Synchronization of traveling waves in a dispersive system of weakly coupled equations

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Abstract. The system of weakly coupled differential equations describing traveling waves in dispersive media is considered. The Lyapunov — Schmidt construction is used to study the branching of cnoidal-type periodic solutions. The analysis of bifurcation equations uses the group symmetry and cosymmetry of original equations. Sufficient condition for existence of the phase-shifted modes of cnoidal waves is formulated in terms of the Pontryagin's function determined by the nonlinear perturbation terms.

1. Introduction

We study periodic solutions for weakly coupled nonlinear differential equations describing traveling waves in the multi-modal dispersive systems. For a coupled model, one often observes the phase shift arising among the modes of non-perturbed systems. In this context, the synchronization means the existence of phase-shifted solutions which can be constructed from decoupled modes by an appropriate perturbation procedure. Using the Lyapunov — Schmidt construction, we reduce the original problem to an equivalent implicit system of bifurcation equations. The asymptotic analysis of these equations results in the sufficient condition of synchronization which can be formulated via coupling nonlinear terms. In fact, this analytical condition reduces generic problem to the search for a simple root of a special Pontryagin type function which depends on the unknown phase shift c. We refer to the original paper [1] where this construction was suggested by the analysis of periodic solutions of planar systems close to the Hamiltonian ones. Bruno [2] considered the case of multiple roots of Pontryagin's function, and Malkin [3] extended the result known for periodic solutions of planar systems to the class of Lyapunov systems with n degrees of freedom. Similar existence condition involving Pontryagin's function was obtained in [4, 5] for solitary-wave solutions of the bimodal KdV-type systems. In present work, we use this bifurcation technique by the study of synchronized cnoidal wave solutions.

2. Statement of the problem

Let us consider the system of ordinary differential equations

$$\frac{d^2u}{dx^2} = \Phi_u(u, v, \varepsilon), \quad \frac{d^2v}{dx^2} = \Phi_v(u, v, \varepsilon),$$

(1)

where $u(x)$, $v(x)$ are unknown scalar functions, ε is small parameter and the potential function Φ has the form

$$\Phi(u, v, \varepsilon) = \frac{1}{2}\left(u^2 + v^2 - u^3 - v^3\right) + \varepsilon\Omega(u, v, \varepsilon).$$

For a simplicity, the perturbation term Ω is assumed to be analytic, and the following condition is also satisfied

$$\Omega(0, 0, \varepsilon) = \Omega_u(0, 0, \varepsilon) = \Omega_v(0, 0, \varepsilon) = 0.$$

Equations (1) arise for example by describing traveling waves in the system of two weakly coupled Korteweg — de Vries equations [5, 6]. For $\varepsilon = 0$, decomposed system has a solution in the form of a cnoidal wave

$$u_0(x) = \alpha_2 + (\alpha_3 - \alpha_2)\,cn^2(rx; \text{æ}), \quad v_0(x) = u_0(x + c), \tag{2}$$

where the parameters r and æ are defined as

$$r = \frac{\sqrt{\alpha_3 - \alpha_1}}{2}, \quad \text{æ}^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1},$$

and parameters $\alpha_3 > \alpha_2 > \alpha_1$ are simple roots of the cubic polynomial $-u^3 + u^2 + 2h = 0$ with given constant h. The phase shift c is arbitrary for decoupled system at leading order in ε due to the translation invariance of the model equations. The problem is to find the values of c providing bifurcation of the solution by the perturbation in ε. It is well-known that the function u_0 formulated above approaches to the solitary wave solution $u_0(x) = \alpha_1 + (\alpha_3 - \alpha_1)\,ch^{-2}(rx)$ when $\alpha_2 \to \alpha_1$. This case was studied in [4, 5].

