Conservation laws and symmetries of the shallow water system above rough bottom

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Abstract. The system of one-dimensional shallow water equations above the rough bottom is considered. All its hydrodynamic conservation laws are found, and a group classification is performed. A new conservation law additional to the two basic conservation laws is found. It is shown that the system of shallow water equations can be linearized by a point change of variables only in cases of constant and linear bottom profiles.

1. Introduction

In dimensionless variables, the system of one-dimensional shallow water equations above the rough bottom has the form [1]

$$
\begin{aligned}
&u_t + u u_x + \eta_x = 0\,,\\
&\eta_t + [(\eta + h)u]_x = 0\,.
\end{aligned}
\tag{1}
$$

Here $y = -h(x)$, $h(x) \geqslant 0$ is the bottom profile, $u = u(x,t)$ is the average speed on the horizontal depth, and $\eta = \eta(x,t)$ is the deviation of the free surface.

It was observed in [2] that the system of shallow water equations with the bottom profile $h(x) = -x$ can be linearized in characteristic variables. Later in [3], it was shown that in this case the original system (1) can be reduced to the following form via a point change of variables,

$$
\begin{aligned}
&U_T + N_X = 0\,,\\
&N_T - X U_X - U = 0\,,
\end{aligned}
\tag{2}
$$

this form can be seen as formally discarding the nonlinear terms in the system (1). Classes of exact solutions of linear system (2) are given in [4, 5].

In this paper, we find all hydrodynamic conservation laws for all bottom profiles, and solve the problem of group classification.

2. Conservation laws

We will seek hydrodynamic conservation laws of system (1) as a pair of functions $P(x,t,u,\eta)$, $Q(x,t,u,\eta)$ which satisfy the equation

$$
D_t(Q) + D_x(P) = 0
\tag{3}
$$

on solutions of the system (1). Here

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \eta_t \frac{\partial}{\partial \eta}, \qquad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \eta_x \frac{\partial}{\partial \eta}$$

are operators of the total derivative.

Comment 1. Note that both equations of the system (1) have the form (3). The corresponding conservation laws,

$$
\begin{aligned}
Q_1 &= u, & P_1 &= \frac{u^2}{2} + \eta, \\
Q_2 &= \eta, & P_2 &= (\eta + h)u,
\end{aligned}
\tag{4}
$$

have the meaning of the conservation laws of momentum and mass.

Comment 2. From (3), it follows that $Q' = Q + f_x$, $P' = P - f_t$ and $Q'' = CQ$, $P'' = CP$ are also conservation laws for arbitrary function $f = f(x,t)$ and constant C. Our classification is performed modulo this natural equivalence.

Using the expression (3) and the system of equations (1), one can obtain the following overdetermined system for the unknown function P and Q,

$$
\begin{aligned}
P_x &= uh'Q_\eta - Q_t, \\
P_u &= uQ_u + (\eta + h)Q_\eta, \\
P_\eta &= Q_u + uQ_\eta.
\end{aligned}
\tag{5}
$$

Eliminating P from equations (5) one can get the system for the function Q,

$$
\begin{aligned}
Q_{xu} &= uh'Q_{\eta\eta} - uQ_{x\eta} - Q_{t\eta}, \\
Q_{tu} &= -u^2 h'Q_{\eta\eta} + uh'Q_{u\eta} + \left(u^2 - \eta - h\right)Q_{x\eta} + uQ_{t\eta}, \\
Q_{uu} &= (\eta + h)Q_{\eta\eta}.
\end{aligned}
\tag{6}
$$

The system (6) can be tested for consistency. Here are the results of this test.

2.1. The case of an arbitrary function $h(x)$. In this case, the solution of system (5) has the form

$$
\begin{aligned}
P &= C_1(\eta + h)\left(u^3 + 2u\eta\right) + C_2\left(\frac{u^2}{2} + \eta\right) + C_3 u(\eta + h), \\
Q &= C_1\left[u^2(\eta + h) + \eta^2\right] + C_2 u + C_3 \eta.
\end{aligned}
\tag{7}
$$

Here, C_1, C_2, C_3 are arbitrary constants. In view of comments 2 and 2, we find a new conservation law from (7),

$$Q_3 = u^2(\eta + h) + \eta^2, \qquad P_3 = (\eta + h)(u^3 + 2u\eta),$$

supplementing the basic conservation laws (4).

