On solvability of inverse coefficient problems for nonlinear convection–diffusion–reaction equation

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Abstract. An inverse coefficient problem is considered for a stationary nonlinear convection–diffusion–reaction equation, in which reaction coefficient has a rather common dependence on substance concentration and on spacial variable. The solvability of the considered nonlinear boundary value problem is proved in a general case. The existence of solutions of the inverse problem is proved for the reaction coefficients, which are defined by the product of two functions. The first function depends on a spatial variable, the second one depends nonlinearly on the solution of the boundary value problem. The mentioned inverse problem consists in reconstructing the first function with the help of additional information provided by the solution of the boundary value problem.

1. Introduction. Solvability of the boundary value problem

Inverse problems research for linear and nonlinear heat-and-mass transfer models has been an urgent issue during a long period of time. One of the main problems is an identification problem of unknown densities of boundary and distributed sources or of models differential equations coefficients or boundary conditions with the help of additional information about systems conditions, which is described by the model. The studying of such inverse problems can be reduced to the studying of corresponding extremal problems using the optimization method [1]. A number of papers is dedicated to the description of this method for heat-and-mass transfer and other hydrodynamic models [2]–[10].

In this paper an inverse problem is studied in the case when the recovering reaction coefficient depends nonlinearly on both the boundary value problems solution and the spatial variable. The last one generalizes the statements of inverse coefficient problems from [5]–[9], where coefficients are searched in case of spatial variable dependence, and also of identification problem studied in [11, 12], where reaction coefficients depending only on solution were examined and at the same time the right part of the equation was recovered.

In a bounded domain $\Omega \subset \mathbf{R}^3$ with the boundary Γ the following boundary value problem is considered

$$-\lambda\Delta\varphi + \mathbf{u}\cdot\nabla\varphi + \tilde{k}(\varphi,\mathbf{x})\varphi = f \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Gamma. \tag{1}$$

Here function φ means polluting substance concentration, $\mathbf{u}$ is a given vector of velocity, f is a volume density of external sources of substance, λ is a constant diffusion coefficient, function $\tilde{k}(\varphi,\mathbf{x})$, where $\mathbf{x}\in\Omega$, is a reaction coefficient. This problem (1) will be called problem 1 below.

In this paper firstly a global solvability and local uniqueness of the problem 1 solution is proved, when the reaction coefficient depends on either substance concentration or spatial variables and functions belonging to a rather wide class. Then an identification problem is formulated for the reaction coefficients in the form of $\tilde{k} = \beta(\mathbf{x})k(\varphi)$, which consists of function $\beta(\mathbf{x})$ recovered using the measured substance concentration in a subdomain $Q \subset \Omega$. At that, the conditions on functions $\beta(\mathbf{x})$ and $k(\varphi)$ are also rather general. It should be mentioned that the multiplicative structure of the reaction coefficient gives an opportunity to take into account a rather wide-spread dependence of a reaction velocity on a concentration of a reacting substance from the one side, and from the other one allows us to take into consideration the effect of different types of chemical reaction behavior in the considered domain.

While studying problem 1 and optimal control problems, Sobolev spaces will be used: $H^s(D)$, $s \in \mathbf{R}$ $L^r(D)$, $1 \le r \le \infty$, where D is either a domain Ω or its boundary Γ. Inner products in $L^2(\Omega)$ and $H^1(\Omega)$ are denoted by $(\cdot,\cdot)$ $(\cdot,\cdot)_1$, inner products in $L^2(\Gamma)$ is denoted by $(\cdot,\cdot)_\Gamma$, norm in $L^2(\Omega)$ is denoted by $\|\cdot\|_\Omega$, norm or semi-norm in $H^1(\Omega)$ is denoted by $\|\cdot\|_{1,\Omega}$ or $|\cdot|_{1,\Omega}$. Also let $\mathbf{Z} = \{\mathbf{v} \in \mathbf{L}^4(\Omega) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}$.

