One-dimensional mixing layer model for a shear Hele-Shaw flow

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Abstract. A shear flow of a viscosity-stratified fluid in a Hele-Shaw cell is considered. The long-wave approximation is applied to the governing equations. To describe the evolution of the mixing layer, a special flow with a three-layered structure is considered. A one-dimensional model is derived by averaging the motion equations over the cell width, taking into account the flow structure. For a stationary flow, solutions of motion equations are constructed. The influence of viscosity on the mixing layer evolution is investigated by performing a numerical experiment for a flow with different viscosities in the layers and for a flow with always zero viscosity. It is shown that viscosity has a significant influence on the flow evolution.

1. Introduction

The process of interaction of two fluids with different viscosities often appears to be unstable and leads to formation of different structures. In the case of a more viscous fluid displacing a less viscous one, a so-called viscous fingering occurs [1]. The classical mathematical model which describes evolution of the Saffman-Taylor instability in the Hele-Shaw cell is described in the papers [2, 3]. However, the need to incorporate inertial forces leads to development of more complex non-linear models [4], which makes it possible to consider flows with different velocities in the layers, when describing the formation and evolution of instabilities. In its turn, in shear flows the evolution of instabilities results in formation of such structures as mixing layers. The instabilities occur either on the interface between the fluids or on the free surface, which is then known as a turbulent bore. The relation of the vertical to horizontal characteristic flow size is usually small for those flows, which makes it possible to describe the instability evolution in the framework of a two-dimensional shallow water model, and additionally to consider mixing effects between the layers [5, 6]. In [7] a model describing the horizontal-shear flow was proposed, which allowed to generalize the concept of subcritical and supercritical flow. Further development of this model resulted in a method, which also takes into account the turbulent mixing effect and the mass transfer between the layers and is based on the multi-layered fluid theory. This method was successfully applied in works [8, 9] for describing evolution of a subsurface turbulent layer in plane-parallel flows. Using this method one can determine the size of the mixing area staying in the framework of a one-dimensional model, which simplifies the initially two-dimensional calculations drastically. The aim of this work is to derive equations for a three-layered shear flow in the Hele-Shaw cell under the assumption that the viscosity varies in the flow and to study the evolution of the mixing layer. The work is based on a model of a flow averaged through the gap

with quadratic velocity and in the framework of the shallow water theory [9, 10]. The stationary solutions of the governing equations are sought and analyzed. It is shown that viscosity has a significant influence on the flow evolution.

2. Mathematical model

Gap-averaged shear flows of a weakly-compressible fluid in the Hele-shaw cell are described by the following system of equations [9, 10]:

$$\rho_t + (u\rho)_x + (v\rho)_y = 0,$$
$$(\rho u)_t + (\beta\rho u^2 + p)_x + (\beta\rho uv)_y = -\mu u,$$
$$(\rho v)_t + (\beta\rho uv)_x + (\beta\rho v^2 + p)_y = -\mu v,$$
$$(c\rho)_t + (uc\rho)_x + (vc\rho)_y = 0.$$

