Junction problems for thin inclusions in elastic bodies

A M Khludnev[1,2]

[1] Lavrentyev Institute of Hydrodynamics, 630090 Novosibirsk, Russia
[2] Novosibirsk State University, 630090 Novosibirsk, Russia

E-mail: khlud@hydro.nsc.ru

Abstract. Equilibrium problems for a 2D elastic bodies with thin Euler-Bernoulli and Timoshenko elastic inclusions are considered. It is assumed that inclusions have a joint point, and a junction problem for these inclusions is analyzed. Existence of solutions is proved, and different equivalent formulations of problems are discussed. In particular, junction conditions at the joint point are found. A delamination of the elastic inclusions is also assumed. In this case, inequality type boundary conditions are imposed at the crack faces to prevent a mutual penetration between crack faces. A convergence to infinity of a rigidity parameter of the elastic inclusions is investigated. Limit problems are analyzed.

1. Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ. Denote $\gamma_b = (0,1) \times \{0\}$, $\gamma_t = (1,2) \times \{0\}$, $\gamma = \gamma_b \cup \gamma_t \cup \{(1,0)\}$ assuming $\bar{\gamma} \subset \Omega$, see Figure 1 Denote by $\nu = (0,1)$ a unit normal vector to γ; $\tau = (0,1)$, $\Omega_\gamma = \Omega \setminus \bar{\gamma}$.

The domain Ω_γ represents a region with an elastic material, γ_b and γ_t are thin elastic Euler-Bernoulli and Timoshenko inclusions, respectively, incorporated in the elastic material. This means that a behavior of the inclusions is described by the Euler- Bernoulli and Timoshenko equations.

Let $B = \{b_{ijkl}\}, i,j,k,l = 1,2,$ be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$b_{ijkl} = b_{jikl} = b_{klij}, \quad i,j,k,l = 1,2,\ b_{ijkl} \in L^\infty(\Omega),$$
$$b_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij},\ c_0 = const > 0\ .$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in these indices. Let $f = (f_1, f_2) \in L^2(\Omega)^2$ be a given function.

We first provide a variational formulation of the equilibrium problem for the elastic body with thin inclusions γ_b, γ_t. A space is introduced

$$W = \{(u,v,w,\varphi)\,|\, u \in H_0^1(\Omega)^2, (v,w) \in H^1(\gamma)^2, v \in H^2(\gamma_b),$$
$$\varphi \in H^1(\gamma_t);\ v = u_\nu,\ w = u_\tau \text{ on } \gamma;\ v_x(1-) + \varphi(1+) = 0\}$$

with the norm

$$\|(u,v,w,\varphi)\|_W^2 = \|u\|^2_{H_0^1(\Omega)^2} + \|(v,w)\|^2_{H^1(\gamma)^2} + \|v\|^2_{H^2(\gamma_b)} + \|\varphi\|^2_{H^1(\gamma_t)}.$$

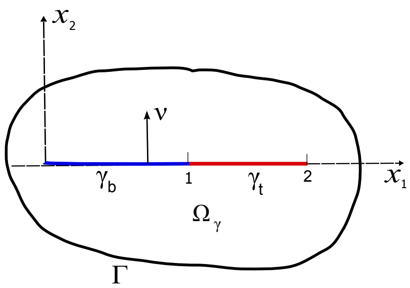

Figure 1. Elastic body with thin inclusions

Here, $u = (u_1, u_2)$, $u_\nu = u\nu$, $u_\tau = u\tau$. The standard notations $H_0^1(\Omega)$, $H^1(\gamma)$, etc., are used for Sobolev spaces. We identify functions defined on γ with functions of the variable x; $x = x_1$, $h_x = \frac{dh}{dx}$, $(x_1, x_2) \in \Omega$. For a simplicity, we write $\sigma(u)\varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u)$.

