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Dynamic response of mechanical systems to impulse process stochastic excitations: Markov approach

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Abstract. Methods for determination of the response of mechanical dynamic systems to Poisson and non-Poisson impulse process stochastic excitations are presented. Stochastic differential and integro-differential equations of motion are introduced. For systems driven by Poisson impulse process the tools of the theory of non-diffusive Markov processes are used. These are: the generalized Itô's differential rule which allows to derive the differential equations for response moments and the forward integro-differential Chapman-Kolmogorov equation from which the equation governing the probability density of the response is obtained. The relation of Poisson impulse process problems to the theory of diffusive Markov processes is given. For systems driven by a class of non-Poisson (Erlang renewal) impulse processes an exact conversion of the original non-Markov problem into a Markov one is based on the appended Markov chain corresponding to the introduced auxiliary pure jump stochastic process. The derivation of the set of integro-differential equations for response probability density and also a moment equations technique are based on the forward integro-differential Chapman-Kolmogorov equation. An illustrating numerical example is also included.

1. Introduction

In some problems in structural dynamics the loads may be adequately idealized as discontinuous stochastic processes: trains of loading pulses, or impulses, occurring at random times and characterized by random magnitudes. Such models are pertinent e.g. to: behaviour of a vehicle travelling over a very rough road (impacts or shocks, due to sudden humps or holes) [1,2], moving loads on a bridge due to highway traffic [3,4], dynamic loading due to wind gusts [5-7]. It is known that if the excitation is a Poisson impulse process, the state vector of the dynamic system is non-diffusive (Poisson-driven)a Markov process. However, if the excitation is a non-Poisson (e.g. renewal) impulse process, the state vector of the dynamic system is not a Markov process. Conversion of the original non-Markov pulse problem into a Markov one is in some cases possible owing to the introduction of auxiliary state variables in form of pure-jump stochastic processes governed by Poisson-driven stochastic differential equations. Then the original state vector augmented by those auxiliary variables becomes a non-diffusive Markov process and differential equations for moments may be derived [8-12]. At the same time, the explicitly introduced, Poisson-driven, pure-jump stochastic processes are characterized by a chain of Markov states. Consequently the original state variables and the states of the auxiliary pure-jump stochastic process are jointly Markovian and the problem is characterized by the set of mixed-type, joint probability density - discrete distribution functions governed by forward integro-differential Chapman-Kolmogorov equations. From the forward equation, with the aid of suitably defined jump probability intensity functions, the explicit integro-differential equations for the response probability density function are derived [13]. The response statistical moments are defined as integrals with respect to the mixed-type, joint probability density - discrete distribution function. Based on the forward integro-differential Chapman-Kolmogorov equation the differential equations for moments are obtained [14]. An illustrating numerical example is also included.

2. State-space formulation of the problem: stochastic differential equations of motion

When the pulse duration is much shorter that the fundamental natural period of the system the train of short-duration pulses is idealized as an impulse process

$$F(t) = \sum_{i=1}^{\mathcal{N}(t)} P_i \delta(t - t_i), \tag{1}$$

where $\mathcal{N}(t)$ is a stochastic point (random counting) process (e.g. a Poisson or a renewal process.), $t_i \in [0, t)$, i.e. excluding the event (impulse) that may occur at t. The impulse magnitudes P_i are independent random variables, identically distributed as a random variable P and statistically independent of the random times t_i, or of the counting process $\mathcal{N}(t)$. The usual differential

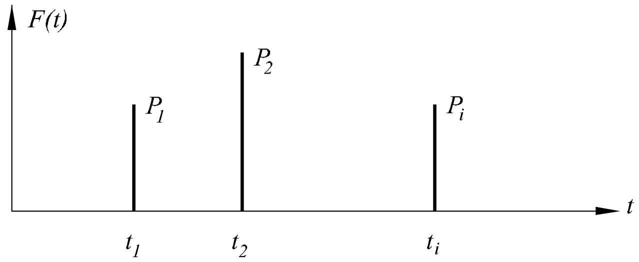

Figure 1. Stochastic impulse process

equation of motion of a linear SDOF system is

$$\ddot{Y}(t) + 2\zeta\omega\dot{Y}(t) + \omega^2 Y(t) = \sum_{i=1}^{\mathcal{N}(t)} P_i \delta(t - t_i) \tag{2}$$