$$(1)$$

Here $\rho(t, x, y)$ is the density, $u(t, x, y)$ and $v(t, x, y)$ are the horizontal velocity vector components, $p = p(t, x, y)$ is the pressure and $c = c(t, x, y)$ is the concentration, which is scaled such that if the flow divides into two layers with different properties, it is equal to unity in one of the fluid layers and zero in the other layer. The constant β comes from integration over the cell gap, and in general $\beta = 6/5$, but since it has a negligibly small effect on the flow evolution [4], in this work $\beta = 1$ is considered. The spatial configuration of the flow is shown in Figure 1. The cell gap, which stretches over the z-axis is considered to be small relative to the horizontal cell sizes. Motions of the fluid along the z-axis are restrained by solid plates.
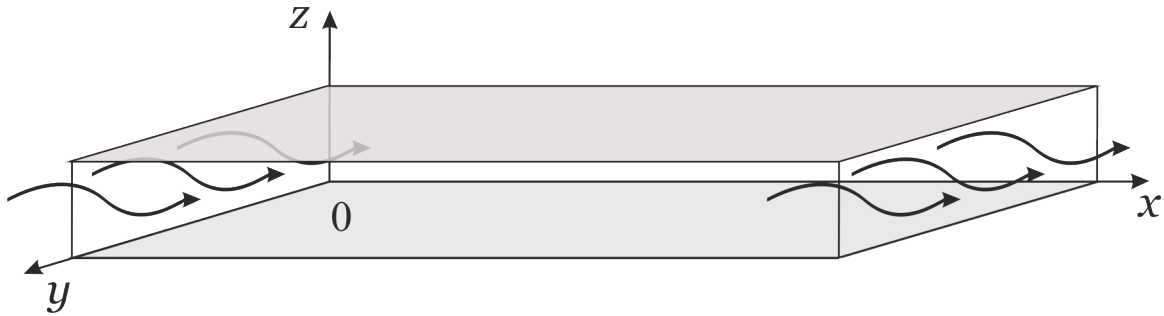
Figure 1. The spatial configuration of the flow

To perform the long-wave approximation in the motion equations (1), one has to assume that the flow is essentially parallel and apply the scaling

$$t \to \varepsilon^{-1}t, \quad x \to \varepsilon^{-1}x, \quad v \to \varepsilon v, \quad \mu \to \varepsilon\mu$$

and then, discard the terms of power two (ε^2) for being negligibly small, where $\varepsilon = L_y/L_x \ll 1$ is the relation of characteristic size of the cell over the y-axis to the characteristic size over the x-axis. As a result, the following system of equations is derived:

$$u_t + uu_x + vu_y + a\rho_x = -\mu u/\rho, \quad \rho_y = 0,$$
$$c_t + uc_x + vc_y = 0, \quad \rho_t + (u\rho)_x + (v\rho)_y = 0.$$

$$(2)$$

Moreover, a closing relation for the pressure is used to obtain the system (2). Namely, $p(\rho) = a\rho^2/2$, where $a^2 = c_0^2/\rho_0$, and the constants c_0 and ρ_0 determine the characteristic sound speed and density of the fluid.

In this work a flow between two solid plates parallel to the x-axis is considered. The plates are located on the levels $y = 0$ and $y = H$, and therefore the following additional impermeability conditions can be written:

$$v\big|_{y=0} = v\big|_{y=H} = 0.$$

In case when $\mu = 0$, a similar approach leads to the classical long-wave theory equations for an ideal fluid [12], if one excludes from consideration the equation for the concentration $c(t, x, y)$.

Furthermore, from the system (2), the conservation laws of momentum ρu and energy $E = (u^2 + a\rho)\rho/2$ can be derived:

$$
\begin{aligned}
&(\rho u)_t + (\rho u^2 + a\rho^2/2)_x + (\rho uv)_y = -\mu u, \\
&E_t + ((E + a\rho^2/2)u)_x + ((E + a\rho^2/2)v)_y = -\mu u^2.
\end{aligned}
\tag{3}
$$

3. Multi-layered model

Hereinafter the problem is considered in the framework of a special flow structure [8]. Let the fluid flow into the cell from the left side. The shear flow consists of two layers with different viscosities and velocities. Then at some point in space (which will be further considered to be position $x = 0$) a mixing layer starts to develop, as shown in Figure 2.
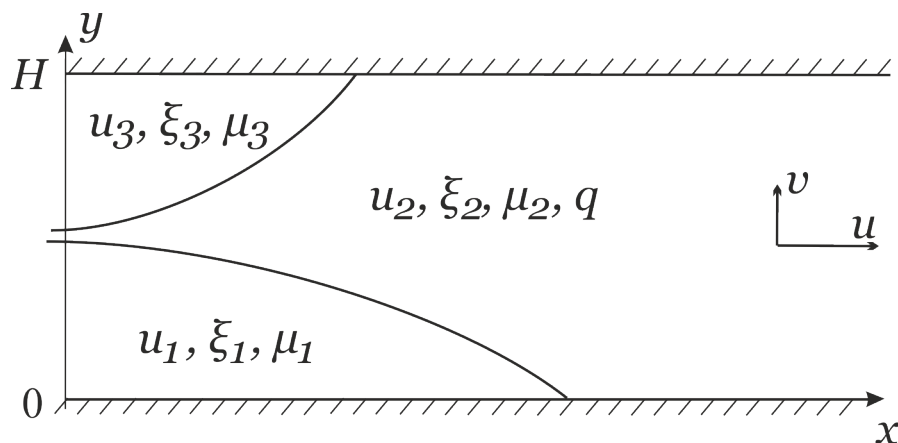
Figure 2. A three-layered flow in the Hele-Shaw cell