3. Bifurcation of harmonic solutions
The problem under consideration is closely related to a problem of the perturbation of a limit cycle to the dynamical systems

$$\frac{du}{dx} = -H_v(u, v) + \varepsilon p(u, v, \varepsilon), \quad \frac{dv}{dx} = H_u(u, v) + \varepsilon q(u, v, \varepsilon) \tag{3}$$

being close to the Hamiltonian systems which were first studied in a two-dimensional case by Pontryagin [1]. Specifically, it was proved the following

Theorem 3.1 *Let C_0 be a closed curve whose points satisfy the equation $H(u, v) = h_0$ and functions H_u and H_v are not equal to zero simultaneously. Then there exist a closed curve C_h near C_0 whose points satisfy the equation $H(u, v) = h$ when $|h - h_0|$ is sufficiently small. Let*

$$\psi(h) = \iint\limits_{D} (p_{\varepsilon u} + q_{\varepsilon v})\,du dv, \tag{4}$$

where D is the region inside a curve C_h. If $\psi'(h_0) = e \neq 0$ and $\psi(h_0) = 0$ then there exist a unique limit cycle of (2) which depends continuosly on ε and it's characteristic number has the same sign as εe. Moreover $C_h \to C_0$ when $\varepsilon \to 0$.

At first step, we demonstrate how the theorem 3.1 can be extended to some simple class of multi-dimensional systems of non-linear oscillators having rationally independent basic frequencies. Let us consider the autonomous system of differential equations

$$\frac{dw}{dx} = \mathcal{A}w + f(w; \varepsilon) \tag{5}$$

where $w = (w_1, \ldots, w_n) \in \mathbf{R}^n$ is an unknown vector function with even number of components $n = 2s$, and the matrix $\mathcal{A}$ is the block-diagonal matrix

$$\mathcal{A} = \begin{pmatrix} \omega_1 J & & & \\ & \omega_2 J & & \\ & & \ddots & \\ & & & \omega_s J \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The frequencies $\omega_1, \ldots, \omega_s$ are assumed to be rationally independent, and the analytic vector function $f(w; \varepsilon) = (f_1, f_2, ..., f_n)$ is assumed to satisfy the condition $f(w, 0) = 0$. Let us note that the system (5) is linear at $\varepsilon = 0$, and it has in this case the Hamiltonian

$$H(w) = \frac{1}{2} \sum_{j=1}^{s} \omega_j (w_{2j-1}^2 + w_{2j}^2).$$

Therefore, the periodic 1-mode solution of (5) should have the form of harmonically oscillate solution

$$w_0(x) = \varrho_0 \left(\zeta e^{i\omega_1 x} + \overline{\zeta} e^{-i\omega_1 x} \right) \tag{6}$$

as $\varepsilon \to 0$. Here $\zeta = (1, i, 0, ..., 0)$ is the complex eigenvector of the real-valued matrix $\mathcal{A}$ corresponding to its eigenvalue $i\omega_1$, bar denotes complex conjugate and the real-valued amplitude parameter ϱ_0 depends on the constant level $H(w_0) = h_0$. Now we seek the solution w of a nonlinear system (5) to be periodic in x with an unknown period $T(\varepsilon) = 2\pi/\omega(\varepsilon)$ where the frequency $\omega(\varepsilon)$ is analytic in ε and $\omega(0) = \omega_1$. Changing independent variable $x = \omega(\varepsilon)\tau$, we consider an unknown solution in the form

$$w(\tau; \varepsilon, \omega, \varrho_0) = w_0(\tau; \varrho_0) + \varepsilon\theta(\tau; \varepsilon, \omega, \varrho_0).$$

By that way, the problem reduces to the finding 2π-periodic solution θ of equivalent operator equation

$$A\theta = R(\theta; \omega, \varepsilon) \tag{7}$$

where A is the linear differential operator acting by the formula

$$A\theta = \frac{d\theta}{d\tau} - \mathcal{C}\theta, \quad \mathcal{C} = \omega_1^{-1}\mathcal{A}, \tag{8}$$

and nonlinear mapping R is defined as follows:

$$R(\theta; \varepsilon, \omega) = \frac{1}{\varepsilon\omega_1} \left\{ (\omega_1 - \omega)\frac{dw_0}{d\tau} + \varepsilon(\omega_1 - \omega)\frac{d\theta}{d\tau} + f(w_0 + \varepsilon\theta; \varepsilon) \right\}. \tag{9}$$