The identity is considered

$$(u, v, w, \varphi) \in W, \tag{1}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}) - \int_{\Omega_\gamma} f\bar{u} + \int_{\gamma_b} v_{xx}\bar{v}_{xx} + \int_{\gamma} w_x\bar{w}_x + \tag{2}$$

$$+ \int_{\gamma_t} \{(v_x + \varphi)(\bar{v}_x + \bar{\varphi}) + \varphi_x\bar{\varphi}_x\} = 0 \quad \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in W.$$

Theorem 1. The problem (1)-(2) has a unique solution.

We are able to provide a differential formulation of the problem (1)-(2). It is necessary to find functions $u = (u_1, u_2), v, w, \varphi$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, as follows

$$-\mathrm{div}\ \sigma = f,\ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{3}$$

$$v_{xxxx} = [\sigma_\nu] \text{ on } \gamma_b,\ -w_{xx} = [\sigma_\tau] \text{ on } \gamma, \tag{4}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu],\ -\varphi_{xx} + v_x + \varphi = 0 \quad \text{on} \quad \gamma_t, \tag{5}$$

$$(u, v, w, \varphi) \in W;\ v_{xxx} = v_{xx} = w_x = 0 \ \text{ for } x = 0, \tag{6}$$

$$w_x(1-) = w_x(1+);\ v_x + \varphi = \varphi_x = w_x = 0 \ \text{ for } x = 2, \tag{7}$$

$$-v_{xxx}(1-) = (v_x + \varphi)(1+);\ v_{xx}(1-) = -\varphi_x(1+). \tag{8}$$

Here, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma_{ij}\nu_j\tau_i$, and $[h] = h^+ - h^-$ is a jump of a function h on γ, where $h^\pm$ are the traces of h on the faces $\gamma^\pm$. The signs $\pm$ correspond to positive and negative directions of ν.

The function $u = (u_1, u_2)$ describes a displacement field of the elastic body; functions w, v fit to displacements of the inclusions γ_b, γ_t along the axis x_1 and axis x_2 respectively; the function φ describes a rotation angle of the inclusion γ_t. Observe that a part of boundary conditions for functions u, v, w, φ is included in the condition $(u, v, w, \varphi) \in W$.

Relations (3) are the equilibrium equations for the elastic body and Hooke's law; (4)-(5) are the Euler-Bernoulli and Timoshenko equilibrium equations for the inclusions γ_b and γ_t. According to the condition $(u, v, w, \varphi) \in W$, the vertical (along the axis x_2) and tangential (along the axis x_1) displacements of the elastic body coincide with the inclusion displacements at γ. The right-hand sides $[\sigma_\nu], [\sigma_\tau]$ in (4), (5) describe forces acting on γ from the surrounding elastic media.

Theorem 2. *Problem formulations* (3)-(8) *and* (1)-(2) *are equivalent provided that the solutions are smooth.*

By (8) and the first relations of (6), (7), we can write a complete system of junction conditions at the joint point $(1,0)$:

$$w(1-) = w(1+),\ v(1-) = v(1+),\ v_x(1-) = -\varphi(1+),$$
$$w_x(1-) = w_x(1+),\ -v_{xxx}(1-) = (v_x+\varphi)(1+),\ v_{xx}(1-) = -\varphi_x(1+).$$

2. Delaminated elastic inclusion

Assume that the Euler-Bernoulli part γ_b of the inclusion γ is delaminated, please refer to Figure 1. This means that a crack is located between γ_b and the elastic matrix. To fix a situation, the delamination is assumed to be at the positive side of γ_b. In this case, displacements v, w of the inclusion γ_b should coincide with displacements of the elastic body at γ_b^-. In our model, inequality type boundary conditions are considered at the crack faces to prevent a mutual penetration between the faces.

