Then the function U_3 has the following asymptotic behavior as $t \rightarrow \infty$ at the point $x = X(\xi, t)$:*

$$U_3 = -t \left(\hat{u}_0(\Phi^0/\varepsilon, \xi), \nabla V_0(\xi) \right) + O(1) \quad \text{for } \nu = 0,$$

$$U_3 = -\sqrt{\frac{t}{4\pi\nu}} \frac{e^{-\frac{(\Phi^0)^2(\xi)}{4t\varepsilon^2\nu_0(\nabla\Phi^0)^2}}}{|\nabla\Phi^0(\xi)|} \int_{-\infty}^{\infty} (\hat{u}_0(y, \xi), \nabla V_0(\xi)) dy + O(1) \quad \text{for } \nu > 0.$$

The functions U_1, U_2 remain bounded.

Thus, we obtain the following picture of the evolution of a localized perturbation. The leading term of the asymptotic behavior of the localized solution remains bounded on the trajectories of the field V issuing from the points ξ at which $k_3^0 = \partial\Phi^0/\partial x_3 \neq 0$. The vertical component of the asymptotic solution increases on the trajectories of points at which $k_3^0 = \partial\Phi^0/\partial x_3 = 0$ as \sqrt{t} if the viscosity is high enough (if the fluid is perfect, i.e., $\nu = 0$, then the growth is linear). The horizontal components remain bounded.

It is clear that, for $\xi \in M_t$, the growth condition $\partial\Phi^0/\partial x_3 = 0$ means that the tangent plane to M at the given point contains the vertical direction, i.e., the velocity vector of the external flow is tangent to M . For example, if the initial perturbation is concentrated on the standardly embedded sphere, then the growth will be observed along the trajectories issued from the points on the equator (and also from the points placed at a distance of the order of ε from the equator).

4.2. Plane-parallel flows with a shift of the direction

Let the external flow be of the form $V = (V_1(x_3), V_2(x_3), 0)$. Its trajectories are straight lines lying in the horizontal planes $x_3 = \text{const}$; not only the velocity, but also the direction of the flow changes from plane to plane (a shear).

Assertion 2. *Let V be a plane-parallel flow with a shift of the direction, and let $(V', k^0) = (V', \nabla\Phi^0) = 0$ at some point ξ . Then the function U has the following asymptotic behavior:*

$$U = -tV'(\xi)\hat{u}_{03}(\Phi^0(\xi)/\varepsilon, \xi) + O(1) \quad \text{for } \nu = 0;$$

$$U = -V'(\xi)\sqrt{\frac{t}{4\pi\nu}} \frac{1}{|\nabla\Phi^0(\xi)|} e^{-(\Phi^0)^2/(4t\varepsilon^2(\nabla\Phi^0)^2\nu)} \int_{-\infty}^{\infty} \hat{u}_{03}(y, \xi) dy + O(1) \quad \text{for } \nu > 0.$$

At the points at which $(V', \nabla\Phi^0) \neq 0$, the function U remains bounded, while, for $\nu_0 \geq \text{const} > 0$, U decreases as $O(t^{-3/2})$. The condition ensuring the growth of the perturbation also determines a curve (in general) on the initial surface; in particular, if M is a sphere centered at the origin, the growth will occur along the trajectories issuing from the line of intersection of the sphere with the helical surface

$$x_1V'_1 + x_2V'_2 = 0.$$

The rate of growth on this curve is still $O(\sqrt{t})$ if ν is bounded away from zero and $O(t)$ if $\nu = 0$.

4.3. Two-dimensional flows with closed stream lines

Consider now the third class of stationary two-dimensional Euler flows, namely, two-dimensional vector fields $V = (V_1(x_1, x_2), V_2(x_1, x_2), 0)$. The condition that such a field on a plane is divergence-free means that the field is Hamiltonian,

$$V = \left(-\frac{\partial\Psi}{\partial x_2}, \frac{\partial\Psi}{\partial x_1}\right);$$

in hydrodynamics, the Hamilton function Ψ is referred to as the *stream function*. As is well known, the curl $\Delta\Psi$ of the two-dimensional Euler flow is constant on its trajectories, and the Euler equations are reduced to the equation $\{\Psi, \Delta\Psi\} = 0$, where $\{\cdot, \cdot\}$ stands for the Poisson bracket.

Consider a nonsingular closed trajectory $X(\xi, t)$ of this field,

$$X(\xi, t+T) = X(\xi, t),$$

where $T(\xi)$ stands for the period of the trajectory. Assume that, as t increases, the trajectory is going around counterclockwise. Since V is divergence-free, it follows that the nearby trajectories of this field are also closed, and thus, the curve $X(\xi, t)$ belongs to the domain fibered into periodic trajectories.

In this case, the matrix A can grow exponentially; the set on which the growth takes place is defined by the following conditions (see [9]).