From the impulse-momentum principle it follows that at the impulses occurrence times t_i the velocity response process $\dot{Y}(t)$ changes by jumps, it is piece-wise continuous (continuous-jump). Consequently the displacement response process $Y(t)$ is continuous but it is only piece-wise continuously differentiable. As the jump change in the velocity may occur at any point, the usual rules of calculus do not apply. The usual notation of the differential equation of motion is **not mathematically meaningful**. The stochastic counterparts of the usual differential equations are the **stochastic differential equations**

$$\begin{aligned} dY(t) &= \dot{Y}(t)dt, \\ d\dot{Y}(t) &= -2\zeta\omega\dot{Y}(t)dt - \omega^2 Y(t)dt + P(t)d\mathcal{N}(t), \end{aligned} \tag{3}$$

where $P(t)$ is the magnitude of the impulse which occurs in the time interval $[t, t + dt)$.

The impulse process excitation may be also parametric, i.e. multiplicative to the displacement or to the velocity response process.

Consider the stochastic impulse process $F(t)$ acting as an axial force on a beam-column (Figure 2). The governing equation of motion is

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+c\frac{\partial w(x,t)}{\partial t}+\mu\frac{\partial^2 w(x,t)}{\partial t^2}+F(t)\frac{\partial^2 w(x,t)}{\partial x^2}=0,\quad F(t)=\sum_{i=1}^{\mathcal{N}(t)}P_i\delta(t-t_i) \tag{4}$$

If the normal mode approach is used

$$w(x,t)=\sum_{j=1}^{n}Y_j(t)\varphi_j(x), \tag{5}$$

where $\varphi_j(x)$ are the normal modes then the set of usual differential equation of motion is

$$\ddot{Y}_j(t)+2\zeta_j\omega_j\dot{Y}_j(t)+\omega_j^2Y_j(t)+\beta_jY_j(t)\sum_{i=1}^{\mathcal{N}(t)}P_i\delta(t-t_i)=0,\quad j=1,2,3,...n, \tag{6}$$

where $\beta_j=\dfrac{\int_o^\ell \varphi_j''(x)\varphi_j(x)dx}{\mu\int_o^\ell \varphi_j^2(x)dx}$ and $(...)''=\dfrac{d^2}{dx^2}(...)$.

The stochastic differential equations of motion are obtained as

$$\begin{aligned} dY_j(t) &= \dot{Y}_j(t)dt,\\ d\dot{Y}_j(t) &= -2\zeta_j\omega_j\dot{Y}_j(t)dt-\omega_j^2Y_j(t)dt-\beta_jY_j(t)P(t)d\mathcal{N}(t),\quad j=1,2,3,...n. \end{aligned} \tag{7}$$

hence the impulse process excitation term is multiplicative to the displacement response process.

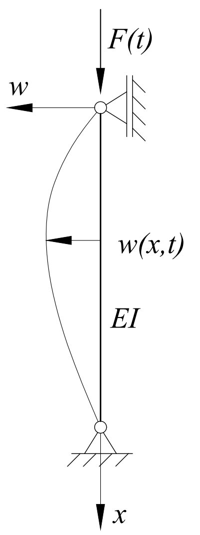

Figure 2. Stochastic impulse process $F(t)$ acting as an axial force on a beam-column.

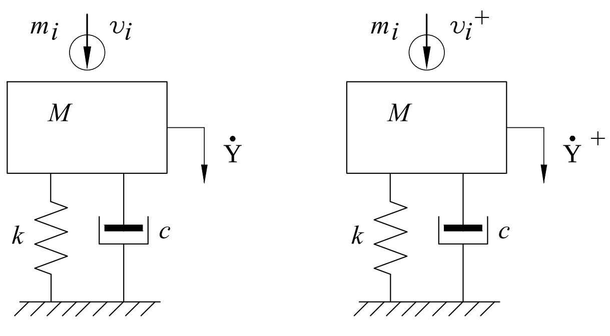

Figure 3. Linear oscillator of mass M subjected to a random train of impacts with particles of random masses m_i, arriving at random times t_i with random velocities v_i.

