Localized vortices in a nonlinear shallow water model: examples and numerical experiments

S A Beisel[1] and A A Tolchennikov[2]

[1] Institute of Computational Technologies SB RAS, Novosibirsk, Russia
[2] Institute for Problems in Mechanics RAS, Moscow, Russia

E-mail: beisels@gmail.com, tolchennikovaa@gmail.com

Abstract. Exact solutions of the system of nonlinear shallow water equations on paraboloid are constructed by the method of group analysis. These solutions describe fast wave motion of the fluid layer and slow evolution of symmetric localized vortices. Explicit formulae are obtained for asymptotic solution related to the linear shallow water approximation. Numerical methods are used by the modeling the trajectory of the vortex center in the case of asymmetric vortices.

1. Introduction

A system of shallow water equations is used for models of atmospheric and oceanic motion with large length scales [1]. For example, this system can describe propagation of tsunami waves and vortices caused by the uplift of the ocean surface localized in the area of a characteristic size l [2],[3]. It is assumed that l is small in comparison with the size of the basin L (but l is greater than the average depth of the basin). Thus, in dimensionless variables, there is a small parameter $\mu = \frac{l}{L}$ and the solution can be called localized.

In the work [4], the authors concerned the localized vortices (the characteristic size of vortex — μ is a small parameter) moving along the background flow trajectory (the background is the certain solution of the system of shallow water equations). The authors constructed explicit formulas for the asymptotic solution in linear approximation. Also, it was noticed that the vortex has the same form as the initial condition if the Cauchy-Riemann conditions are satisfied along the trajectory of background (i.e., the phase flow is conformal). Nonlinear localized vortices are concerned. For a background, the famous exact solution of the system of shallow water equations on paraboloid from the paper of Thacker [5] was chosen (see also the papers of Ball [6], [7] for common properties of motion of a shallow rotating liquid lying on a paraboloid; and the paper of Rogers [8] for more general exact solutions). In order to construct such a moving vortex, remarkable work [9] is used. Using theorem 2 from [9] (applying one-parametric group of transformations to the stationary localized vortex), exact moving symmetric vortex localized in a vicinity of the center of gravity of liquid is obtained and its properties are described. One applies numerical methods in case the center of vortex does not coincide with the center of gravity and when the vortex is asymmetric.

Journal of Physics: Conference Series **722** (2016) 012038 doi:10.1088/1742-6596/722/1/012038

2. Formulation of problem and results

2.1. Formulation of problem

Let $z = D(x)$ (where $x = (x_1, x_2)$) be the equation of the basin surface underlying the liquid. Let $V(x,t) = (V_1(x,t), V_2(x,t))$ (velocity field) and $H(x,t)$ (free surface elevation) be a solution of the shallow water system:

$$\begin{cases} H_t + ((H + D(x))V_1)_{x_1} + ((H + D(x))V_2)_{x_2} = 0, \\ (V_1)_t + V_1(V_1)_{x_1} + V_2(V_1)_{x_2} + H_{x_1} = 0, \\ (V_2)_t + V_1(V_2)_{x_1} + V_2(V_2)_{x_2} + H_{x_2} = 0. \end{cases} \tag{1}$$

Let $u = (u_1, u_2), \eta$ be a perturbation of (1), i.e. $V + u$, $H + \eta$ also satisfy (1). Then there is a system for u, η:

$$\begin{cases} \eta_t + (\rho u_1 + \eta u_1 + \eta V_1)_{x_1} + (\rho u_2 + \eta u_2 + \eta V_2)_{x_2} = 0, \\ (u_1)_t + (u_1 + V_1)(u_1)_{x_1} + (u_2 + V_2)(u_1)_{x_2} + u_1(V_1)_{x_1} + u_2(V_1)_{x_2} + \eta_{x_1} = 0, \\ (u_2)_t + (u_1 + V_1)(u_2)_{x_1} + (u_2 + V_2)(u_2)_{x_2} + u_1(V_2)_{x_1} + u_2(V_2)_{x_2} + \eta_{x_2} = 0. \end{cases} \tag{2}$$