Denote $\Omega_b = \Omega \setminus \bar{\gamma}_b$ and introduce a set of admissible functions

$$K = \{(u,v,w,\varphi) \mid u \in H^1_\Gamma(\Omega_b)^2, (v,w) \in H^1(\gamma)^2, v \in H^2(\gamma_b),$$
$$\varphi \in H^1(\gamma_t),\ u|_{\gamma^-} = (w,v),\ [u_\nu] \geq 0 \text{ on } \gamma_b;\ v_x(1-) + \varphi(1+) = 0\},$$

where $H^1_\Gamma(\Omega_b) = \{\phi \in H^1(\Omega_b) \mid \phi = 0 \text{ on } \Gamma\}$. Notice that the inequality $[u_\nu] \geq 0$ included in the definition of K provides a mutual nonpenetration between the crack faces $\gamma_b^\pm$. An equilibrium problem for the elastic body with the delaminated Euler-Bernoulli inclusion γ_b and the Timoshenko inclusion γ_t can be formulated as follows. We have to find functions $u = (u_1, u_2), v, w, \varphi$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, as follows

$$-\text{div}\ \sigma = f,\ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{9}$$
$$v_{xxxx} = [\sigma_\nu] \text{ on } \gamma_b,\ -w_{xx} = [\sigma_\tau] \text{ on } \gamma, \tag{10}$$
$$-v_{xx} - \varphi_x = [\sigma_\nu],\ -\varphi_{xx} + v_x + \varphi = 0 \quad \text{on} \quad \gamma_t, \tag{11}$$
$$(u,v,w,\varphi) \in K;\ v_{xxx} = v_{xx} = w_x = 0 \text{ for } x = 0, \tag{12}$$
$$w_x(1-) = w_x(1+);\ v_x + \varphi = \varphi_x = w_x = 0 \text{ for } x = 2, \tag{13}$$
$$\sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u_\nu] = 0 \text{ on } \gamma_b, \tag{14}$$
$$-v_{xxx}(1-) = (v_x+\varphi)(1+);\ v_{xx}(1-) = -\varphi_x(1+). \tag{15}$$

As before, a part of boundary conditions is included in the relation $(u,v,w,\varphi) \in K$.

Remark that the problem (9)-(15) admits an equivalent variational formulation. Indeed, a solution (u,v,w,φ) satisfies a variational inequality

$$\int_{\Omega_b} \sigma(u)\varepsilon(\bar{u}-u) - \int_{\Omega_b} f(\bar{u}-u) + \int_{\gamma_b} v_{xx}(\bar{v}_{xx} - v_{xx}) + \tag{16}$$
$$+ \int_{\gamma_t} \{\varphi_x(\bar{\varphi}_x - \varphi_x) + (v_x+\varphi)(\bar{v}_x + \bar{\varphi} - v_x - \varphi)\} +$$
$$+ \int_{\gamma} w_x(\bar{w}_x - w_x) \geq 0 \quad \forall\, (\bar{u},\bar{v},\bar{w},\bar{\varphi}) \in K;\ (u,v,w,\varphi) \in K.$$

Journal of Physics: Conference Series **722** (2016) 012043 doi:10.1088/1742-6596/722/1/012043

3. Rigidity of Euler-Bernoulli beam goes to infinity

In practice, a solution of the problem (9)-(15) should depend on the rigidity parameters of the elastic inclusions. In the model (9)-(15), these parameters were taken to be equal to 1. In this section we introduce a rigidity parameter $\delta > 0$ in the Euler-Bernoulli equations of the problem (9)-(15) and analyze its passage to infinity. For a fixed parameter δ, we have to solve the following problem: to find $u^\delta, v^\delta, w^\delta, \varphi^\delta$, $\sigma^\delta = \{\sigma^\delta_{ij}\}, i, j = 1, 2$, such that

$$-\text{div}\ \sigma^\delta = f,\ \sigma^\delta - B\varepsilon(u^\delta) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{17}$$

$$\delta v^\delta_{xxxx} = [\sigma^\delta_\nu] \text{ on } \gamma_b,\ -\ \text{div}(a^\delta w^\delta_x) = [\sigma^\delta_\tau] \text{ on } \gamma, \tag{18}$$