Consider now a linear oscillator of mass M is subjected to a random train of impacts with particles of random masses m_i, arriving at random times t_i with random velocities v_i [15] as shown in the Figure 3. Before the impact the oscillator has the velocity $\dot{Y}$. Random times t_i are assumed to be independent of m_i and v_i. After the impact the velocities of the oscillator and of the particle are $\dot{Y}^+$ and v_i^+, respectively.

The law of conservation of linear momentum states that

$$M\dot{Y} + m_i v_i = M\dot{Y}^+ + m_i v_i^+ \tag{8}$$

At the time of impact the jump change in the oscillator velocity equals

$$\dot{Y}^+ - \dot{Y} = \frac{m_i(1+K)}{M+m_i}(v_i - \dot{Y}). \tag{9}$$

With due account for the impulse of restoring and damping forces during the time interval dt the stochastic equations of motion are

$$\begin{aligned} dY &= \dot{Y}dt, \\ d\dot{Y} &= (-2\zeta\omega\dot{Y} - \omega^2 Y)dt + (V(t) - \dot{Y})P(t)d\mathcal{N}(t), \end{aligned} \tag{10}$$

where the random variable equals $P(t) = \dfrac{m(t)(1+K)}{M+m(t)}$ and $m(t)$ and $V(t)$ denote, respectively, the mass and the velocity of the particle which impacts in the time interval $[t, t+dt)$ and K is the coefficient of restitution. Hence the impulse process excitation terms are additive and multiplicative to the velocity response process.

In general the state vector of the system, $\mathbf{Y}(t)$, consisting of the generalized coordinates and velocities, is governed by the set of stochastic differential equations of motion

$$d\mathbf{Y}(t) = \mathbf{c}(\mathbf{Y}(t), t)dt + \mathbf{b}(\mathbf{Y}(t), t))P(t)d\mathcal{N}(t), \quad \mathbf{Y}(0) = \mathbf{y}_0, \tag{11}$$

3. Non-diffusive Markov problem

3.1. Stochastic integro-differential equation of motion

If $\mathcal{N}(t)$ is a Poisson counting process $N(t)$ (hence the increments $dN(t)$ are independent), $N(t)$ and P_i are independent, P_i are mutually independent and both $N(t)$ and P_i are independent of the initial conditions $\mathbf{y}_0$, then the state vector $\mathbf{Y}(t)$ is a Markov process. It is then a **non-diffusive** (**Poisson-driven**) **Markov process** governed by the set of stochastic integro-differential equations [16, 17]

$$d\mathbf{Y}(t) = \mathbf{c}(\mathbf{Y}(t), t)dt + \mathbf{b}(\mathbf{Y}(t), t)) \int_{\mathcal{P}} p\Pi(dt, dp), \quad \mathbf{Y}(0) = \mathbf{y}_0, \tag{12}$$

where $\mathbf{c}(\mathbf{Y}(t), t)$ is the drift vector, $\mathbf{b}(\mathbf{Y}(t), t))$ is an analogue of the diffusion vector, $\Pi(dt, dp)$ is the Poisson random measure, interpreted as the random number of points (impulses) in the interval $[t, t+dt)$ with the values of the random impulse magnitude P in $(p, p+dp)$ and $\mathcal{P}$ is the sample space of the random impulse magnitude P.

3.2. Generalized Itô's differential rule

For a function $V[t, \mathbf{Y}(t)]$, bounded for t and $\mathbf{Y}(t)$ finite, and once continuously differentiable with respect to all its arguments, the following **generalized Itô's differential rule** is valid for

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a Poisson-driven Markov process [16, 17]

$$
\begin{aligned}
dV(t,\mathbf{Y}(t)) &= \frac{\partial V(t,\mathbf{Y}(t))}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial V(t,\mathbf{Y}(t))}{\partial Y_i}c_i\left(\mathbf{Y}(t),t\right)dt \\
&+ \int_{\mathcal{P}}\left[V\left[t,\mathbf{Y}(t)+\mathbf{b}\Big(\mathbf{Y}(t),t\Big)p\right] - V\left[t,\mathbf{Y}(t)\right]\right]\Pi(dt,dp)
\end{aligned}
\tag{13}
$$