The linearization of this system:

$$\begin{cases} \eta_t + (\rho u_1 + \eta V_1)_{x_1} + (\rho u_2 + \eta V_2)_{x_2} = 0, \\ (u_1)_t + V_1(u_1)_{x_1} + V_2(u_1)_{x_2} + u_1(V_1)_{x_1} + u_2(V_1)_{x_2} + \eta_{x_1} = 0, \\ (u_2)_t + V_1(u_2)_{x_1} + V_2(u_2)_{x_2} + u_1(V_2)_{x_1} + u_2(V_2)_{x_2} + \eta_{x_2} = 0. \end{cases} \tag{3}$$

$H + D = \rho$ is denoted.

2.2. Background solution

As a background, one takes the exact solution from [8], where depth is the circular paraboloid and velocity field is linear. The Cauchy-Riemann conditions for velocity field are satisfied.

$$D(x_1, x_2) = 1 - \bar{A}(x_1^2 + x_2^2), \;\; \rho(x_1, x_2, t) = \begin{pmatrix} x_1 - c_1(t) & x_2 - c_2(t) \end{pmatrix} F(t) \begin{pmatrix} x_1 - c_1(t) \\ x_2 - c_2(t) \end{pmatrix} + \frac{\bar{C}}{W(t)},$$

$$V(x_1, x_2, t) = L(t) \begin{pmatrix} x_1 - c_1(t) \\ x_2 - c_2(t) \end{pmatrix} + \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix},$$

$$F(t) = \frac{1}{W^2(t)} \begin{pmatrix} \frac{\bar{B}}{2} & 0 \\ 0 & \frac{\bar{B}}{2} \end{pmatrix},$$

$$L(t) = \frac{1}{W(t)} \begin{pmatrix} \frac{W'(t)}{2} & -1 \\ 1 & \frac{W'(t)}{2} \end{pmatrix},$$

$$W(t) = \bar{\lambda} \cos^2(\sqrt{2\bar{A}}\; t) + 2\bar{\mu} \cos(\sqrt{2\bar{A}}\; t) \sin(\sqrt{2\bar{A}}\; t) + \bar{\nu} \sin^2(\sqrt{2\bar{A}}\; t).$$

Constants $\bar{A}, \bar{B}, \bar{C}, \bar{\lambda}, \bar{\mu}, \bar{\nu}$ are as follows

$$\bar{\lambda}\bar{\nu} - \bar{\mu}^2 = \frac{1 - \bar{B} - \bar{\beta}^2}{2\bar{A}}.$$

$(c_1(t), c_2(t))$ (center of gravity) moves around the ellipse

$$c_1(t) = \bar{C}_1 \sin(\sqrt{2\bar{A}}\; t) + \bar{C}_2 \cos(\sqrt{2\bar{A}}\; t), c_2(t) = \bar{C}_3 \sin(\sqrt{2\bar{A}}\; t) + \bar{C}_4 \cos(\sqrt{2\bar{A}}\; t).$$

Two cases will be considered:

example 1: $\bar{B} \neq 0$ (fluid volume is finite, water height is a circular paraboloid).

example 2: $\bar{B} = 0, c_1(t) = 0, c_2(t) = 0$ (fluid volume is infinite, water height depends on time only).

2.3. Numerical experiments

In the following examples, one solves numerically systems (2), (3) with the Cauchy data

$$u|_{t=0} = \ sgrad\ a \exp\left(-\frac{(x_1 - x_1^0)^2}{\mu_1^2} - \frac{(x_2 - x_2^0)^2}{\mu_2^2}\right),\ \eta|_{t=0} = 0,$$

where $sgrad = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$. We use the finite-difference MacCormack scheme of the second order approximation.

2.3.1. Asymmetrical vortex, background from example 2. Let $\mu_1 = 0.025, \mu_2 = 2\mu_1$. In Fig. 1 and Fig. 2 one can see graphics of the free surface elevation η at times $t = 0.16, 0.31, 0.59, 1.06, 2.17$ obtained by linear and nonlinear models (3), (2) respectively. The nonlinearity parameter $a = -0.25$ (i.e., background and vortex twist in the opposite directions).