It will be assumed that the following conditions hold:

(i) Ω is a bounded domain in $\mathbf{R}^3$ with boundary $\Gamma \in C^{0,1}$;

(ii) $\mathbf{u} \in \mathbf{Z}$, $f \in L^2(\Omega)$;

In the papers [11, 12] the reaction coefficient $k(\varphi)$ was considered in case, when it was not a function of φ in ordinary sense, as in [13], but was denoted as an operator $k(\varphi) : H^1(\Omega) \to L^p_+(\Omega)$, where $p \ge 3/2$, such as for any $w_1, w_2 \in B_r = \{w \in H^1(\Omega) : \|w\|_{1,\Omega} \le r\}$ an estimate takes place:

$$\|k(w_1) - k(w_2)\|_{L^p(\Omega)} \le L\|w_1 - w_2\|_{L^4(\Omega)}, \tag{2}$$

were L is a constant, depending on r, but not depending on w_1, w_2.

Let us mention that power functions $k(w) = w^2$ and $k(w) = w^2|w|$ also satisfy the inequality (2). Indeed, for the function $k(w) = w^2|w|$ from [14, 15] the estimate is true:

$$\|k(w_1) - k(w_2)\|^{3/2}_{L^{3/2}(\Omega)} = \int_\Omega (w_1^2|w_1| - w_2^2|w_2|)^{3/2} d\Omega \le \int_\Omega |w_1 - w_2|^{3/2}(w_1^2 + |w_1 w_2| + w_2^2)^{3/2} d\Omega \le$$

$$\le \|w_1 + w_2\|^3_{L^6(\Omega)}\|w_1 - w_2\|^{3/2}_{L^3(\Omega)} \le (2C_6 r)^3\|w_1 - w_2\|^{3/2}_{L^3(\Omega)} \le (2C_6 r)^3\|w_1 - w_2\|^{3/2}_{L^4(\Omega)},$$

which means that this function satisfy the inequality (2). For $k(w) = w^2$ the reasoning is the same.

Further let us make an example of the operator $k(\varphi)$, satisfying (2) and not being an ordinary function of φ: $k(\varphi) = \varphi^2$ in a subdomain $Q \subset \Omega$ and $k(\varphi) = k_0$ in $\Omega \setminus \overline{Q}$, where $k_0 \in L^{3/2}_+(\Omega)$.

In the present paper the reaction coefficient dependence on either problem 1 solution or the spatial variable $\mathbf{x} \in \Omega$ is emphasized. Let us suppose that it is true for the reaction coefficient $\tilde{k}(w, \mathbf{x})$:

(iii) for any $w_1, w_2 \in B_r = \{w \in H^1(\Omega) : \|w\|_{1,\Omega} \le r\}$ the estimate takes place:

$$\|k(w_1, \mathbf{x}) - k(w_2, \mathbf{x})\|_{L^p(\Omega)} \le L\|w_1 - w_2\|_{L^4(\Omega)}, \ \ \mathbf{x} \in \Omega,$$

where L is a constant, which depends on r, but does not depend on w_1, w_2.

Let us note that in the condition (iii) the constant L can depend not only on r. For example, the constant L also depends on $\|\beta_1(\mathbf{x})\|_{H^2(\Omega)}$ for the reaction coefficient $\tilde{k}(\varphi, \mathbf{x}) = \beta_1(\mathbf{x})k(\varphi) + \beta_2(\mathbf{x})$, where $\beta_1(\mathbf{x}) \in H^2_+(\Omega)$, $\beta_2(\mathbf{x}) \in L^2_+(\Omega)$, and the operator $k(\varphi)$ was denoted above.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable, finite functions in Ω, $H^1_0(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $H^{-1}(\Omega) = H^1_0(\Omega)^*$. The Poincare-Friedrichs inequality holds:

$|\varphi|^2_{1,\Omega} \geq \delta\|\varphi\|^2_{1,\Omega}$ $\forall\varphi \in H^1_0(\Omega)$, where $\delta = \text{const} > 0$, $L^p_+(\Omega) = \{k \in L^p(\Omega) : k \geq 0\}$, $p \geq 3/2$, $H^s_+(\Omega) = \{h \in H^s(\Omega) : h \geq 0\}$, $s \geq 0$.