Let E and F be a Banach spaces of 2π-periodic real-valued functions $\theta(\tau)$ being smooth and continuous respectively, which can be presented by a uniformly convergent Fourier series

$$\theta(\tau) = \sum_{m=-\infty}^{+\infty} \theta^{(m)} e^{im\tau}, \quad \theta^{(m)} \in \mathbf{C}^n : \theta^{(-m)} = \bar{\theta}^{(m)}$$

having finite norms

$$\|\theta\|_E = \sum_{m=-\infty}^{\infty} (1 + |m|)|\theta^{(m)}|, \qquad \|\theta\|_F = \sum_{m=-\infty}^{\infty} |\theta^{(m)}|.$$

Lemma 3.2 *Let the frequencies $\omega_1, ..., \omega_s$ satisfy the inequalities $\left|\omega_1^{-1}\omega_j\right| < 2$ $(j = 2, \ldots, s)$. Then the non-homogeneous equation $Aw = g$ with a given vector function $g(\tau) \in F$ has a 2π-periodic solution $w(\tau) \in E$ if and only if the function g satisfies the orthogonality condition*

$$(g, \zeta e^{i\tau}) \overset{def}{=} \int_0^{2\pi} g(\tau) \cdot \zeta \, e^{i\tau} \, d\tau = 0 \tag{10}$$

where ζ is the complex eigenvector from (6) and the symbol $\cdot$ denotes Hermitian scalar product.

This lemma imply that operator $A : E \to F$ is the Fredholm operator. It has a two-dimensional null-space $Ker\,A$ generated by harmonic solutions of the form (6), so the Banach spaces E and F split into the direct sums $E = Ker\,A \oplus X$, $F = Y \oplus Im\,A$ where $X \subset E$ is closed infinite-dimensional subspace of E, and Y is the two-dimensional subspace of F. Thus, according to the Lyapunov — Schmidt construction [7]–[9], we can search for the small solution of the equation (7) in the form

$$\theta = \xi\varphi + \overline{\xi}\,\overline{\varphi} + \sigma, \qquad \xi \in \mathbf{C}, \ \ \varphi(\tau) = \zeta e^{i\tau}, \ \ \sigma \in X.$$

By that, the operator equation (7) reduces to equivalent bifurcation equation

$$QR(\xi\varphi + \overline{\xi}\,\overline{\varphi} + \sigma(\tau; \xi, \overline{\xi}, \varepsilon, \omega); \varepsilon, \omega) = 0 \tag{11}$$

where $Q : F \to Y$ is the projection associated with the orthogonality condition (10), and the nonlinear mapping $\sigma = \sigma(\tau; \xi, \overline{\xi}, \varepsilon, \omega)$ is specified by implicit equation

$$\sigma = \widetilde{A}^{-1}(I - Q)R(\xi\varphi + \overline{\xi}\,\overline{\varphi} + \sigma; \varepsilon, \omega) \tag{12}$$

where the isomorphism $\widetilde{A} : X \to Im\,A$ is generated by the restriction of the operator A. The system (11) at leading-order $\varepsilon = 0$ results in the pair of independent scalar relations

$$\varrho_0 \, \omega'(0) + \Gamma(\varrho_0) = 0, \qquad \Psi(\varrho_0) = 0 \tag{13}$$

with smooth functions Γ and Ψ. More precisely, the function Ψ has the form

$$\Psi(\varrho_0) = \int_0^{2\pi} \left\{ \cos\tau \, \frac{\partial f_1}{\partial \varepsilon}(w_0; 0) - \sin\tau \, \frac{\partial f_2}{\partial \varepsilon}(w_0; 0) \right\} \, d\tau \tag{14}$$