2.2. The case of a linear function $h(x) = a_1 x + a_2$. In this case, the solution of the system (5) has the form

$$
\begin{aligned}
P = C_1 &\left(-a_1 xt\beta - x\alpha(\beta + a_2) + \frac{3}{2}a_1 t^2 \alpha(\beta + a_2) + \frac{1}{2}t\beta^2 + a_2\beta t + \right. \\
&\left. + \frac{1}{2}a_1^2 t^3 \beta + a_2 t\alpha^2 + t\alpha^2\beta \right) + a_1 tQ_1 + P_1, \\
Q = C_1 &\left(a_2 t\alpha + t\alpha\beta - x\beta + \frac{1}{2}a_1 t^2 \beta \right) + Q_1.
\end{aligned}
$$

Here, $\alpha = u - a_1 t$, $\beta = \eta + a_1 x$; C_1 is an arbitrary constant; $P_1 = P_1(\alpha, \beta)$, $Q_1 = Q_1(\alpha, \beta)$ is an arbitrary solution of the system

$$
\begin{aligned}
P_{1\alpha} &= \alpha Q_{1\alpha} + (\beta + a_2) Q_{1\beta}\,, \\
P_{1\beta} &= Q_{1\alpha} + \alpha Q_{1\beta}\,.
\end{aligned}
$$

2.3. The case $h(x) = \dfrac{b_1}{2} x^2 + b_2 x + b_3,\ b_1 > 0.$ In this case, the solution of the system (5) has the form

$$
\begin{aligned}
P =\ & C_1(\eta + h(x))\left(u^3 + 2u\eta\right) + C_2\left(\frac{u^2}{2} + \eta\right) + C_3 u(\eta + h(x)) + \\
& + C_4 e^{-\sqrt{b_1}t}\left[\left(\sqrt{b_1}u^2 + uh'(x)\right)(\eta + h(x)) + \frac{\sqrt{b_1}}{2}\eta(\eta + 2h(x))\right] + \\
& + C_5 e^{\sqrt{b_1}t}\left[\left(-\sqrt{b_1}u^2 + uh'(x)\right)(\eta + h(x)) - \frac{\sqrt{b_1}}{2}\eta(\eta + 2h(x))\right], \\
Q =\ & C_1\left(u^2(\eta + h(x)) + \eta^2\right) + C_2 u + C_3 \eta + \\
& + C_4 e^{-\sqrt{b_1}t}\left(\sqrt{b_1}u(\eta + h(x)) + \eta h'(x)\right) + \\
& + C_5 e^{\sqrt{b_1}t}\left(-\sqrt{b_1}u(\eta + h(x)) + \eta h'(x)\right).
\end{aligned}
\tag{8}
$$

Here, C_1, C_2, C_3, C_4, C_5 are arbitrary constants. From the solution (8) we find that in this case there are two additional conservation laws,

$$
\begin{aligned}
Q_4 &= e^{-\sqrt{b_1}t}\left(\sqrt{b_1}u(\eta + h(x)) + \eta h'(x)\right), \\
P_4 &= e^{-\sqrt{b_1}t}\left[\left(\sqrt{b_1}u^2 + uh'(x)\right)(\eta + h(x)) + \frac{\sqrt{b_1}}{2}\eta(\eta + 2h(x))\right],
\end{aligned}
$$

and

$$
\begin{aligned}
Q_5 &= e^{\sqrt{b_1}t}\left(-\sqrt{b_1}u(\eta + h(x)) + \eta h'(x)\right), \\
P_5 &= e^{\sqrt{b_1}t}\left[\left(-\sqrt{b_1}u^2 + uh'(x)\right)(\eta + h(x)) - \frac{\sqrt{b_1}}{2}\eta(\eta + 2h(x))\right].
\end{aligned}
$$