Let us also remind that on the strength of Sobolev embedding theorem the space $H^1(\Omega)$ embeds in the space $L^s(\Omega)$ continuously at $s \leq 6$ and compactly at $s < 6$, and the estimate takes place with some constant C_s, depending on s and Ω

$$\|\varphi\|_{L^s(\Omega)} \leq C_s\|\varphi\|_{1,\Omega} \;\; \forall\varphi \in H^1(\Omega). \tag{3}$$

The following Green's theorem will be used (see [10, p. 128] for more details about it):

$$(\Delta\varphi, \eta) = -(\nabla\varphi, \nabla\eta) + \langle\partial\varphi/\partial n, \eta\rangle_\Gamma \quad \forall\varphi \in H^1(\Delta, \Omega),\ \eta \in H^1(\Omega), \tag{4}$$

$$(\mathbf{u} \cdot \nabla\varphi, \eta) = -(\mathbf{u} \cdot \nabla\eta, \varphi) \;\; \forall\mathbf{u} \in \mathbf{Z},\ \varphi, \eta \in H^1_0(\Omega). \tag{5}$$

Here and below $\langle\cdot, \cdot\rangle_\Gamma$ means the relation of duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

For any function $k_0 \in L^p_+(\Omega)$, $p > 1$ Holder's inequality for three functions holds:

$$|(k_0\varphi, \eta)| \equiv |\int_\Omega k_0\, \varphi\, \eta d\mathbf{x}| \leq \|k_0\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)}\|\eta\|_{L^r(\Omega)},\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,\ q > 1,\ r > 1. \tag{6}$$

Let us denote bilinear forms $a_\mathbf{u}$ and a_0: $H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbf{R}$ by formulae

$$a_\mathbf{u}(\varphi, \eta) = (\mathbf{u} \cdot \nabla\varphi, \eta) = \int_\Omega (\mathbf{u} \cdot \nabla\varphi)\eta d\mathbf{x},\ a_0(\varphi, \eta) = \lambda(\nabla\varphi, \nabla\eta) + a_\mathbf{u}(\varphi, \eta) + (k_0\varphi, \eta). \tag{7}$$

Using (3), (5), (6), it is not difficult to check that when conditions (i), (ii) and $k_0 \in L^p_+(\Omega), p \geq 3/2$ hold, then the forms, introduced in (7), are continuous and

$$|a_\mathbf{u}(\varphi, \eta)| \leq |(\mathbf{u} \cdot \nabla\varphi, \eta)| \leq \gamma_1\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \quad \forall(\varphi, \eta) \in H^1_0(\Omega)^2, \tag{8}$$

$$|(k_0\varphi, \eta)| \leq \gamma_p\|k_0\|_{L^p(\Omega}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \;\; \forall(\varphi, \eta) \in H^1_0(\Omega)^2, \tag{9}$$

$$a_\mathbf{u}(\varphi, \varphi) = 0,\ a_0(\varphi, \varphi) = \lambda(\nabla\varphi, \nabla\varphi) + (k_0\varphi, \varphi) \geq \lambda_*\|\varphi\|^2_{1,\Omega} \quad \forall\varphi \in H^1_0(\Omega),\ \lambda_* \equiv \delta\lambda. \tag{10}$$

Here γ_1 or γ_p are some constants, which depend on Ω or on Ω and p correspondingly. From (8)–(10) it follows that for any function $k_0 \in L^p_+(\Omega), p \geq 3/2$ the form $a_0(\cdot, \cdot)$ is continuous and coercitive with the constant λ_* on $H^1_0(\Omega)$. In its turn, this estimate results from (9) and (iii):

$$|((k(\varphi_1, \mathbf{x}) - k(\varphi_2, \mathbf{x}))\varphi, \eta)| \leq \gamma_p L\|\varphi_1 - \varphi_2\|_{L^4(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \forall\varphi_1, \varphi_2 \in B_r,\ \varphi, \eta \in H^1_0(\Omega). \tag{11}$$

Let us multiply the equation in (1) by $h \in H^1_0(\Omega)$ and integrate over Ω, applying the formula (4). Then this will be obtained:

$$\lambda(\nabla\varphi, \nabla h) + (k(\varphi, \mathbf{x})\varphi, h) + (\mathbf{u} \cdot \nabla\varphi, h) = (f, h) \;\; \forall h \in H^1_0(\Omega). \tag{12}$$

The function $\varphi \in H^1_0(\Omega)$, which satisfies (12), will be called a weak solution of problem 1.