with the components f_1, f_2 of the perturbation term f from (5), this is analog of the Pontryagin's function (4) for the ODE system (5). Let us remark that nonlinear bifurcation equation (11) is completely degenerated as far as a direct application of the implicit function theorem is concerned. However, the presence of group symmetry allows to reduce the number of parameters. Specifically, the kernel of the linear operator A is invariant with respect to the representation T_γ of the translation group $\tau \mapsto \tau + \gamma$. This representation acts on the null space $Ker\,A$ in accordance with the formula $T_\gamma \varphi(\tau) = e^{i\gamma} \varphi(\tau)$. Therefore, T_γ induces the representation of a compact group $SO(2)$ acting in a complex parametric plane $\xi \in \mathbf{C}$ by the formula $S_\gamma \xi = e^{i\gamma}\xi$. Thus, we can use the reduction theorem which was proved in [10]. According to this theorem, all solutions of the equations (11) can be presented at $\varepsilon = 0$ in the form

$$\theta_0 = T_\gamma \left\{ |\xi|(\varphi + \overline{\varphi}) + \widetilde{A}^{-1}(R_0) \right\} \tag{15}$$

with appropriate $\gamma \in [0, 2\pi]$. Thus, we can fix here the phase shift by choosing $\gamma = 0$ without loss of generality. In this case, the system (11) simplifies to the system

$$\begin{cases} |\xi|\Psi'(\varrho_0) + \Pi_1(\omega'(0), \varrho_0) + \varepsilon\chi_1(|\xi|; \varepsilon, \varrho_0, \omega) = 0 \\ |\xi|\Pi_2(\varrho_0, \omega'(0)) - \varrho_0\omega''(0) + \Pi_3(\omega'(0), \varrho_0) + \varepsilon\chi_2(|\xi|; \varepsilon, \varrho_0, \omega) = 0 \end{cases} \tag{16}$$

where explicit form of smooth functions Π_j ($j = 1, 2, 3$) and χ_j ($j = 1, 2$) is not essential for analysis. Note that the parameters ϱ_0 and $\omega'(0)$ have already known from the leading-order equations (13). Thus, if the condition $\Psi'(\varrho_0) \neq 0$ holds and the parameter ε is sufficiently small then we can apply the implicit function theorem in order to determine the magnitude $|\xi|$ and the parameter $\omega''(0)$ from the equations (16). Finally, we obtain the following result.

Theorem 3.3 *If ϱ_0 is a simple root of the function Ψ from (14) and ε is sufficiently small, then there exists $2\pi/\omega(\varepsilon)$-periodic solution to (5) of the form $w = w_0 + \varepsilon\theta$ such that $\omega(\varepsilon) \to \omega_1$ as $\varepsilon \to 0$.*

4. Bifurcation of cnoidal wave solutions

Let us return to the original system (1). Similarly, we can seek the periodic solution

$$u(x; \omega, c, \varepsilon) = u_0(x) + \varepsilon\, u_1(x; \omega, c, \varepsilon), \quad v(x; \omega, c, \varepsilon) = u_0(x + c) + \varepsilon\, v_1(x; \omega, c, \varepsilon) \tag{17}$$

having unknown period $T(\varepsilon) = 2K(\ae)/r\omega(\varepsilon)$ such that $\omega(0) = 1$ where u_0 is the cnoidal-wave solution (2) and $K(\ae)$ is the complete elliptic integral of the first kind. In the same way, this problem can be reduced by changing independent variable $x = \omega(\varepsilon)\tau$ to the problem of finding $2K/r$-periodic solution $w = (u_1, v_1)$ to the equivalent operator equation

$$Aw = R(w; \omega, \varepsilon) \tag{18}$$

which has the linear part

$$Aw = \left(u_1'' + (3u_0 - 1)u_1,\ v_1'' + (3v_0 - 1)v_1 \right).$$

Accordingly, the nonlinear operator $R(w; \omega, \varepsilon) = (f_1(u_1, v_1; \omega, \varepsilon), f_2(u_1, v_1; \omega, \varepsilon))$ has the components

$$\begin{aligned} f_1 &= \varepsilon^{-1}(1 - \omega^2)(u_0'' + \varepsilon u_1'') - \frac{3}{2}\varepsilon u_1^2 + \Omega_u(u_0 + \varepsilon u_1, v_0 + \varepsilon v_1, \varepsilon) \\ f_2 &= \varepsilon^{-1}(1 - \omega^2)(v_0'' + \varepsilon v_1'') - \frac{3}{2}\varepsilon v_1^2 + \Omega_v(u_0 + \varepsilon u_1, v_0 + \varepsilon v_1, \varepsilon). \end{aligned} \tag{19}$$