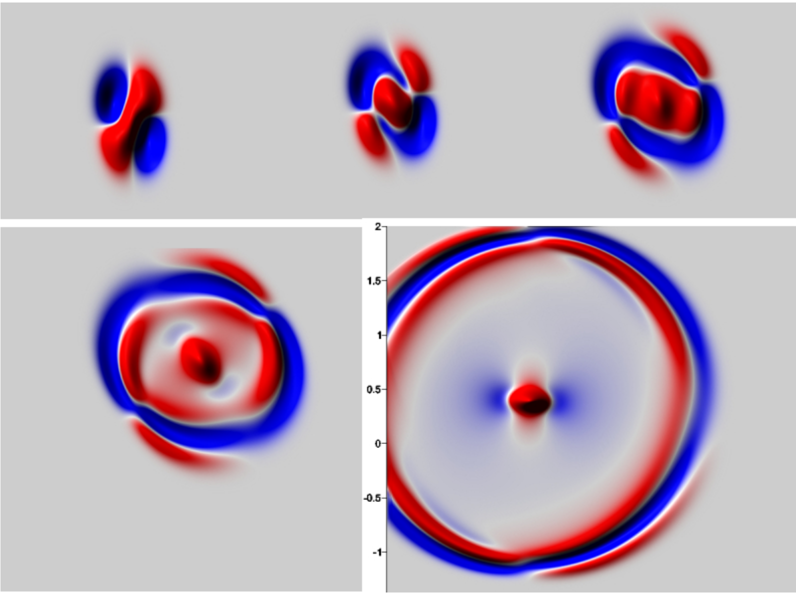

Figure 1. Free surface elevation η (obtained from linear system (3)) at various times. Case $a < 0$

In case of positive a, the free surface elevation graphics for $t = 2.17, a = 0.05, 0.25$ are in Fig. 3.

2.3.2. Symmetrical vortex, background from example 1. Let $\mu_1 = \mu_2 = 0.025$. The main difference from the previous paragraph is in the vortex trajectory. When one solves the linear system (3), then the vortex trajectory coincides with the trajectory of background velocity (in Fig. 4 one can see such a close trajectory). But in the nonlinear model, these two trajectories are not coincide (Fig. 5). The bigger is the nonlinearity parameter, the bigger is the difference between trajectories.

2.4. Model localized vortex

Using method from [9], one can construct a localized vortical solution for backgrounds from examples 1 and 2. Namely:

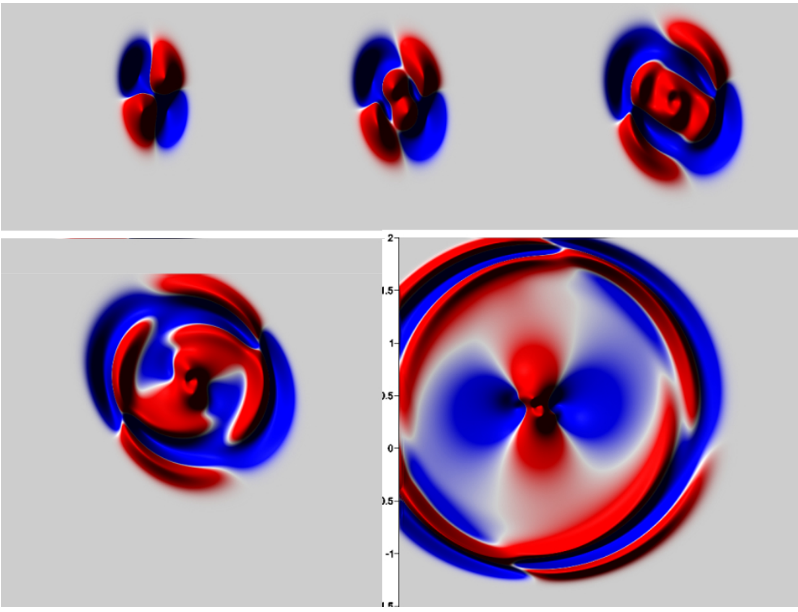

Figure 2. Free surface elevation η (obtained from nonlinear system (2)) at various times. Case $a < 0$

