Presentation of special series with computed recurrently coefficients of solutions of nonlinear evolution equations

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Abstract. One of the analytical methods of presenting solutions of nonlinear partial differential equations is the method of special series in powers of specially selected functions called basis functions. The coefficients of such series are found successively as solutions of linear differential equations. To find recurrence, the coefficient is achieved by the choice of basis functions, which may also contain arbitrary functions. By using such functional arbitrariness, it allows in some cases to prove the global convergence of the corresponding constructed series, as well as the solvability of the boundary value problem.

1. Introduction

Method of special series [1, 2, 3] is a method of representation of solutions of nonlinear partial differential equations in the form of series by the powers of one or several functions chosen in a special way which allows the series' coefficients to be calculated recurrently without applying any truncation procedures. In the sequel, these functions are called as *basic functions* [4]. When the well-known Fourier method is used for linear equations, the coefficients are also calculated recurrently. For nonlinear case, the variables cannot be usually separated and the superposition principle is inapplicable. If the solutions are represented by Fourier series, one or another truncation procedure is usually used to derive the final system of equations, from which the coefficients are to be determined. In the most cases, a characteristic feature of special series is that the construction is independent of the type of nonlinear equation being solved, although the study of convergence of these series is sure to be closely related to the type of the equation. The approaches more closed to the special series method of obtaining solutions of partial differential equations are related to characteristic expansions investigated in the works of R. Courant [5].

In contrast to the power Taylor series, which converge only for the Cauchy-Kovalevskaya equations under the conditions of analyticity of the problem initial data, the constructed series can converge for wider classes of equations and systems.

In some cases, it is possible to exactly satisfy zero boundary conditions by choosing the basis functions (for example, for description of nonlinear vibrating string with fixed end points [1], or membrane with fixed edges [6]). In other cases, it is possible to satisfy a predetermined boundary condition by using the functional arbitrariness contained in the basic functions [7].

In some cases, it is possible to prove global convergence of the constructed series, including in unlimited domains, where the use of numerical methods meets essential difficulties.

Construction of solutions in the form of series with recurrently calculated coefficients allows to investigate properties for meaningful problems. For example, these series were used for study new properties of the audio line for Lin–Reissner–Tsien equation [8]. Note that the various representations of the solutions of this equation were also obtained in the form of convergent series in powers of special basic functions with functional arbitrariness [9].

In some cases, it was possible to construct the basis functions that take into account presence of known exact solutions, as well as the specific character of non-linear equations. For example, these series were built and studied in [10, 11]. Sometimes, constructed series turn into finite sums and then obtain the exact solution [12], which allows their use for testing numerical methods [13, 14], and for numerical algorithms, taking into account the asymptotics of solutions in these problems [15]. In this paper, new results are obtained related to investigation of the convergence of series for a class of nonlinear evolution equations.

2. General scheme of the series with recurrently calculated coefficients

Let us consider one of constructions of special series for solving the Cauchy problem for nonlinear partial differential equations of the form

$$u_t = F\left(t, u, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^m u}{\partial x^m}\right), \quad u(0,x) = u^0(x), \tag{1}$$

where F is a polynomial of the unknown function $u(t,x)$ and its derivatives with respect to the space variable. The solution is represented by the series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t) P^n(t,x) \tag{2}$$

by the powers of basic function $P(t,x)$ satisfying the overdetermined system

$$P_x = A(t,P), \quad P_t = B(t,P) \tag{3}$$

with functions $A(t,P)$ and $B(t,P)$ being analytic respect to P and such that $A(t,0) \equiv 0$ and $B(t,0) \equiv 0$. It was shown that if the initial conditions are written in the form

$$u^0(x) = \sum_{n=0}^{\infty} u_n^0 P^n(0,x), \tag{4}$$

then substituting series (2) into equation (1), collecting similar terms, and taking into account relations (3), we obtain the sequence of first-order ordinary differential equations for the coefficients $u_n(t)$

$$u_n^{'} = F_n(t, u_n, \ldots, u_0), \qquad u_n(0) = u_n^0, \quad n = 1, 2, \ldots \tag{5}$$

where the right-hand sides F_n include only u_j with $j \leq n$ and the coefficients u_n may be linearly contained only.

3. Investigation of convergence of special series to the solution of nonlinear evolution equations

The use of special series with basic functions with functional arbitrariness to construction of solutions of various types of nonlinear partial differential equations is considered. Proof of convergence of the constructed series depends on the type of equations to be solved.