The proof of problem (12) solvability will be conducted with the help of Schauder fixed point theorem according to the scheme, which is described in [10] for nonlinear hydrodynamic models. For this purpose let us define a mapping $G : H^1_0(\Omega) \to H^1_0(\Omega)$, which sticks with formula $G(w) = \varphi$ for $w \in H^1_0(\Omega)$. Here function $\varphi \in H^1_0(\Omega)$ is the solution of the linear problem

$$a_w(\varphi, h) = \lambda(\nabla\varphi, \nabla h) + (k(w, \mathbf{x})\varphi, h) + (\mathbf{u} \cdot \nabla\varphi, h) = (f, h) \;\; \forall h \in H^1_0(\Omega). \tag{13}$$

Since $k(w, \mathbf{x}) \in L^p_+(\Omega)$, $p \geq 3/2$ for any $w \in H^1(\Omega)$ on the strength of conditions (iii), then according to (10) the form $a_w : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbf{R}$, introduced in (13), is continuous and coercitive with the constant $\lambda_* = \delta\lambda$, and the problem (13) has a unique solution $\varphi \in H_0^1(\Omega)$, for which the estimate holds

$$\|\varphi\|_{1,\Omega} \leq M_\varphi \equiv C_* \|f\|_\Omega. \tag{14}$$

Let us inject in the space $H_0^1(\Omega)$ the sphere $B_r = \{w \in H_0^1(\Omega) : \|w\|_{1,\Omega} \leq r\}$, where $r = M_\varphi$. From the construction of the sphere B_r and from (14) follows that the operator G, which was defined above, is a mapping of B_r into itself. Let us prove that G is continuous and compact on B_r. Let $\{w_n\}_{n=1}^\infty$ be a random sequence from B_r. On the strength of reflexivity of the space $H_0^1(\Omega)$ and of embedding compactness $H_0^1(\Omega) \subset L^4(\Omega)$ there is a subsequence of the subsequence $\{w_n\}_{n=1}^\infty$, which can be again denoted by $\{w_n\}_{n=1}^\infty$, and there is also a function $w \in B_r$ such as $w_n \to w$ weakly in $H^1(\Omega)$, $w_n \to w$ strongly in $L^4(\Omega)$ at $n \to \infty$. Let $\varphi_n = G(w_n)$, $\varphi = G(w)$. These equalities mean that $\varphi \in H_0^1(\Omega)$ is the solution of problem (13) and $\varphi_n \in H_0^1(\Omega)$ is the solution of problem

$$\lambda(\nabla\varphi_n, \nabla h) + (k(w_n, \mathbf{x})\varphi_n, h) + (\mathbf{u} \cdot \nabla\varphi_n, h) = (f, h) \ \ \forall h \in H_0^1(\Omega), \tag{15}$$

which is obtained from (13) by substituting w for w_n, φ for φ_n. Let us demonstrate that $\varphi_n \to \varphi$ strongly in $H^1(\Omega)$ at $n \to \infty$. For that purpose let us subtract (13) from (15). Taking into consideration that $(k(w_n, \mathbf{x})\varphi_n, h) - (k(w, \mathbf{x})\varphi, h) = (k(w, \mathbf{x})(\varphi_n - \varphi), h) + ((k(w_n, \mathbf{x}) - k(w, \mathbf{x}))\varphi_n, h)$, the following will be got

$$\lambda(\nabla(\varphi_n - \varphi), \nabla h) + (k(w, \mathbf{x})(\varphi_n - \varphi), h) + (\mathbf{u} \cdot \nabla(\varphi_n - \varphi), h) =$$
$$= -((k(w_n, \mathbf{x}) - k(w, \mathbf{x}))\varphi_n, h) \ \ \forall h \in H_0^1(\Omega). \tag{16}$$

Using the estimate (11) at $\varphi_1 = w_n$, $\varphi_2 = w$, $\varphi = \varphi_n$ and the estimate $\|\varphi_n\|_{1,\Omega} \leq M_\varphi$ for $n = 1, 2...$, resulting from (14), it can be obtained that

$$|((k(w_n, \mathbf{x}) - k(w, \mathbf{x}))\varphi_n, h)| \leq \gamma_p L M_\varphi \|w_n - w\|_{L^4(\Omega)} \|h\|_{1,\Omega} \to 0 \ \ n \to \infty \ \forall h \in H_0^1(\Omega). \tag{17}$$

Then from (16) on the strength of (17) and (10) follows that $\|\varphi_n - \varphi\|_{1,\Omega} \to 0$ at $n \to \infty$. This implies continuity and compactness of the operator G. In such case it follows from the Schauder fixed point theorem that the operator G has a fixed point $\varphi = G(\varphi) \in H_0^1(\Omega)$. That is the point φ, which is the searched solution of problem (12).