Hereinafter the index "1" corresponds to the lower flow layer, the index "2" corresponds to the mixing layer and the index "3" corresponds to the upper layer. The width of the i-th layer is ξ_i, velocity is u_i, and a constant viscosity μ_i is also given. The flow in the upper and lower layers is considered to be of the special class $u_y = 0$. Then the momentum equation in this layers can be written as:

$$u_{1t} + u_1 u_{1x} + a\rho_x = -\mu_1 u_1/\rho,$$

$$u_{3t} + u_3 u_{3x} + a\rho_x = -\mu_3 u_3/\rho.$$

Further on, the last equation of (2) and the equations (3) are averaged over the cell width

$y \in [0, H]$:

$$(\xi_1\rho)_t + (u_1\xi_1\rho)_x = -\sigma q\rho, \quad (\xi_2\rho)_t + (u_2\xi_2\rho)_x = 2\sigma q\rho, \quad (\xi_3\rho)_t + (u_3\xi_3\rho)_x = -\sigma q\rho,$$

$$Q_t + ((u_2^2 + q^2)\xi_2\rho + u_1^2\xi_1\rho + u_3^2\xi_3\rho + aH\rho^2/2)_x = -\mu_1\xi_1 u_1 - \mu_2\xi_2 u_2 - \mu_3\xi_3 u_3,$$

$$((u_1^2\xi_1 + (u_2^2 + q^2)\xi_2 + u_3^2\xi_3)\rho + aH\rho^2)_t + ((u_1^3\xi_1 + (u_2^2 + 3q^2)u_2\xi_2 + u_3^3\xi_3 + 2aQ)\rho)_x =$$

$$= -2(\mu_1\xi_1 u_1^2 + \mu_2\xi_2(u_2^2 + q^2) + \mu_3\xi_3 u_3^2) - \theta\rho q^3.$$

Here, the velocities in the mixing layer are described with the following quantities

$$u_2 = \frac{1}{\xi_2} \int_{\xi_1}^{H-\xi_3} u \, dy, \quad q^2 = \frac{1}{\xi_2} \int_{\xi_1}^{H-\xi_3} (u - u_2)^2 \, dy,$$

where the total flow rate is $Q = \rho(u_1\xi_1 + u_2\xi_2 + u_3\xi_3)$, and the empirical constants σ, θ define the mass transfer and the energy dissipation in the flow, respectively. From these averaged equations, one can derive, as consequence, the following one-dimensional model of the flow:

$$(\xi_1\rho)_t + (u_1\xi_1\rho)_x = -\sigma q\rho, \quad (\xi_2\rho)_t + (u_2\xi_2\rho)_x = 2\sigma q\rho, \quad (\xi_3\rho)_t + (u_3\xi_3\rho)_x = -\sigma q\rho,$$

$$u_{1t} + u_1 u_{1x} + a\rho_x = -\frac{\mu_1 u_1}{\rho}, \quad u_{3t} + u_3 u_{3x} + a\rho_x = -\frac{\mu_3 u_3}{\rho},$$

$$u_{2t} + u_2 u_{2x} + \frac{(q^2\xi_2\rho)_x}{\xi_2\rho} + a\rho_x = -\frac{\mu_2 u_2}{\rho} + \frac{\sigma q}{\xi_2}\left(u_1 - 2u_2 + u_3\right), \tag{4}$$

$$q_t + (u_2 q)_x = -\frac{\mu_2 q}{\rho} + \frac{\sigma}{2\xi_2}\left((u_1 - u_2)^2 + (u_3 - u_2)^2 - (2 + \frac{\theta}{\sigma})q^2\right).$$