3.1. Generalized Korteweg–de Vries equations

As an example, we mention the construction of the series (2) for Korteweg-de Vries (KdV) equation with initial condition $u(0,x){=}u^0(x)$ in the form of (4). Basic function

$$P(x,t){=}\frac{1}{\exp bx{+}f(t)}, \quad f(t)\in C^1[0,\infty), \quad b>0 \tag{6}$$

is used, which satisfies system (3) for

$$A(t,P){=}{-}bP{+}bf(t)P^2, \quad B(t,P){=}{-}f(t)'P^2$$

with an arbitrary function $f(t)$. Convergence of series (2), (6) is proved for all $x{\geq}0$, $t{\geq}0$, if $0{\leq}f(t){\leq}(\sqrt[3]{2}{-}1)/(2{-}\sqrt[3]{2})$ [2]. At $x{=}0$, the arbitrary function $f(t)$ gives rise to a new boundary condition $u(0,t){=}h(t)$. We can try to find the function $f(t)$ from the function $h(t)$ [7].

Consider an initial-boundary value problem for the nonlinear evolutionary equation

$$\frac{\partial u}{\partial t}+\gamma\frac{\partial^{2r+1}u}{\partial x^{2r+1}}+F\left(t,u,\ldots,\frac{\partial^{2r}u}{\partial x^{2r}}\right)=0, \quad \gamma=\mathrm{const} \tag{7}$$

with initial conditions (4) and boundary condition at $x=0$

$$u(0,t)=h(t), \quad h(t)\in C^1[0,\infty), \quad h(t)-u_0\neq 0, \quad t\geq 0. \tag{8}$$

The well-posedness of the initial-boundary value problem for the generalized KdV equation ($r{=}1$, $F{=}(g(u))_x$) was considered for the class of generalized functions in [16, 17]. One of the conditions of solvability of the mixed problem is the condition

$$|g(u)|\leq(|u|^{10/3}+|u|), \quad \forall u\in R. \tag{9}$$

We find a solution of equation (7) in the form of series (2) with basic function (6) and coefficient $u_0(t)=u_0=\mathrm{const}$. Then the solution of equations (5) has the form

$$u_n(t){=}\exp(b_nt)\left\{u_n^0{+}\int_0^t\exp(-b_n\tau)[(n{-}1)f'(\tau)u_{n-1}{+}R_n(\tau,u_\nu)]d\tau\right\},$$

where $b_n=\mathrm{const}$ and the expressions $R_n(\tau,u_\nu)$ include only functions u_ν with $\nu<n$ [7].

For $x=0$ and $t=0$ the compatibility condition

$$h(0)=h_0=u_0+\sum_{n\geq1}\frac{u_n^0}{(1+f_0)^n}, \quad f(0)=f_0$$

must be fulfilled. To satisfy boundary condition (8) for $t\geq0$, it is necessary to show that there exists a function $f(t)$ such that the equality

$$h(t)=u_0+\sum_{n\geq1}\frac{u_n(t,f(t))}{(1+f(t))^n}$$

is valid. Taking into account the estimates obtained for the proof of the convergence of the constructed series in powers of the basic functions with a functional arbitrariness to construct the solution of a generalized KdF equation [18], we can prove the following theorem.

Theorem 1. *Let the following conditions be fulfilled:*

1. $|u_n^0| \leq Mn^{-m}, \quad 0 < M \leq M_0, \quad n \geq 1, \quad m = 2r + 4, \quad u_0 > 0;$
2. $0 < f(t) \leq M_1, \quad t \geq 0.$

Then there exists a unique function $f(t) \in C^1[0, T_0]$, $M_0, M_1, T_0 = \text{const}$ *such that problem* (7), (4) *and* (8) *has a solution for* $-\infty < x \leq 0$ *and* $0 \leq t \leq T_0$.

Remark 1. Thus, it is shown that using special series with functional arbitrariness one may solve initial-boundary value problems for an important class of evolutionary equations (7), for which restriction (9) on the nonlinear function increasing may be not fulfilled.