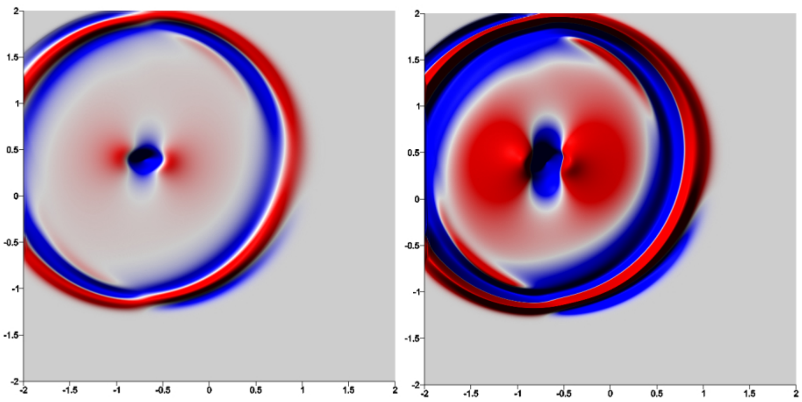

Figure 3. Free surface elevation η (obtained from nonlinear system (2)) for positive $a = 0.05$ (left) and $a = 0.25$ (right)

Let (V, E) be the background from example 1 or 2. $R(t)$ be the trajectory of center of gravity from example 1 or any trajectory from example 2. Let $\mu, a \in \mathbb{R}_+$ (a can be ∞) and functions $f, g \in C^1(\mathbb{R}_+)$ be as follows

$$g(y) = \int_a^y (f^2(x) + 2f(x))dx,$$

then system (2) has the solution

$$u(x,t) = \frac{1}{W(t)} f\left(\frac{|x - R(t)|^2}{2\mu^2 W(t)}\right) \begin{pmatrix} -x_2 + R_2(t) \\ x_1 - R_1(t) \end{pmatrix},$$

$$\eta(x,t) = \frac{\mu^2}{W(t)} g\left(\frac{|x - R(t)|^2}{2\mu^2 W(t)}\right).$$

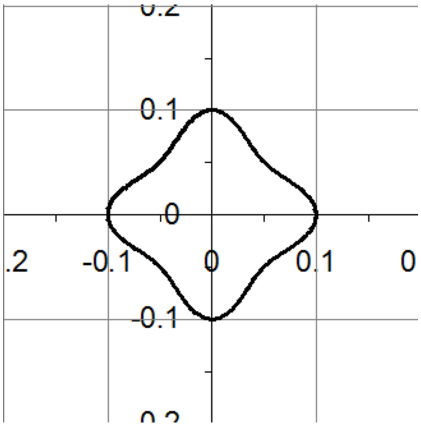

Figure 4. Trajectory of vortex center in linear model

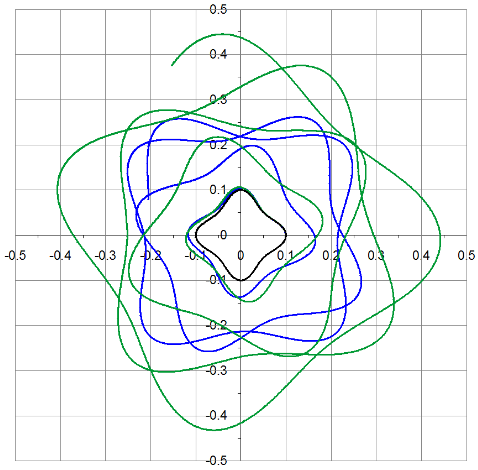

Figure 5. Trajectory of vortex center in nonlinear model. $a = 0.1$ - green line, $a = -0.1$ - blue line

One notes the following:

1. This vortex pulsates, periodically stretches and shrinks.

2. Let $a = \infty$, $\int_0^\infty f^2(x)dx < \infty$, $\int_0^\infty f(x)dx < \infty$, then u, η tend to zero when the distance from the vortex center tends to infinity.