The generating equation for moments is obtained as

$$
\begin{aligned}
&\frac{d}{dt}E\left[V(\mathbf{Y}(t))\right] = \sum_{i=1}^{n}E\left[\frac{\partial V(\mathbf{Y}(t))}{\partial Y_i}c_i\left(\mathbf{Y}(t),t\right)\right] \\
&+\nu\int_{\mathcal{P}}E\left[V\left[\mathbf{Y}(t)+\mathbf{b}\Big(\mathbf{Y}(t)\Big)p\right]\right]f_P(p)dp - \nu E\left[V\left[\mathbf{Y}(t)\right]\right],
\end{aligned}
\tag{14}
$$

where $f_P(p)$ is the probability density function of the random impulse magnitude P. Explicit equations for moments are obtained for $V[\mathbf{Y}(t)] = Y_i(t)Y_j(t)$, $V[\mathbf{Y}(t)] = Y_i(t)Y_j(t)Y_k(t)$, $V[\mathbf{Y}(t)] = Y_i(t)Y_j(t)Y_k(t)Y_l(t)$, etc.

The equations for the mean values, the second-, third- and fourth-order joint central moments of the response, are obtained as

$$
\dot{\mu}_i(t) = E\left[c_i(\mathbf{Y}(t),t)\right] + \nu(t)E[P]E\left[b_i(\mathbf{Y}(t),t)\right] \tag{15}
$$

$$
\begin{aligned}
\dot{\kappa}_{ij}(t) \;&=\; 2\left\{E\left[Y_i^0\left(c_j^0(\mathbf{Y}^0(t),t)+\nu(t)b_j(\mathbf{Y}(t),t)P\right)\right]\right\}_s \\
&\quad +\nu(t)E[P^2]E\left[b_i(\mathbf{Y}(t),t)b_j(\mathbf{Y}(t),t)\right]
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\dot{\kappa}_{ijk}(t) \;&=\; 3\left\{E\left[Y_i^0Y_j^0\left(c_k^0(\mathbf{Y}^0(t),t)+\nu(t)b_k(\mathbf{Y}(t),t)P\right)\right]\right\}_s \\
&\quad +3\nu(t)E[P^2]\left\{E\left[Y_i^0b_j(\mathbf{Y}(t),t)b_k(\mathbf{Y}(t),t)\right]\right\}_s \\
&\quad +\nu(t)E[P^3]E\left[b_i(\mathbf{Y}(t),t)b_j(\mathbf{Y}(t),t)b_k(\mathbf{Y}(t),t)\right]
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\dot{\kappa}_{ijkl}(t) \;&=\; 4\left\{E\left[Y_i^0Y_j^0Y_k^0\left(c_l^0(\mathbf{Y}^0(t),t)+\nu(t)b_l(\mathbf{Y}(t),t)P\right)\right]\right\}_s \\
&\quad +6\nu(t)E[P^2]\left\{E\left[Y_i^0Y_j^0b_k(\mathbf{Y}(t),t)b_l(\mathbf{Y}(t),t)\right]\right\}_s \\
&\quad +4\nu(t)E[P^3]\left\{E\left[Y_i^0b_j(\mathbf{Y}(t),t)b_k(\mathbf{Y}(t),t)b_l(\mathbf{Y}(t),t)\right]\right\}_s \\
&\quad +\nu(t)E[P^4]E\left[b_i(\mathbf{Y}(t),t)b_j(\mathbf{Y}(t),t)b_k(\mathbf{Y}(t),t)b_l(\mathbf{Y}(t),t)\right],
\end{aligned}
\tag{18}
$$

where $Y_i^0(t) = Y_i(t) - \mu_i(t)$ and $c_j^0(\mathbf{Y}^0(t),t) = c_j(\mathbf{Y}(t),t) - E\left[c_j(\mathbf{Y}(t),t)\right]$ denote the zero-mean (centralized) state variables and drift vector, and $\{...\}_s$ denotes the Stratonovich symmetrizing operation, e.g.

$$
\{Y_iY_jc_k\}_s = \frac{1}{3}(Y_iY_jc_k + Y_iY_kc_j + Y_jY_kc_i) \tag{19}
$$

Equations for response moments of a linear system form always a closed set and can be directly solved numerically.