In the process of flow evolution, the width of the upper or lower layer (for example, "3") can eventually become equal to zero. In this case, the flow becomes two-layered, and a similar approach leads to the model of a two-layered flow:

$$(\xi_1\rho)_t + (u_1\xi_1\rho)_x = -\sigma q\rho, \quad (\xi_2\rho)_t + (u_2\xi_2\rho)_x = \sigma q\rho,$$

$$u_{1t} + u_1 u_{1x} + a\rho_x = -\frac{\mu_1 u_1}{\rho},$$

$$u_{2t} + u_2 u_{2x} + \frac{(q^2\xi_2\rho)_x}{\xi_2\rho} + a\rho_x = -\frac{\mu_2 u_2}{\rho} + \frac{\sigma q}{\xi_2}(u_1 - u_2), \tag{5}$$

$$q_t + (u_2 q)_x = -\frac{\mu_2 q}{\rho} + \frac{\sigma}{2\xi_2}\left((u_1 - u_2)^2 - (1 + \frac{\theta}{\sigma})q^2\right).$$

When both the upper and the lower layers vanish (their width becomes equal to zero), the model (4) is rewritten in a form which is close to equations describing the plane-parallel flows [13]:

$$\rho_t + (u_2\rho)_x = 0, \quad (u_2\rho)_t + ((u_2^2 + q^2)\rho + a\rho^2/2)_x = -\mu u_2,$$

$$((u_2^2 + q^2)\rho + a\rho^2)_t + ((u_2^2 + 3q^2)u_2\rho + 2au_2\rho^2)_x = -2\mu_2(u_2^2 + q^2) - \theta h q^3. \tag{6}$$

For the sake of brevity, the characteristics of the two-layered model (5) are sought. The characteristics of the three- and one-layered models can be found using a similar approach. To do this, the model (5) is rewritten in the matrix form

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{F},$$

where $\mathbf{U} = (\rho, \xi_2, u1, u2, q)^{\mathrm{T}}$ is a vector of the unknown quantities,

$$\mathbf{F} = \left(0, \sigma q, -\frac{\mu_1 u_1}{\rho}, -\frac{\mu_2 u_2}{\rho} + \frac{\sigma q}{\xi_2}(u_1 - u_2), -\frac{\mu_2 q}{\rho} + \frac{\sigma}{2\xi_2}\left((u_1 - u_2)^2 - (1 + \frac{\theta}{\sigma})q^2\right)\right)^{\mathrm{T}}$$

is the right part of the equation, and the matrix $\mathbf{A}$ ca be written as follows:

$$\mathbf{A} = \begin{pmatrix} (\xi_1 u_1 + \xi_2 u_2)/H & \rho(u_2 - u_1)/H & \xi_1\rho/H & \xi_2\rho/H & 0 \\ \xi_1\xi_2(u_2 - u_1)/(H\rho) & (\xi_1 u_2 + \xi_2 u_1)/H & -\xi_1\xi_2/H & \xi_1\xi_2/H & 0 \\ a & 0 & u_1 & 0 & 0 \\ a + q^2/\rho & q^2/\xi_2 & 0 & u_2 & 2q \\ 0 & 0 & 0 & q & u_2 \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are sought from the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{E}) = (u_2 - \lambda)((u_2 - \lambda)^2 - 3q^2)\left((u_1 - \lambda)^2 - \frac{a\xi_1\rho}{H}\right) - \frac{a\xi_2\rho}{H}(u_1 - \lambda)^2 = 0.$$

It is obvious that this equation has a root $\lambda = u_2$ which corresponds to the characteristic $dx/dt = u_2$. It is also clearly seen then when the flow is close to the one-layered case, all other characteristics also become real.