Let $f(x) = \alpha f^0(x), f^0(x) > 0$, α – a nonlinearity parameter.

3. If $\alpha > 0$ (both the vortex and the background twist counterclockwise), then the free surface has a form of cavity. The bigger is α, the deeper is the cavity.

4. If $\alpha < 0$ (the vortex twists clockwise, but the background twists counterclockwise), then for small $|\alpha|$, the free surface has a hump form, and for big $|\alpha|$, the free surface turns to the cavity.

2.4.1. Example. Let $\bar{A} = \frac{1}{4}, \bar{C} = 1, \bar{\lambda} = 2, \bar{\nu} = 1, \bar{\mu} = 0, \mu = 0.1$. Then $W(t) = 1 + \cos^2(\frac{t}{\sqrt{2}})$. Let $a = \infty, f(x) = \alpha e^{-x}$, $g(x) = -\frac{1}{2}\alpha^2 e^{-2y} - 2\alpha e^{-y}$. Then u, η are the fast decaying functions at infinity. The free surface elevation η as function of r (distance to the vortex center) for various nonlinearity parameters is in Fig. 6, 7.

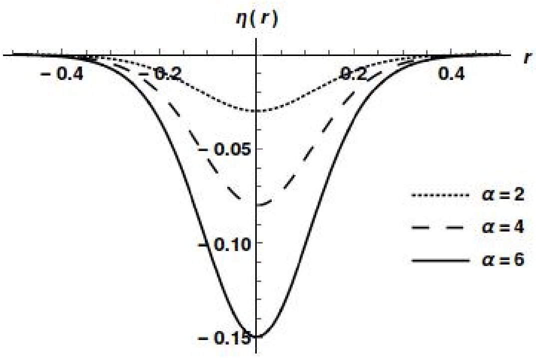

Figure 6. $\eta(r)|_{t=0}, \alpha > 0$

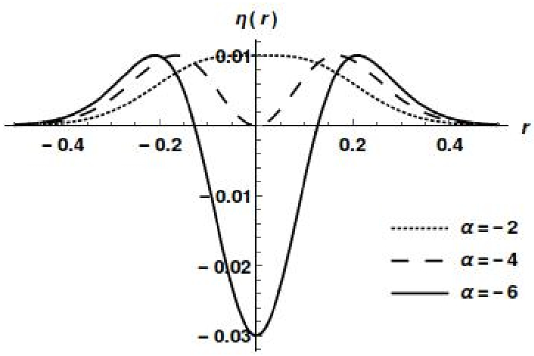

Figure 7. $\eta(r)|_{t=0}, \alpha < 0$

Acknowledgments
The authors are grateful to S.Yu.Dobrokhotov, A.I.Shafarevich, A.A.Chesnokov for useful discussions. This work is supported by the Russian Foundation for Basic Research (Grant No. 16-31-00442).

References
[1] Pedlosky J 1979 *Geophysical Fluid Dynamics* (New York: Springer US)
[2] Pelinovskii E N 1996 *Hydrodynamics of Tsunami Waves*
[3] Berry M V 2005 *New J. Phys.* **7** 129
[4] Dobrokhotov S Y, Shafarevich A I and Tirozzi B 2008 *Russ. J. Math. Phys.* **15** 192–221
[5] Thacker W C 1981 *Journal of Fluid Mechanics.* **107** 499–508
[6] Ball F K 1963 *J. Fluid Mech.* **17** 240–56
[7] Ball F K 1965 *J. Fluid Mech.* **22** 529–45
[8] Rogers C and An H 2010 *Studies in Appl. Math.* **125** 275–99
[9] Chesnokov A 2011 *Appl. Math. and Mech.* **75** 350–6