Let $k \geq 1$ be an integer. Denote by $H_{2K/r}^k$ the class of $2K/r$-periodic function which belongs to the Sobolev space $W_2^k[0, 2K/r]$. Let us consider the linear differential equation

$$u'' + \left(3u_0(\tau + c) - 1\right)u = f(\tau) \tag{20}$$

with $2K/r$-periodic function f. Let G_c be the linear integral operator which is defined by following formula

$$\begin{aligned} (G_c f)(\tau) = &-u_0'(\tau + c) \int_0^{\tau+c} u_*(s)f(s - c)\, ds - u_*(\tau + c) \int_{\tau+c}^{2K/r} u_0'(s)f(s - c)\, ds - \\ &-\frac{r}{4KL} u_*(\tau + c) \int_{-2K/r}^{0} u_*(s)f(s - c)\, ds, \end{aligned} \tag{21}$$

where u_* is a non-periodic solution of homogeneous equation (20) with $c = 0$ which is given by the Liouville's formula

$$u_*(\tau) = u_0'(\tau) \int\limits_0^\tau \frac{ds}{u_0'^2(s)}$$

and the constant L is defined via complete elliptic integrals [11].

Lemma 4.1 *Let $f \in H_{2K/r}^k$ and following condition holds*

$$\int\limits_0^{2K/r} f(\tau)\, u_0'(\tau + c)\, d\tau = 0. \tag{22}$$

Then $G_c f \in H_{2K/r}^{k+2}$ and inequality $\|G_c f\|_{H_{2K/r}^{k+2}} \le C\|f\|_{H_{2K/r}^k}$ is valid.

According to this lemma the general solution of nonhomogeneous equation (20) is of the form

$$u(\tau) = \xi u_0'(\tau + c) + G_c f(\tau), \quad \xi \in \mathbf{R}.$$

Let us return to the equation (18). The linear operator $A : H_{2K/r}^{k+2} \times H_{2K/r}^{k+2} \to H_{2K/r}^k \times H_{2K/r}^k$ is Fredholm and it has a two-dimensional kernel with the basis $e_1 = (u_0', 0)$, $e_2 = (0, v_0')$. The solvability conditions of nonhomogeneous equation $Aw = (f_1, f_2)$ reduces to that of two scalar equations of the form (20). It is important to note that the kernel invariance of linearized operator is cruicial to the reduction of bifurcation equation for the system (5) considered above in the Section 3. In contrast, this property is not held for the original ODE system (1) even this system is also autonomuos. However, the group symmetry also plays a key role while the system (1) possesses the potential

$$l(w; \varepsilon) = \int\limits_0^{T(\varepsilon)} \left\{ \frac{1}{2} u'^2(x) + \frac{1}{2} v'^2(x) + \Phi(u(x), v(x), \varepsilon) \right\} dx$$

which is invariant with respect to the translation group $T_\gamma w = w(x + \gamma)$ where $w = (u, v)$. In this case, the following relation holds $\langle \nabla_w l(w, \varepsilon), Xw \rangle = 0$ where $X = \partial_x$ is the infinitesimal operator of the translation group, and $\langle \cdot, \cdot \rangle$ denotes a scalar product in $L_2[0, T(\varepsilon)] \times L_2[0, T(\varepsilon)]$. The operator X here is the cosymmetry operator for the system (1) in the sense of the paper [12]. More generally [4], if a potential operator has the invariant Lagrangian under action of the Lie group, then cosymmetry is given by the Lie algebra of infinitesimal generators of the symmetry group factorized with respect to the isotropy subgroup of unperturbed solution (u_0, v_0). Yudovich [12, 13] explained by the presence of cosymmetry the branching of solution families near the known non-cosymmetric solution w_0 $(Xw_0 \neq 0)$. Therefore, the reduction theorem proved in [4] reduces a two-dimensional system of bifurcation equations for the system (1) to single scalar equation having the form