4. Stationary solutions

Traveling waves can be investigated in a coordinate system moving with the same speed as the waves do. Therefore, to describe the mixing layer structure, stationary flow should be considered. Then the system (4) is rewritten as:

$$(u_1\xi_1\rho)_x = -\sigma q\rho, \quad (u_2\xi_2\rho)_x = 2\sigma q\rho, \quad (u_3\xi_3\rho)_x = -\sigma q\rho,$$

$$u_1 u_{1x} + a\rho_x = -\frac{\mu_1 u_1}{\rho}, \quad u_3 u_{3x} + a\rho_x = -\frac{\mu_3 u_3}{\rho},$$

$$u_2 u_{2x} + \frac{(q^2\xi_2\rho)_x}{\xi_2\rho} + a\rho_x = -\frac{\mu_2 u_2}{\rho} + \frac{\sigma q}{\xi_2}\left(u_1 - 2u_2 + u_3\right),$$ (7)

$$(u_2 q)_x = -\frac{\mu_2 q}{\rho} + \frac{\sigma}{2\xi_2}\left((u_1 - u_2)^2 + (u_3 - u_2)^2 - (2 + \frac{\theta}{\sigma})q^2\right),$$

and can be solved with respect to the derivative terms, which yields a system of ODEs:

$$\rho' = \frac{G}{\Delta}, \quad u_1' = -\frac{\mu_1}{\rho} - \frac{a}{u_1}\rho', \quad u_3' = -\frac{\mu_3}{\rho} - \frac{a}{u_3}\rho',$$

$$\xi_1' = -\frac{\sigma q}{u_1} - \frac{\xi_1}{u_1}u_1' - \frac{\xi_1}{\rho}\rho', \quad \xi_{3x} = -\frac{\sigma q}{u_3} - \frac{\xi_3}{u_3}u_3' - \frac{\xi_3}{\rho}\rho',$$

$$u_2' = \frac{2\sigma q}{\xi_2} - \frac{u_2}{\xi_2\rho}\left(\xi_2\rho' - \rho\xi_1' - \rho\xi_3'\right), \quad \xi_2' = \frac{2\sigma q}{u_2} - \frac{\xi_2}{u_2}u_2' - \frac{\xi_2}{\rho}\rho',$$ (8)

$$q' = -\frac{\mu_2 q}{\rho u_2} - \frac{q}{u_2}u_2' + \frac{\sigma}{2\xi_2 u_2}\left((u_1 - u_2)^2 + (u_3 - u_2)^2 - (2 + \frac{\theta}{\sigma})q^2\right),$$

where

$$G = \frac{\mu_2\xi_2(u_2^2 - 2q^2)}{u_2(u_2^2 - 3q^2)} + \frac{\mu_1\xi_1}{u_1} + \frac{\mu_3\xi_3}{u_3} - \frac{\sigma q\rho}{u_1} - \frac{\sigma q\rho}{u_3} +$$

$$+ \frac{\sigma q\rho((u_1 - u_2)^2 + (u_3 - u_2)^2 + u_2(u_1 - 4u_2 + u_3) - (6 + \frac{\theta}{\sigma})q^2)}{u_2(u_2^2 - 3q^2)},$$

$$\Delta = H - \frac{a\xi_1\rho}{u_1^2} - \frac{a\xi_3\rho}{u_3^2} - \frac{a\xi_2\rho}{u_2^2 - 3q^2}.$$

The sign of the discriminant Δ indicates the flow type: subcritical when $\Delta < 0$, and supercritical when $\Delta > 0$.

The two- and singe-layered models (5) and (6) can also be rewritten in the form of systems of ODEs.