However, equations for response moments of a non-linear system with polynomial non-linearity form an infinite hierarchy. Then special closure approximations (truncation procedures) must be used, which express higher-order moments (above the level of truncation) in terms of lower-order moments (up to the level of truncation). For other types of non-linearity the equations involve unknown expectations of non-linear functions of state variables and some tentative forms of the joint probability density must be used.

For example if the fifth- and sixth-order cumulants are neglected, the fifth- and sixth-order central moments are expressed in terms of lower-order moments as

$$\begin{aligned} \kappa_{ijklm}(t) &= 10\left\{\kappa_{ij}(t)\kappa_{klm}(t)\right\}_s \\ \kappa_{ijklmn}(t) &= 15\left\{\kappa_{ij}(t)\kappa_{klmn}(t)\right\}_s + 10\left\{\kappa_{ijk}(t)\kappa_{lmn}(t)\right\}_s - 30\left\{\kappa_{ij}(t)\kappa_{kl}(t)\kappa_{mn}(t)\right\}_s \end{aligned} \tag{20}$$

The above closure approximations are equivalent to assuming the tentative joint probability density function of the state variables in the form of the suitable Gram-Charlier expansion, which is the expansion in terms of generalized Hermite polynomials. The accuracy of the results obtained with the help of closure approximations depends on whether or not the truncated pertinent Gram-Charlier expansion is an adequate approximation to the actual density function.

Consider the dynamical system subjected to a random train of impulses, with zero initial conditions $\mathbf{Y}(0) = \mathbf{0}$. If in the time interval $[0, t)$ no impulse occurred, the system is still at rest at the time t.

If the train of impulses is driven by a homogeneous Poisson process, the probability P_0 of no impulse occurrence in the time interval $[0, t)$ is expressed as

$$P_0 = \Pr\{N(t) = 0\} = \exp(-\nu t). \tag{21}$$

The probability P_0 may be high, close to the unity, if the length t of the time interval is small, i.e. at the early transient stage, especially if also the mean arrival rate ν is small.

Joint probability density function of the state vector $\mathbf{Y}(t)$ can be represented in form of the sum of the continuous and discrete parts as (Figure 4) [18]

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}, t) &= f_{\mathbf{Y}}(\mathbf{y}, t \mid N(t) = 0)\Pr\{N(t) = 0\} + f_{\mathbf{Y}}(\mathbf{y}, t \mid N(t) > 0)\Pr\{N(t) > 0\} \\ &= P_0 \prod_{i=1}^{n} \delta\Big(y_i\Big) + (1 - P_0) f^0_{\mathbf{Y}}(\mathbf{y^0}, t), \end{aligned} \tag{22}$$

where n is the number of state variables.

In the first, discrete, part the Dirac delta spike $\delta(y_i)$ represents the finite probability of the system being at rest, i.e. $y_i = 0$. This probability is, of course, concentrated at the displacement $y_i = 0$. The second, continuous, part i.e. $f^0_{\mathbf{Y}}\left(\mathbf{y^0}, t\right)$ is the conditional density, given that at least one impulse occurred.

In the situations when the first, discrete, term in the assumed discrete-continuous density function is predominant, it can be predicted that the probability density function $f_{\mathbf{Y}}(\mathbf{y}, t)$ will be difficult to approximate by the truncated Gram-Charlier expansion. The reason is that the Dirac delta is difficult to approximate in terms of Hermite polynomials. The usual cumulant neglect closure approximations may be then very inaccurate for the unconditional central moments $\kappa_{i_1 \cdots i_n}$.

The Gram-Charlier expansion can be used for the continuous part of the discrete-continuous pdf only, i.e. for the conditional density $f^0_{\mathbf{Y}}(\mathbf{y^0}, t)$. This is equivalent to assuming the closure approximations for the conditional moments.

With the aid of the relationships (identities) between the unconditional moments and conditional moments the final modified cumulant neglect closure approximations are obtained.

The use of modified cumulant neglect closure approximations allows to improve the results for the problems with sparse pulse trains, i.e. with a low mean arrival rate ν [18].