$$\int\limits_0^{2K/r} \left\{ \varepsilon^{-1}(1 - \omega^2)(u_0'' + \varepsilon u_1'') - \frac{3}{2} \varepsilon u_1^2 + \Omega_u(u_0 + \varepsilon u_1, v_0 + \varepsilon v_1, \varepsilon) \right\} u_0'(\tau) d\tau = 0. \tag{23}$$

Here the implicit mapping $(\xi_1, \xi_2; c, \omega) \to (u_1, v_1)$ is specified at leading order $\varepsilon = 0$ by the formulae

$$u_1 = \xi_1 u_0' + G_0 f_1^0, \quad v_1 = \xi_2 v_0' + G_c f_2^0 \tag{24}$$

where the operator G_c acts by the formula (21), and the functions f_1^0, f_2^0 are defined by the formula (19) taken at the limit $\varepsilon = 0$. Taking into account these notations, we obtain the limit form of the equation (23) at $\varepsilon = 0$ which involves the Pontryagin's function as follows:

$$\widetilde{\Psi}(c) \overset{def}{=} \int\limits_0^{2K/r} \Omega_u(u_0(\tau), u_0(\tau + c), 0)\, u_0'(\tau) d\tau = 0.$$

Using this fact and substituting explicit formulae (24) into equation (23) we arrive at the final form of the bifurcation equation

$$\widetilde{\Psi}'(c)(\xi_1 - \xi_2) + \nu(c) + \varepsilon\Theta(\xi_1, \xi_2; \omega, c, \varepsilon) = 0, \tag{25}$$

where explicit form of smooth functions ν and Θ is not essential for analysis. It is clear that equation (25) uniquely determines the parameter ξ_2 by the implicit function theorem when the condition $\widetilde{\Psi}'(c) \neq 0$ holds. Let us remark that the similar sufficient condition was obtained in [4, 5] in the limit case of solitary-wave solutions. In addition, it turns out that the stability of these solutions considered in the context of the time evolution model can be also checked in terms of the Pontryagin's function. Specifically, the stability of the bifurcate solution depends on a sign of $\widetilde{\Psi}'(c)$ in accordance with [5].

Acknowledgments

References

[1] Pontryagin L S 1934 *Zh. Exp. Theor. Phys.* **4** 234–38
[2] Bruno A D 1990 *Math. Notes* **48** 1100–08
[3] Malkin I G 2004 *Methods of Lyapunov and Poincare in the theory of nonlinear oscillations* (Editorial: Moscow) (in Russian)
[4] Makarenko N I 1996 *Dokl. Math.* **53** 369–71
[5] Wright J D and Scheel A 2007 *Z. Angew. Math. Phys.* **58** 535–70
[6] Gear J A and Grimshaw R 1984 *Stud. Appl. Math.* **70** 235–58
[7] Vainberg M M and Trenogin V A 1974 *Theory of branching of solutions of non-linear equations* (Leyden: Noordhoff)
[8] Iooss G and Joseph D D 1980 *Elemenatry stability and bifurcation theory* (New York: Springer)
[9] Vanderbauwhede A 2009 *Encyclopedia of Complexity and Systems Science* (New York: Springer) 5299–315
[10] Loginov B V and Trenogin V A 1971 *Math. USSR Sb.* **14** 438–52
[11] Littman W 1957 *Comm. Pure Appl. Math.* **10** 241–70
[12] Yudovich V I 1991 *Math. Notes* **49** 540–45
[13] Kurakin L G and Yudovich V I 2000 *Sib. Math. J.* **41** 114–24