2.4. The case $h(x) = \dfrac{b_1}{2} x^2 + b_2 x + b_3,\ b_1 < 0.$ In this case, the solution of the

system (5) has the form

$$
P = C_1(\eta + h(x))\left(u^3 + 2u\eta\right) + C_2\left(\frac{u^2}{2} + \eta\right) + C_3 u(\eta + h(x)) +
$$

$$
+ C_4\left[\sin(\sqrt{-b_1}t)\left(\sqrt{-b_1}u^2(\eta + h(x)) + \frac{\sqrt{-b_1}}{2}\eta(\eta + 2h(x))\right) + \right.
$$

$$
\left. + \cos(\sqrt{-b_1}t)\cdot uh'(x)(\eta + h(x))\right] + C_5\left[-\cos(\sqrt{-b_1}t)\left(\sqrt{-b_1}u^2(\eta + h(x)) + \right.\right.
$$

$$
\left.\left. + \frac{\sqrt{-b_1}}{2}\eta(\eta + 2h(x))\right) + \sin(\sqrt{-b_1}t)\cdot uh'(x)(\eta + h(x))\right], \tag{9}
$$

$$
Q = C_1\left(u^2(\eta + h(x)) + \eta^2\right) + C_2 u + C_3\eta +
$$

$$
+ C_4\left(\sin(\sqrt{-b_1}t)\cdot\sqrt{-b_1}u(\eta + h(x)) + \cos(\sqrt{-b_1}t)\cdot\eta h'(x)\right) +
$$

$$
+ C_5\left(-\cos(\sqrt{-b_1}t)\cdot\sqrt{-b_1}u(\eta + h(x)) + \sin(\sqrt{-b_1}t)\cdot\eta h'(x)\right).
$$

From the solution (9) we find that in this case there are two additional conservation laws,

$$
Q_4 = \sin(\sqrt{-b_1}t)\cdot\sqrt{-b_1}u(\eta + h(x)) + \cos(\sqrt{-b_1}t)\cdot\eta h'(x),
$$

$$
P_4 = \sin(\sqrt{-b_1}t)\left(\sqrt{-b_1}u^2(\eta + h(x)) + \frac{\sqrt{-b_1}}{2}\eta(\eta + 2h(x))\right) +
$$

$$
+ \cos(\sqrt{-b_1}t)\cdot uh'(x)(\eta + h(x)),
$$

and

$$
Q_5 = -\cos(\sqrt{-b_1}t)\cdot\sqrt{-b_1}u(\eta + h(x)) + \sin(\sqrt{-b_1}t)\cdot\eta h'(x),
$$

$$
P_5 = -\cos(\sqrt{-b_1}t)\left(\sqrt{-b_1}u^2(\eta + h(x)) + \frac{\sqrt{-b_1}}{2}\eta(\eta + 2h(x))\right) +
$$

$$
+ \sin(\sqrt{-b_1}t)\cdot uh'(x)(\eta + h(x)).
$$

3. Group classification

The task of the group classification is to find Lie symmetries admitted by the considered system of equations depending on the unknown function entering the system [6]. We seek symmetry operators of the system (1) as

$$
X = \xi^1(x, t, u, \eta)\frac{\partial}{\partial x} + \xi^2(x, t, u, \eta)\frac{\partial}{\partial t} + \eta^1(x, t, u, \eta)\frac{\partial}{\partial u} + \eta^2(x, t, u, \eta)\frac{\partial}{\partial \eta}.
$$

Applying the criterion of invariance [6], we obtain the following overdetermined linear homogeneous system of determining equations,