3.2. Nonlinear wave equations
Let consider the Cauchy problem for nonlinear wave equation

$$u_{tt} = u_{xx} + G(t, u), \tag{10}$$

$$u_t(x, 0) = u_1(x), \quad u(x, 0) = u_0(x). \tag{11}$$

Here, $G(t, u)$ is a polynomial by the unknown function $u(x, t)$ with the coefficients which are continuous functions of t. Let initial conditions (11) be represented in the form of convergent series by the powers of function (6)

$$u_\nu(x) = \sum_{n=0}^{\infty} u_{n,\nu} P^n(0, x), \quad u_{n,\nu} = \text{const}, \quad \nu = 1, 2. \tag{12}$$

Then a solution of problem (10), (10), (12) also may be constructed in the form of the series (2), (6). The coefficients $u_n(t)$ after substitution of series (2), (6) into (10) and equating the expressions of the same powers $P(x, t)$, will be found from a sequence of linear second order differential equations

$$u_n'' - b^2 n^2 u_n = L_n(t, f) + R_n(t), \quad n \geq 0, \tag{13}$$

where $L_n(t, f)$ is the following expression

$$L_n(t, f) = f'' u_{n-1} + 2(n-1) f' u_{n-1}' - b^2 f (n-1) n u_{n-1} - b^2 (n-1)^2 u_{n-1}$$
$$+ [b^2 f^2 - (f')^2](n-2)(n-1) u_{n-2},$$

related to the linear terms in equation (10), and $R_n(t)$ is an expression, related to nonlinear function $G(t, u)$. By this, the coefficients $u_k(t)$ of series (2), (6) are included into expressions $L_n(t, f)$ and $G(t, u)$ with $k \leq n - 1$, that allows to find the coefficients of the series successively.

Initial conditions for the equation (13) are determined by constant $u_{n,\nu}$, $\nu = 1, 2$ in initial conditions (11)

$$u_n(0) = u_{n,0}, \quad u_n'(0) = u_{n,1} + (n-1) f'(0), \quad n \geq 1. \tag{14}$$

The following theorem is valid.

Theorem 2. *Let the following conditions be fulfilled:*
1. $|u_{n,0}| \leq \frac{M}{6n^4}, \quad |u_{n,1}| \leq \frac{Mb}{6n^3}, \quad M > 0, \quad n \geq 1;$
2. $f(t) \in C^2[0, \infty)$, $0 < q_1 \leq f(t) \leq q$, $|f'(t)| \leq q$, $|f''(t)| \leq q$, $\quad q_1, q > 0, \quad t \geq 0.$

Then there are exist constants $M_0 > 0$ *and* $q_0 > 0$, *that for* $M \leq M_0$ *and* $q \leq q_0$ *series* (2), (6) *converges to a solution of the Cauchy problem* (10), (11), (12) *for all* $x \geq 0$ *and* $0 \leq t \leq T$, *where* $T = \frac{\ln(1+q_0)}{2b}$.

Proof. Solutions of the sequence of linear second order differential equations (13), (14) have the form

$$u_n(t) = A_n \exp(bht) + B_n \exp(-bnt) + \frac{1}{bn} \int_0^t \operatorname{sh}(bn(\tau - t))(L_n(\tau, f) + R_n(\tau)) d\tau, \quad n \geq 1, \tag{15}$$

where constants A_n and B_n are determined from initial conditions (14)

$$A_n + B_n = u_{n,0}, \quad A_n bn - B_n bn = u_{n,1}.$$

To investigate convergence of series (2), (6) we estimate the coefficients of the series $u_n(t)$. In contrast to estimations of the series which is used to solve equation (7), for nonlinear wave equation (10) it is necessary to estimate and the derivatives of these coefficients up to the second order. At the same time in the assessments, we cannot use the same inductive assumption that for equation (7). It is related to the difference of types of ordinary differential equations for the series coefficients $u_n(t)$.

In the considered case of nonlinear wave equation the following estimations are valid:

$$|u_n(t)| \leq \frac{M\exp(2bnt)}{n^4}, \quad n \geq 1, \tag{16}$$

$$|u_n'(t)| \leq \frac{Mb\exp(2bnt)}{n^3}, \quad n \geq 1, \tag{17}$$

$$|u_n''(t)| \leq \frac{4Mb^2\exp(2bnt)}{n^2}, \quad n \geq 1, \tag{18}$$

which are proved by induction.

We carry out the proof on the base of example of inequality (16). One notes that for $n = 0$ $u_0(t) = A_0 + B_0 t$. Obviously, an estimation like (16) is valid for $u_0(t)$. We can assume that inequality $|u_0(t)| \leq M\exp(2bt)$ is valid.