- (i) $X(\xi, t)$ is a convex curve,
- (ii) the circulation of the field V along the trajectory $X(\xi, t)$ decreases outwards,

$$\Delta\Psi(\xi) < 0;$$

(iii)

$$(\nabla\Phi^0, V(\xi)) = 0, \quad \partial\Phi^0/\partial x_3(\xi) \neq 0.$$

Let us suppose that these conditions are fulfilled. The equations for A now have periodic coefficients and the corresponding monodromy operator has two distinct real eigenvalues of the form $e^{\pm\sigma(\xi)T}$, $\sigma > 0$, where T is the period of the trajectory. We denote by $e_{\pm}(\xi, t)e^{\pm\sigma(\xi)t}$, $e_{\pm}(\xi, t+T) = e(\xi, t)$ the basis of the Floquet solutions.

Assertion 3. *Let V be a two-dimensional Eulerian field with closed current lines and let its trajectory $X(\xi, t)$ issuing from a point ξ of a neighborhood of the surface M satisfy the requirements (i) – (iii). Then the function U has the following asymptotic behavior as $t \rightarrow \infty$ at the point $X(\xi, t)$ of the neighborhood of M_t :*

$$U(\xi, t) = e^{\sigma t} \left(e_+(\xi, t) e^+ \left[\frac{\Phi^0(\xi)}{\varepsilon}, \xi \right] + o(1) \right) \quad \text{for } \nu = 0,$$

$$U(\xi, t) = \frac{e^{\sigma t}}{\sqrt{t}} \left(\frac{e^{-(\Phi^0)^2/(4\varepsilon^2\nu_0 < k^2 >)}}{\sqrt{4\pi\nu < k^2 >}} e_+(\xi, t) \int_{-\infty}^{\infty} e^+[\hat{u}_0(z, \xi)] dz + O(1/\sqrt{t}) \right) \quad \text{for } \nu > 0,$$

where $e_0^\pm(\xi)$ are the elements of the dual basis to the basis $e_\pm^0(\xi)$, and

$$\langle k^2 \rangle(\xi) = \frac{1}{T} \int_0^T k^2(\xi, t) dt.$$

Remark 2. *Thus, in the present case, we observe a strong (exponential) growth of a localized perturbation, and the viscous decay cannot hinder this growth. The sufficient conditions for growth, as in the case of flows with constant direction, define a set of points of the initial surface for which the tangent plane contains a vector of the external flow V (generally speaking, this is a curve). For example, consider a radially symmetrical field as an external flow: $\Psi = \Psi(r)$, $V = \omega(r)\partial/\partial\varphi$, where r, φ are the polar coordinates on the plane x_1, x_2 . Let the initial surface M be the sphere $|x - x^0|^2 = a^2$, be contained entirely in an area in which the trajectories satisfy condition (ii), which, in this case, coincides with the Rayleigh instability condition $\Phi(r) < 0$, where Φ stands for the Rayleigh function*

$$\Phi(r) = \frac{1}{r^3} \frac{d}{dr} (r^4 \omega^2(r)).$$

The field V is tangent to the sphere M at the points of the meridian obtained as the intersection of the sphere with the vertical plane containing the radius vector of the center of the sphere; on the trajectories issuing from this meridian, the initial perturbation grows exponentially.

5. The MHD equations

Now we discuss briefly the linearized MHD equations

$$\begin{cases} \frac{\partial v}{\partial t} + (V, \nabla)v + (v, \nabla)V + \nabla P = (B, \nabla)b + (b, \nabla)B + \varepsilon^2 \nu \Delta u, \\ \frac{\partial b}{\partial t} + (V, \nabla)b + (v, \nabla)B = (B, \nabla)v + (b, \nabla)V + \varepsilon^2 \mu \Delta b, \\ (\nabla, u) = (\nabla, b) = 0. \end{cases} \quad (3)$$

We suppose that V, B have the same analytic properties as the vector field V from the previous sections; moreover, we assume that these vector fields satisfy stationary ideal MHD equations

$$(V, \nabla)V - (B, \nabla)B + \nabla P_0 = 0, \quad (V, \nabla)B - (B, \nabla)V = 0, \quad (\nabla, V) = (\nabla, B) = 0.$$

Let us pose the Cauchy problem with a localized initial condition

$$v|_{t=0} = v_0\left(\frac{\Phi^0(x)}{\varepsilon}, x\right), \quad b|_{t=0} = b_0\left(\frac{\Phi^0(x)}{\varepsilon}, x\right), \quad (4)$$

where the properties of the initial functions are analogous to those from the previous sections. During the evolution of the perturbation, the initial surface is divided into two surfaces M_1^t ,