Let us determine the sufficient conditions for the uniqueness of problem (12) solution. Let $\varphi_1, \varphi_2 \in H_0^1(\Omega)$ be two solutions of problem (12), for which the estimate (14) takes place. Obviously, the difference $\varphi = \varphi_1 - \varphi_2$ satisfies the relation

$$\lambda(\nabla\varphi, \nabla h) + (k(\varphi_2, \mathbf{x})\varphi, h) + (\mathbf{u} \cdot \nabla\varphi, h) = -((k(\varphi_1, \mathbf{x}) - k(\varphi_2, \mathbf{x}))\varphi_1, h) \ \ \forall h \in H_0^1(\Omega). \tag{18}$$

Reasoning like above, when the estimate (17) was obtained, it can be shown that

$$|((k(\varphi_1, \mathbf{x}) - k(\varphi_2, \mathbf{x}))\varphi_1, h)| \leq \gamma_p C_4 L M_\varphi \|\varphi_1 - \varphi_2\|_{1,\Omega} \|h\|_{1,\Omega}. \tag{19}$$

Here C_4 is a constant, which is included in (3) at $s = 4$. As on the strength of (iii) $k(\varphi_2, \mathbf{x}) \in L_+^p(\Omega)$, then it follows from (10) subject to (19) and from the equality $M_\varphi = C_* \|f\|_\Omega$ that $\|\varphi\|_{1,\Omega} \leq C_*^2 \gamma_p C_4 L \|f\|_\Omega \|\varphi\|_{1,\Omega}$. Let the condition hold

$$\gamma_p C_4 L \|f\|_\Omega < \lambda_*^2 = C_*^{-2}. \tag{20}$$

Then from the previous inequality it follows that $\|\varphi\|_{1,\Omega} = 0$, and, accordingly, $\varphi_1 = \varphi_2$. Let us formulate the received results as the following theorem.

Theorem 1. *Let conditions (i)–(iii) hold. Then there is a weak solution $\varphi \in H_0^1(\Omega)$ of the problem 1 and the estimate (14) is met. If, besides, the condition (20) is satisfied, then the problem 1 weak solution is unique.*

While studying extremal problems, it will be assumed that the function $\tilde{k}(\varphi, \mathbf{x})$ satisfies the condition

(iv) $\tilde{k}(\varphi, \mathbf{x}) = \beta(\mathbf{x})k(\varphi)$, where $\beta(\mathbf{x}) \in H_+^1(\Omega)$, while the function $k(\varphi) \in L_+^2(\Omega)$ and on any sphere $B_r = \{\varphi \in H^1(\Omega) : \|\varphi\|_{1,\Omega} \le r\}$ the inequality holds

$$\|k(\varphi_1) - k(\varphi_2)\|_\Omega \le L\|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \ \ \forall \varphi_1, \varphi_2 \in B_r. \tag{21}$$

Here L is a constant, depending on r, but not depending on $\varphi_1, \varphi_2 \in B_r$.

Conditions (iv) describe a particular type of function $\tilde{k}(\varphi, \mathbf{x})$, satisfying the conditions (iii). Indeed,

$$\|\beta(\mathbf{x})(k(\varphi_1) - k(\varphi_2))\|_{L^{3/2}(\Omega)} \le \|\beta(\mathbf{x})\|_{L^6(\Omega)}\|k(\varphi_1) - k(\varphi_2)\|_\Omega \le$$
$$\le L\|\beta(\mathbf{x})\|_{1,\Omega}\|\varphi_1 - \varphi_2\|_{L^4(\Omega)}.$$