$$
\begin{aligned}
&\xi^1_\eta - u\xi^2_\eta + \xi^2_u = 0, \\
&\xi^1_u - u\xi^2_u + (\eta + h)\xi^2_\eta = 0, \\
&\eta^1 - u\xi^1_x + u^2\xi^2_x - (\eta + h)\eta^1_\eta - \xi^1_t + u\xi^2_t + \eta^2_u + (\eta + h)\xi^2_x = 0, \\
&h'\xi^1 + \eta^2 + (\eta + h)\left(\eta^1_u - \xi^1_x + 2u\xi^2_x - \eta^2_\eta + \xi^2_t - 2uh'\xi^2_\eta\right) = 0, \\
&2u\xi^2_x - \eta^1_u + \xi^2_t + \eta^2_\eta - \xi^1_x = 0, \\
&2(\eta + h)\eta^1_\eta - 2\eta^2_u + uh'\left(\xi^1_\eta - \xi^2_u - u\xi^2_\eta\right) = 0, \\
&u\eta^1_x + \eta^1_t + \eta^2_x + uh'\left(\xi^2_x - \eta^1_\eta\right) = 0, \\
&uh''\xi^1 + h'\eta^1 + (\eta + h)\eta^1_x + u\eta^2_x + \eta^2_t + uh'\left(u\xi^2_x - \eta^2_\eta + \xi^2_t - uh'\xi^2_\eta\right) = 0.
\end{aligned} \tag{10}
$$

The system (10) can be tested for consistency. Here are the results of this test.

3.1.　$h(x)$ – arbitrary function.

$$X_1 = \frac{\partial}{\partial t}\,.$$

This operator will be present for all bottom profiles.

3.2.　$h(x) = n_2 e^{n_1 x} + n_3,\ n_1 \neq 0,\ n_2 \neq 0$. In this case, we obtain the additional symmetry operator

$$X_2 = \frac{\partial}{\partial x} - \frac{n_1}{2} t \frac{\partial}{\partial t} + \frac{n_1}{2} u \frac{\partial}{\partial u} + n_1(\eta + n_3)\frac{\partial}{\partial \eta}\,.$$

3.3.　$h(x) = m_2 \ln|x + m_1| + m_3,\ m_2 \neq 0$. In this case, we obtain the additional symmetry operator

$$X_2 = (x + m_1)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - m_2\frac{\partial}{\partial \eta}\,.$$

3.4.　$h(x) = c_3(x + c_1)^{c_2} + c_4,\ c_2 \neq 0,\ c_2 \neq 1,\ c_2 \neq 2,\ c_3 \neq 0$. In this case, we obtain the additional symmetry operator

$$X_2 = 2(x + c_1)\frac{\partial}{\partial x} - (c_2 - 2)t\frac{\partial}{\partial t} + c_2 u \frac{\partial}{\partial u} + 2c_2(\eta + c_4)\frac{\partial}{\partial \eta}\,.$$

3.5.　$h(x) = b_1 x^2/2 + b_2 x + b_3,\ b_1 > 0$. In this case, we obtain the additional symmetry operators

$$X_2 = \left(x + \frac{b_2}{b_1}\right)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \left(2\eta - \frac{b_2^2}{b_1} + 2b_3\right)\frac{\partial}{\partial \eta}\,,$$

$$X_3 = e^{\sqrt{b_1}\,t}\left(\frac{\partial}{\partial x} + \sqrt{b_1}\frac{\partial}{\partial u} - (b_1 x + b_2)\frac{\partial}{\partial \eta}\right)\,,$$

$$X_4 = e^{-\sqrt{b_1}\,t}\left(\frac{\partial}{\partial x} - \sqrt{b_1}\frac{\partial}{\partial u} - (b_1 x + b_2)\frac{\partial}{\partial \eta}\right)\,.$$

3.6.　$h(x) = b_1 x^2/2 + b_2 x + b_3,\ b_1 < 0$. In this case, we obtain the additional symmetry operators

$$X_2 = \left(x + \frac{b_2}{b_1}\right)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \left(2\eta - \frac{b_2^2}{b_1} + 2b_3\right)\frac{\partial}{\partial \eta}\,,$$

$$X_3 = \cos(\sqrt{-b_1}\,t)\frac{\partial}{\partial x} - \sqrt{-b_1}\sin(\sqrt{-b_1}\,t)\frac{\partial}{\partial u} - (b_1 x + b_2)\cos(\sqrt{-b_1}\,t)\frac{\partial}{\partial \eta}\,,$$

$$X_4 = \sin(\sqrt{-b_1}\,t)\frac{\partial}{\partial x} + \sqrt{-b_1}\cos(\sqrt{-b_1}\,t)\frac{\partial}{\partial u} - (b_1 x + b_2)\sin(\sqrt{-b_1}\,t)\frac{\partial}{\partial \eta}\,.$$