M_2^t defined by the equations $\Phi_1(x, t) = 0$ and $\Phi_2(x, t) = 0$. The surface M_1 moves along the trajectories of the field $V - B$, and the surface M_2 — along the trajectories of the field $V + B$. Namely, let us denote by $x_{1,2}^0(x, t)$ the initial point of the trajectory of $V \pm B$ which comes to the point x during the time t ; the functions $\Phi_{1,2}$ have the form $\Phi_{1,2}(s, t) = \Phi^0(x_{1,2}^0(x, t))$. The interaction of these two modes is described by two functions $\Phi_{12}(x, t, \tau)$ and $\Phi_{21}(x, t, \tau)$ which have the form

$$\Phi_{12}(x, t, \tau) = \Phi^0(x_1^0(x_2^0(x, \tau), t - \tau)), \quad \Phi_{21}(x, t, \tau) = \Phi^0(x_2^0(x_1^0(x, \tau), t - \tau)).$$

Note that for each $\tau \in [0, t]$ the equation $\Phi_{12} = 0$ defines the surface which is obtained from M_0 by the shift during the time τ along the trajectories of $V + B$ and then during the time $t - \tau$ — along the trajectories of $V - B$. The structure of the asymptotic solution has the following form.

Theorem 2. *Solution of the Cauchy problem (3) — (4) has the form $v = v^0 + w, b = b^0 + q$, $w, q = O(\varepsilon)$, where*

$$\begin{aligned} v^0 &= v_1\left(\frac{\Phi_1(x, t)}{\varepsilon}, x, t\right) + v_2\left(\frac{\Phi_2(x, t)}{\varepsilon}, x, t\right) + \\ &+ \int_0^t v_{12}\left(\frac{\Phi_{12}(x, t, \tau)}{\varepsilon}, x, t, \tau\right) d\tau + \int_0^t v_{21}\left(\frac{\Phi_{21}(x, t, \tau)}{\varepsilon}, x, t, \tau\right) d\tau, \\ b^0 &= b_1\left(\frac{\Phi_1(x, t)}{\varepsilon}, x, t\right) + b_2\left(\frac{\Phi_2(x, t)}{\varepsilon}, x, t\right) + \\ &+ \int_0^t b_{12}\left(\frac{\Phi_{12}(x, t, \tau)}{\varepsilon}, x, t, \tau\right) d\tau + \int_0^t b_{21}\left(\frac{\Phi_{21}(x, t, \tau)}{\varepsilon}, x, t, \tau\right) d\tau. \end{aligned}$$

The functions v_1, b_1 and v_2, b_2 satisfy the Cauchy problems for ordinary differential equations along the trajectories of $V - B$ and $V + B$, while the functions v_{12}, b_{12} and v_{21}, b_{21} satisfy the Goursat problems for hyperbolic systems of equations with times τ and $\eta = t - \tau$. The lengths of the vectors v_1, v_2, b_1, b_2 are constant along the corresponding trajectories (i.e. the growth of the solutions can be produced only by $v_{12}, b_{12}, v_{21}, b_{21}$).

Remark 3. The vector fields v_1, b_1 and v_2, b_2 describe the Alfvén modes while the vector fields v_{12}, b_{12} and v_{21}, b_{21} describe the interaction of the modes.

6. Localization and growth of the solutions

Let us consider three different situations.

- (i) $(B, \nabla \Phi_0)|_{M_0} \neq 0$ on the support of the initial vector fields. In this case the interaction is small: $\int_0^t v_{12} = O(\varepsilon)$, $\int_0^t v_{21} = O(\varepsilon)$, $\int_0^t b_{12} = O(\varepsilon)$, $\int_0^t b_{21} = O(\varepsilon)$. The leading term of asymptotics is localized near two surfaces M_1^t and M_2^t and there is no growth of the perturbations (the lengths of the corresponding vectors are conserved).
- (ii) $(B_0, \nabla \Phi_0)|_{M_0} \equiv 0$. The interaction is very strong — the leading term of asymptotics has the form $\hat{v}\left(\frac{\Phi(x, t)}{\varepsilon}, x, t\right)$ and is localized near the surface M^t , obtained from M^0 via shift along the trajectories of V . For certain fields B, V the function \hat{v} can grow in time (exponentially or as some power of t).
- (iii) $(B_0, \nabla \Phi_0)|_{M_0} = 0$ on a curve $\gamma_0 \subset M_0$. In this case $\int_0^t v_{12}, \int_0^t v_{21}, \int_0^t b_{12}, \int_0^t b_{21} = O(\sqrt{\varepsilon})$. These fields are localized near surfaces

$$M_{12} = \cup_{\tau \in [0, t]} g^{t-\tau} h^\tau \gamma_0$$

and

$$M_{21} = \cup_{\tau \in [0, t]} g^{t-\tau} h^{\tau} \gamma_0.$$

Here g^s and h^s denote shifts along the trajectories of $V-B$ and $V+B$ during the time s . The functions $\int_0^t v_{12}$, $\int_0^t v_{21}$, $\int_0^t b_{12}$, $\int_0^t b_{21}$ can grow in time for certain fields V, B (exponentially or as some power of t).

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