It is clear that the function $\tilde{k}(\varphi, \mathbf{x}) = \beta(\mathbf{x})\varphi^2$ satisfies the condition (iv), because for any φ^2 the estimate (21) is true. It is interesting to mention that at $\tilde{k}(\varphi, \mathbf{x}) = \beta(\mathbf{x})\varphi^2$ there is a nonlocal (i.e. without smallness condition (20)) uniqueness of problem 1 solution. Actually, let $\varphi_1, \varphi_2 \in H_0^1(\Omega)$ be two solutions of the problem 1 at $\tilde{k}(\varphi, \mathbf{x}) = \beta(\mathbf{x})\varphi^2$. Then their difference $\varphi = \varphi_1 - \varphi_2$ satisfies the relation

$$\lambda(\nabla\varphi, \nabla h) + (\beta(\mathbf{x})(\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2), \varphi h) + (\mathbf{u} \cdot \nabla\varphi, h) = 0 \ \forall h \in H_0^1(\Omega).$$

As $\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2 \ge 0$ a.e. in Ω and $\beta(\mathbf{x}) \in H_+^1(\Omega)$, then it follows form (10) that $\varphi = 0$.

Let us note another possible kind of dependence $k(\varphi) = \varphi^2|\varphi|$, considered in [14, 15]. However, to deal with such coefficient, stricter conditions for the function $\beta(\mathbf{x})$ should be used, for example, $\beta(\mathbf{x}) \in H^2(\Omega)$.

2. Statement of the optimal control problem and its solvability

There are a lot of situations in practice, when some parameters of problem 1 are unknown and it is required to determine them together with the solution φ according to some information about the solution. On the capacity of the mentioned information about solution, the values $\varphi^d(\mathbf{x})$ of the concentration φ are often taken, which can be measured in points of some set $Q \subset \Omega$.

The case will be studied below, when the function $\beta(\mathbf{x})$ is unknown, should be searched together with the solution φ and is multiplicatively included in reaction coefficient $\tilde{k}(\varphi, \mathbf{x}) = \beta(\mathbf{x})k(\varphi)$, will be studied. To solve this inverse problem the optimization method will be used, according to which the considered problem is reduced to solving the control problem. In compliance with this method the whole set of initial data of problem 1 is divided into two groups: the group of fixed data, where functions $\mathbf{u}$, f and $k(\varphi)$ will be included, and the group of controls, which consists of function $\beta(\mathbf{x})$, assuming that it can be changed in some set K.

Let us introduce an operator $F : H_0^1(\Omega) \times K \to H^{-1}(\Omega)$ by formula

$$\langle F(\varphi, \beta), h\rangle = \lambda(\nabla\varphi, \nabla h) + (\beta(\mathbf{x})\, k(\varphi)\varphi, h) + (\mathbf{u} \cdot \nabla\varphi, h) - (f, h).$$

Then (12) can be rewritten in the following form:

$$F(\varphi, \beta) = 0. \tag{22}$$

Treating (22) as a conditional restriction on the state $\varphi \in H_0^1(\Omega)$ and on the control $\beta \in K$, the problem of conditional minimization can be formulated as follows:

$$J(\varphi, \beta) \equiv \frac{\mu_0}{2}I(\varphi) + \frac{\mu_1}{2}\|\beta\|_{1,\Omega}^2 \to \inf,\ F(\varphi, \beta) = 0,\ (\varphi, \beta) \in H_0^1(\Omega) \times K. \tag{23}$$

Here $I : H_0^1(\Omega) \to \mathbf{R}$ is a weakly semicontinuous below functional.

The set of possible pairs for problem (23) is denoted by $Z_{ad} = \{(\varphi, \beta) \in H_0^1(\Omega) \times K : F(\varphi, \beta) = 0, J(\varphi, \beta) < \infty\}$ and let us suppose that these conditions hold

(j) $K \subset H_+^1(\Omega)$ is a nonempty convex closed set;

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$ and K is a bounded set or $\mu_0 > 0$, $\mu_1 > 0$ and functional I is bounded below.

The following cost functionals can be used in the capacity of the possible ones [10]:

$$I_1(\varphi) = \|\varphi - \varphi^d\|_Q^2 = \int_\Omega |\varphi - \varphi^d|^2 d\mathbf{x}, \;\; I_2(\varphi) = \|\varphi - \varphi^d\|_{1,Q}^2. \tag{24}$$

Here $\varphi^d \in L^2(Q)$ (or $\varphi^d \in H^1(Q)$) is a given function in some subdomain $Q \subset \Omega$.