To calculate the initial values of the velocities, it is assumed that $\xi_2|_{x=0} \to 0$, and that the limits of u_2 and q exist. Then from the equations (7), the following relations can be derived:

$$
\begin{aligned}
q_0^2 &= \frac{1}{2}u_{10}u_{20} - u_{20}^2 + \frac{1}{2}u_{20}u_{30} \\
u_{20} &= \frac{(6+\theta/\sigma)(u_{10}+u_{30})}{4(4+\theta/\sigma)}\left(1 \pm \sqrt{1 - 16\frac{(4+\theta/\sigma)(u_{10}^2+u_{30}^2)}{(6+\theta/\sigma)^2(u_{10}+u_{30})^2}}\right),
\end{aligned}
\tag{9}
$$

where the index "0" denotes the values of the functions at $x = 0$. That root, which provides the relation $u_{10} \le u_{20} \le u_{30}$, is considered for the value u_{20}. This is necessary for the problem to be physically reasonable.

5. Numerical results

To investigate the viscous terms effects on the mixing layer evolution, a stationary flow model (8) is considered. This model can reduce to its two- or singe-layered version in the process of flow evolution.

The numerical experiment is performed by using a simple Euler method on a mesh with resolution $N = 5000$. The computational domain is $x \in [0, 12]$, the width of the cell $h = 1$, viscosity in the mixing layer is given by the formula $\mu_2 = \sqrt{\mu_1\mu_3}$. The initial conditions are: $u_1 = 0.3$, $u_3 = 0.75$, $\xi_1 = 0.599$, $\xi_3 = 0.399$, $\theta = 0.6$, $a = 1500$, the initial velocities u_2 and q are given by the formula (9). The parameter $\theta = 0.6$, and the parameter σ varies from 0.15 to 0.2.
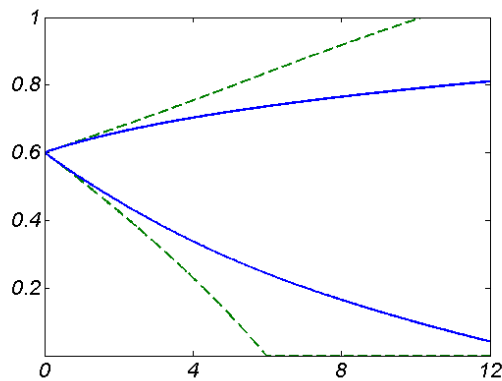
Figure 3. The mixing layer for the subcritical flow with viscosity (solid) and without viscosity (dotted)
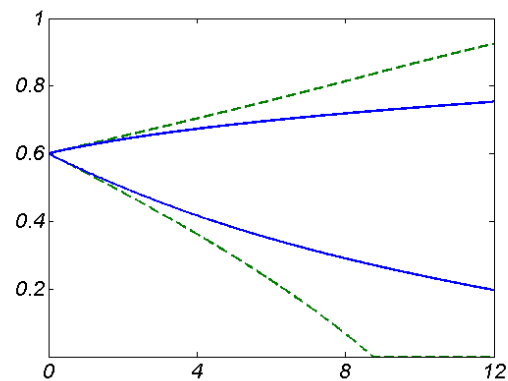
Figure 4. The mixing layer for the supercritical flow with viscosity (solid) and without viscosity (dotted)

The calculations for the subcritical flow are shown in Figure 3, and for the supercritical flow in Figure 4. The dotted lines show the modeling in assumption of a fluid with viscosity $\mu_1 = \mu_3 = 0$ and the solid lines show the modeling of a flow with different viscosities in the layers $\mu_1 = 0.5$ and $\mu_3 = 0.2$ The figures in both subcritical and supercritical case show that the viscous terms significantly slow the process of evolution of the mixing layer.

6. Conclusion

In the current work, a one-dimensional three-layered model of a shear flow in the Hele-Shaw cell is derived in the terms of long-wave approximation. Different viscosities of the fluid in the layers are assumed. A solution for the stationary flow was proposed and a numerical modeling was performed. The numerical results indicate that viscosity slows the process of evolution of the mixing layer significantly. Further work will be devoted to comparing the numerical results to the non-stationary flow solution and to the two-dimensional modeling of equations (1).

Acknowledgments

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