3.7. $h(x) = a_2$. In this case, the basis of symmetry operators can be written as

$$Y_1 = \left(\frac{3}{2}t(\eta + a_2) - \frac{3}{4}tu^2\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}x - \frac{3}{2}tu\right)\frac{\partial}{\partial t} +$$

$$+ \left(\frac{1}{4}u^2 + \eta + a_2\right)\frac{\partial}{\partial u} + u(\eta + a_2)\frac{\partial}{\partial \eta},$$

$$Y_2 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u} + (\eta + a_2)\frac{\partial}{\partial \eta},$$

$$Y_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$Y_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},$$

$$Y_\infty = (u\, w_1(u,\eta) + w_2(u,\eta))\frac{\partial}{\partial x} + w_1(u,\eta)\frac{\partial}{\partial t},$$

where $w_1(u,\eta)$, $w_2(u,\eta)$ are arbitrary solutions of the system

$$w_{1u} + w_{2\eta} = 0,$$

$$w_{2u} + w_1 + (\eta + a_2)w_{1\eta} = 0.$$

Note that the symmetry operator X_1 is contained in the operator Y_∞ if one puts $w_1 = 1$, $w_2 = -u$.

3.8. $h(x) = a_1 x + a_2,\ a_1 \neq 0$. In this case, the basis of the symmetry operators can be written as

$$Y_1 = \left(-4a_1xt - a_2t - a_1{}^2t^3 + \frac{3}{2}tu^2 - 3t\eta\right)\frac{\partial}{\partial x} + \left(3tu - x - \frac{5}{2}a_1t^2\right)\frac{\partial}{\partial t} +$$

$$+ \left(4a_1tu - 3a_1x - 3a_1{}^2t^2 - \frac{1}{2}u^2 - 2\eta\right)\frac{\partial}{\partial u} +$$

$$+ \left(6a_1{}^2xt + 3a_1a_2t + a_1{}^3t^3 - 2a_1xu - \frac{3}{2}a_1tu^2 + 5a_1t\eta - 2a_2u - 2u\eta\right)\frac{\partial}{\partial \eta},$$

$$Y_2 = \left(x + \frac{1}{2}a_1t^2\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + a_1t\frac{\partial}{\partial u} - \left(a_1x + \frac{1}{2}a_1^2t^2\right)\frac{\partial}{\partial \eta},$$

$$Y_3 = 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + 2(\eta + a_2)\frac{\partial}{\partial \eta},$$

$$Y_4 = -t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} + a_1t\frac{\partial}{\partial \eta},$$

$$Y_\infty = \left(u\, w_1(u - a_1t, \eta + a_1x) + w_2(u - a_1t, \eta + a_1x)\right)\frac{\partial}{\partial x} + w_1\frac{\partial}{\partial t} +$$

$$+ a_1\, w_1\frac{\partial}{\partial u} + (a_1u\, w_1 + a_1\, w_2)\frac{\partial}{\partial \eta},$$

where $w_1(u - a_1t, \eta + a_1x)$, $w_2(u - a_1t, \eta + a_1x)$ are arbitrary solutions of the system

$$w_{1\alpha} + w_{2\beta} = 0,$$

$$w_{2\alpha} + w_1 + (\beta + a_2)w_{1\beta} = 0.$$

Here, the situation is the same as in the search for conservation laws of the system of equations with a linear bottom profile,

$$\alpha = u - a_1t,$$

$$\beta = \eta + a_1x.$$

Comment 3. In paper [5], a point change of variables that links the systems with constant and linear bottom profiles was found. Thus, the Lie algebras of symmetry operators of these systems are isomorphic.

On the basis of group classification results, we can conclude that the system (1) can be linearized by a point change of variables only in the case of constant and linear bottom profiles (otherwise, the Lie symmetry algebra of the system is finite).

4. Conclusion

The main results of the paper are the new basic conservation law supplementing the laws of conservation of momentum and mass, as well as the solution to the problem of group classification. The results can be used for the construction of new exact solutions, as well as for numerical modeling of shallow water motion over an uneven bottom.

References

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