Theorem 2. *Let conditions (i), (ii), (iv) and (j), (jj) hold, $k(\varphi) \in L^3(\Omega)$, $I : H_0^1(\Omega) \to \mathbf{R}$ is a weakly semicontinuous below functional, and Z_{ad} is nonempty set. Then there is at least one solution $(\varphi, \beta) \in H_0^1(\Omega) \times K$ of optimal control problem (23).*

Proof. Let $(\varphi_m, \beta_m) \in Z_{ad}$ be a minimizing sequence for the functional J, for which the following is true:

$$\lim_{m\to\infty} J(\varphi_m, \beta_m) = \inf_{(\varphi,\beta)\in Z_{ad}} J(\varphi, \beta) \equiv J^*.$$

That and the conditions of theorem for the functional J from (23) imply the estimate $\|\beta_m\|_{1,\Omega} \leq c_1$. From theorem 1 follows directly that $\|\varphi_m\|_{1,\Omega} \leq c_2$, where constants c_1, c_2 do not depend on m.

Then the weak limits $\varphi^* \in H_0^1(\Omega)$ and $\beta^* \in K$ of some subsequences of the sequences $\{\varphi_m\}$ and $\{\beta_m\}$ exist. Corresponding sequences will be also denoted by $\{\varphi_m\}$ and $\{\beta_m\}$. With this in mind it can be considered that

$$\varphi_m \to \varphi^* \in H_0^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^s(\Omega),\; s < 6, \tag{25}$$

$$\beta_m \to \beta^* \in K \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^s(\Omega),\; s < 6. \tag{26}$$

Let us show that $\mathcal{F}(\varphi^*, \beta^*) = 0$, i.e.

$$\lambda(\nabla\varphi^*, \nabla h) + (\beta^* k(\varphi^*)\varphi^*, h) + (\mathbf{u} \cdot \nabla\varphi^*, h) = (f, h) \;\; \forall h \in H_0^1(\Omega). \tag{27}$$

And it should be taken into account that the pair (φ_m, β_m) satisfies the relations

$$\lambda(\nabla\varphi_m, \nabla h) + (\beta_m k(\varphi_m)\varphi_m, h) + (\mathbf{u} \cdot \nabla\varphi_m, h) = (f, h) \;\; \forall h \in H_0^1(\Omega),\; m = 1, 2, \ldots. \tag{28}$$

Let us pass to the limit in (28) at $m \to \infty$. All linear summands in (28) turn into corresponding ones in (27).

From condition (iv) follows that $k(\varphi_m) \to k(\varphi)$ strongly in $L^2(\Omega)$. Taking into account (26), it is not difficult to show that

$$\beta_m k(\varphi_m) \to \beta k(\varphi) \text{ strongly in } L^{3/2}(\Omega).$$

Really,

$$\int_\Omega (\beta_m k(\varphi_m) - \beta k(\varphi))^{3/2} d\Omega \leq \int_\Omega \tilde{C} k(\varphi)^{3/2} (\beta_m - \beta)^{3/2} d\Omega + \int_\Omega \tilde{C} \beta_m^{3/2} (k(\varphi_m) - k(\varphi))^{3/2} d\Omega \leq$$

$$\leq \tilde{C} \|k(\varphi)\|_{L^3(\Omega)}^{1/2} \|\beta_m - \beta\|_{L^3(\Omega)}^{1/2} + \tilde{C} \|\beta_m\|_{L^6(\Omega)}^{1/4} \|k(\varphi_m) - k(\varphi)\|_{L^2(\Omega)}^{3/4} \to 0 \text{ at } m \to \infty.$$

On the strength of (25) $\varphi_m h \to \varphi h$ weakly in $L^3(\Omega)$ at $m \to \infty$ for all $h \in H_0^1(\Omega)$. This allows to pass to the limit in the nonlinear summand in (28).

As the functional J is weakly semicontinuous below on $H_0^1(\Omega) \times H^1(\Omega)$, then from aforesaid follows that

$$J^* = \lim_{m\to\infty} J(\varphi_m, \beta_m) = \underline{\lim}_{m\to\infty} J(\varphi_m, \beta_m) \geq J(\varphi^*, \beta^*) \geq J^*. \;\blacksquare$$

In more general case, when $k(\varphi) \in L_+^2(\Omega)$, the theorem 2 should be proved further as in [12].

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