$$-v^\delta_{xx} - \varphi^\delta_x = [\sigma^\delta_\nu],\ -\varphi^\delta_{xx} + v^\delta_x + \varphi^\delta = 0 \quad \text{on} \quad \gamma_t, \tag{19}$$

$$(u^\delta, v^\delta, w^\delta, \varphi^\delta) \in K;\ v^\delta_{xxx} = v^\delta_{xx} = w^\delta_x = 0 \ \text{for}\, x = 0, \tag{20}$$

$$\delta w^\delta_x(1-) = w^\delta_x(1+);\ v^\delta_x + \varphi^\delta = \varphi^\delta_x = w^\delta_x = 0 \ \text{for}\ x = 2, \tag{21}$$

$$\sigma^{\delta+}_\nu \le 0,\ \sigma^{\delta+}_\tau = 0,\ \sigma^{\delta+}_\nu [u^\delta_\nu] = 0 \text{ on } \gamma_b, \tag{22}$$

$$-\delta v^\delta_{xxx}(1-) = (v^\delta_x + \varphi^\delta)(1+);\ \delta v^\delta_{xx}(1-) = -\varphi^\delta_x(1+). \tag{23}$$

Here, $a^\delta(x) = 1$ on γ_t, and $a^\delta(x) = \delta$ on γ_b.

The problem (17)-(23) admits an equivalent variational formulation. A unique solution of the variational inequality (with $\sigma(u^\delta) = \sigma^\delta$) exists

$$(u^\delta, v^\delta, w^\delta, \varphi^\delta) \in K, \tag{24}$$

$$\int\limits_{\Omega_b} \sigma(u^\delta)\varepsilon(\bar u - u^\delta) - \int\limits_{\Omega_b} f(\bar u - u^\delta) + \delta \int\limits_{\gamma_b} v^\delta_{xx}(\bar v_{xx} - v^\delta_{xx}) + \tag{25}$$
$$+ \int\limits_{\gamma_t} \{\varphi^\delta_x(\bar\varphi_x - \varphi^\delta_x) + (v^\delta_x + \varphi^\delta)(\bar v_x + \bar\varphi - v^\delta_x - \varphi^\delta)\} +$$
$$+ \int\limits_{\gamma} a^\delta w^\delta_x(\bar w_x - w^\delta_x) \ge 0 \quad \forall\, (\bar u, \bar v, \bar w, \bar\varphi) \in K.$$

Our aim is to justify a passage to the limit as $\delta \to \infty$ in the problem (24)-(25).

Introduce a set of infinitesimal rigid displacements

$$R(\gamma_b) = \{\rho = (\rho_1, \rho_2) \mid \rho(x_1, x_2) = d(-x_2, x_1) + (d^1, d^2),$$
$$(x_1, x_2) \in \gamma_b\},\ d, d^1, d^2 \in \mathbb{R},$$

and a set of admissible functions

$$K_r = \{(u, v, w, \varphi) \mid u \in H^1_\Gamma(\Omega_b)^2, (v, w, \varphi) \in H^1(\gamma_t)^3,\ [u_\nu] \ge 0 \text{ on } \gamma_b;$$
$$u|_{\gamma_t} = (w, v),\ \ u|_{\gamma_b^-} = (\rho_1, \rho_2) \in R(\gamma_b),\ \rho_{2x}(1) + \varphi(1) = 0\}.$$

It can be proved that $\delta \to +\infty$,

$$u^\delta \to u \text{ weakly in } \ H^1_\Gamma(\Omega_b)^2, \varphi^\delta \to \varphi \text{ weakly in } \ H^1(\gamma_t), \tag{26}$$

$$v^\delta \to v \text{ weakly in } H^1(\gamma),\ \text{weakly in } H^2(\gamma_b),\ v_{xx} = 0 \text{ in } \gamma_b, \tag{27}$$

$$w^\delta \to w \quad \text{weakly in} \ \ H^1(\gamma),\ w_x = 0 \ \text{in} \ \ \gamma_b. \tag{28}$$