For $n = 1$, taking into account (15), there is

$$u_1(t) = A_1\exp(bt) + B_1\exp(-bt) + \frac{e^{-bt}}{2b}\int_0^t f''(\tau)(A_0\tau + B_0)e^{b\tau}d\tau - \frac{e^{bt}}{2b}\int_0^t f''(\tau)(A_0\tau + B_0)e^{-b\tau}d\tau.$$

Let estimate the series coefficient $u_1(t)$.

$$|u_1(t)| \leq M\exp(2bt)\Big(\frac{|A_1|e^{-bt}}{M} + \frac{|B_1|e^{-3bt}}{M} + \frac{q}{6b^2} + \frac{q}{2b^2}\Big) \leq M\exp(2bt)\Big(\frac{1}{6} + \frac{1}{6} + \frac{2q}{3b^2}\Big). \tag{19}$$

For $q_0 \leq b^2/2$ from inequality (19), it follows that inequalities (16) are valid for $n = 1$. Thus, induction assumption (16) is proved for $n = 1$. Assuming that the inequality (16) is valid for $n = N$, it is easy to prove this inequality and $n = N + 1$. Let estimate expression $L_{N+1}(t)$, which contains the series coefficients $u_k(t)$ for $k \leq N$ corresponding the linear part of (10).

$$|L_{N+1}(t)| \leq Mqe^{2bNt}\Big(\frac{2b}{N^3} + \frac{b^2qN}{(N-1)^3} + \frac{1}{N^3} + \frac{2b}{N^2} + \frac{qN}{(N-1)^3}\Big). \tag{20}$$

Using inequality (20), we obtain the following estimation for the integral:

$$\frac{1}{b(N+1)}\int_0^t \mathrm{sh}(b(N+1)(\tau - t))|L_{N+1}(\tau, f)|d\tau \leq \frac{M\exp(2b(N+1)t)}{3(N+1)^4}. \tag{21}$$

Note, that for $b \geq 1$ value of q_0 it is possible to estimate. In this case for $q_0 = 1/27$, estimation (21) is valid. T.i. the second condition of Theorem 2 is used only for estimating terms of $L_{N+1}(t, f)$, which correspond to the linear part of the equation (10). These restrictions on arbitrary function $f(t)$ are similar to those of an arbitrary function in Theorem 1.

Estimating expression $R_{N+1}(t)$, we use the condition $M \leq M_0$, where M_0 is determined by the form of nonlinear function $G(t,u)$. Constant M_0 we may choose so that the following inequality will be fulfilled

$$\frac{1}{b(N+1)}\int\limits_0^t \operatorname{sh}(b(N+1)(\tau - t))|R_{N+1}(\tau,f)|d\tau \leq \frac{M\exp(2b(N+1)t)}{3(N+1)^4}. \tag{22}$$

Thus, inequality (21), (22) prove inequality (16). Similarly, inequality (17), (18) can be proved.

Inequalities (16)–(18) allow to prove convergence of series (2), (6). Indeed,

$$|u(x,t)| \leq \sum_{n=0}^{\infty}|u_n(t)|P^n(t,x) \leq |u_0| + \sum_{n=1}^{\infty}\frac{M\exp(2bnt)}{n^4}P^n(t,x) \leq |u_0|$$

$$+M\sum_{n=1}^{\infty}\frac{\exp(2bnt)}{n^4(1+q_0)^n} \leq |u_0| + M\sum_{n=1}^{\infty}\frac{e^{2bTn}}{n^4(1+q_0)^n}.$$

Similarly one can prove the convergence of the series corresponding derivatives u_t, u_{tt}, u_t, u_{xx}.

Consequently, series (2), (6) converge to a solution of the Cauchy problem (10), (11), (12) for all $x \geq 0$ and $0 \leq t \leq T$, where $T = \frac{\ln(1+q_0)}{2b}$. Theorem 2 is proved.

Remark 2. Proven estimates for the coefficients and convergence of (2), (6) to a solution of the Cauchy problem in a semi-infinite region in x allow to hope to prove existence of solutions of the initial-boundary value problem (10), (11), (12) with the given boundary condition at $x = 0$ by choosing an arbitrary function $f(t)$ and the second boundary condition, given in the form $u(+\infty, 0) = u_0$ due to the specificity of the base function (6) tends to zero when $x \to +\infty$. The choice of such a function $f(t)$ is currently still open.

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