Moreover,

$$(u,v,w,\varphi)\in K_r,\ \int\limits_{\Omega_b}\sigma(u)\varepsilon(\bar u-u)-\int\limits_{\Omega_b}f(\bar u-u)+ \tag{29}$$
$$+\int\limits_{\gamma_t}\{\varphi_x(\bar\varphi_x-\varphi_x)+(v_x+\varphi)(\bar v_x+\bar\varphi-v_x-\varphi)+$$
$$+w_x(\bar w_x-w_x)\}\ge 0\quad\forall\,(\bar u,\bar v,\bar w,\bar\varphi)\in K_r.$$

The problem (29) admits an equivalent differential formulation: find displacements $u=(u_1,u_2)$, v,w, a rotation angle φ, a stress tensor $\sigma=\{\sigma_{ij}\}, i,j=1,2$, and $\rho^0\in R(\gamma_b)$ as follows

$$-\mathrm{div}\ \sigma=f,\ \sigma-B\varepsilon(u)=0\quad\text{in}\quad\Omega_\gamma, \tag{30}$$
$$-v_{xx}-\varphi_x=[\sigma_\nu],\ -\varphi_{xx}+v_x+\varphi=0,\ -w_{xx}=[\sigma_\tau]\ \text{on}\ \gamma_t, \tag{31}$$
$$u^-=\rho^0,\ \sigma_\nu^+\le 0,\ \sigma_\tau^+=0,\ \sigma_\nu^+[u_\nu]=0\ \text{on}\ \gamma_b, \tag{32}$$
$$(u,v,w,\varphi)\in K_r;\ v_x+\varphi=\varphi_x=w_x=0\ \text{for}\ x=2, \tag{33}$$
$$\int\limits_{\gamma_b}[\sigma\nu]\rho+(w_x\rho_1)(1)-(\varphi_x\rho_{2x})(1)+ \tag{34}$$
$$+((v_x+\varphi)\rho_2)(1)=0\ \ \forall\rho=(\rho_1,\rho_2)\in R(\gamma_b).$$

Thus, the following statement can be proved.

Theorem 3. *The solutions of the problem* (24)-(25) *converge to the solution of* (29) *in the sense* (26)-(28) *as* $\delta\to\infty$.

The model (30)-(34), or (29), describes an equilibrium state for the elastic body with the rigid inclusion γ_b and elastic Timoshenko inclusion γ_t. The identity (34) provides equilibrium conditions for the rigid inclusion γ_b, i.e. a principal vector of forces and a principal vector of moments acting on γ_b are equal to zero. Indeed, denoting $(\sigma\nu)^\pm$ by $(\sigma^1,\sigma^2)^\pm$ on $\gamma_b^\pm$, the condition (34) can be rewritten in the following form:

$$\int\limits_{\gamma_b}[\sigma^1]=-w_x(1),\ \int\limits_{\gamma_b}[\sigma^2]=-(v_x+\varphi)(1), \tag{35}$$
$$\int\limits_{\gamma_b}([\sigma^2]x_1-[\sigma^1]x_2)=\varphi_x(1). \tag{36}$$

We can also write junction conditions included in the definition of K_r:

$$w(1)=\rho_1^0(1),\ v(1)=\rho_2^0(1),\ \varphi(1)+\rho_{2x}^0(1)=0 \tag{37}$$

and consider (35)-(37) as a complete system of junction conditions at the joint point $(1,0)$. Consequently, nonlocal condition (34) can be seen as a part of junction conditions at the joint point $(1,0)$.

References

[1] Khludnev A M and Negri M 2012 *Z. Angew. Math. Mech.* **92** 341–54
[2] Khludnev A M and Leugering G R 2014 *Math. Mech. Complex Systems* **2** 1–21
[3] Khludnev A M and Leugering G R 2015 *Math. Mech. Solids* **20** 495–511
[4] Itou H and Khludnev A M *Math. Meth. Appl. Sciences* doi 10.1002/mma.3279