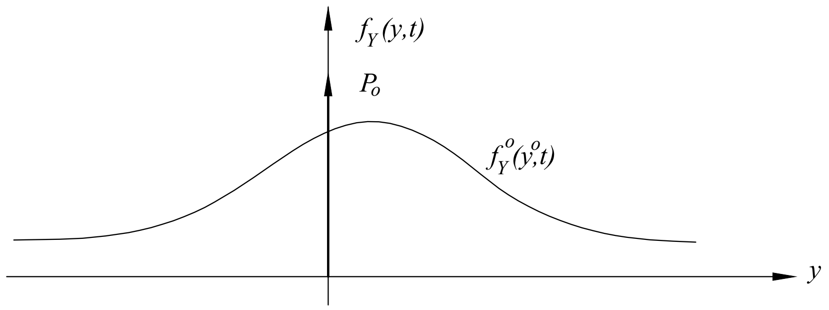

Figure 4. Tentative probability density function of the Poisson-driven response process

3.3. Integro-differential equation governing the response process probability density

The Markov process with continuous-jump sample paths is governed by an **integro-differential Chapman-Kolmogorov equation** (called **master equation** in physics) [9,19]

$$\begin{aligned}\frac{\partial}{\partial t}q_{\{\mathbf{Y}\}}(\mathbf{y},t) &= -\sum_i \frac{\partial}{\partial y_i}[c_i(\mathbf{y},t)q_{\{\mathbf{Y}\}}(\mathbf{y},t)]+ \\ &+\int \Big[J_{\{\mathbf{Y}\}}(\mathbf{y}|\mathbf{x},t)q_{\{\mathbf{Y}\}}(\mathbf{x},t) - J_{\{\mathbf{Y}\}}(\mathbf{x}|\mathbf{y},t)q_{\{\mathbf{Y}\}}(\mathbf{y},t)\Big]d\mathbf{x},\end{aligned} \tag{23}$$

$q_{\{\mathbf{Y}\}}(\mathbf{y},t)$ is the joint probability density of the state vector, $c_i(\mathbf{y},t)$ is the drift term of the equation of motion and

$$J_{\{\mathbf{Y}\}}(\mathbf{y}|\mathbf{x},t) = \lim_{\Delta t\to 0}\frac{q_{\{\mathbf{Y}\}}(\mathbf{y},t+\Delta t|\mathbf{x},t)}{\Delta t}. \tag{24}$$

is the **jump probability intensity function**, which is non-zero only if the state vector changes by a finite jump from $\mathbf{x}$ at t to $\mathbf{y}$ at $t+\Delta t$ as $\Delta t \to 0$.

For a system under an external Poisson impulse process excitation, i.e. $\mathbf{b}(\mathbf{Y}(t),t)$ = const. = $\mathbf{b}$, and with a mean arrival rate ν, the insertion of a pertinent jump probability intensity function into Chapman-Kolmogorov equation, followed by integration with respect to $\mathbf{x}$ results in a **Kolmogorov-Feller** integro-differential equation [9].

In the case of a SDOF oscillator, with $\mathbf{b}(\mathbf{Y}(t),t) = const. = \mathbf{b} = [b_1,b_2]^T = [0,b]^T$, the jump probability intensity function is determined as

$$J_{\{\mathbf{Y}\}}(\mathbf{y}|\mathbf{x},t) = \nu\delta(y_1-x_1)\int_{\mathcal{P}}\delta(y_2-(x_2+b_2p))f_P(p)dp \tag{25}$$

The integration of the Chapman-Kolmogorov equation (23) with respect to $\mathbf{x}$ yields a **Kolmogorov-Feller** equation

$$\frac{\partial}{\partial t}q_{\{\mathbf{Y}\}}(\mathbf{y},t) = -\sum_i\frac{\partial}{\partial y_i}[c_i(\mathbf{y},t)q_{\{\mathbf{Y}\}}(\mathbf{y},t)] + \nu\int_{\mathcal{P}} q_{\{\mathbf{Y}\}}(\mathbf{y}-\mathbf{b}p,t)f_P(p)dp - \nu q_{\{\mathbf{Y}\}}(\mathbf{y},t) \tag{26}$$

3.4. Relation to the classical diffusive Markov processes theory

If the impulses mean arrival rate $\nu\to\infty$ and at the same time the impulses magnitudes tend to zero, i.e. the impulse process becomes an infinitely dense train of infinitely small impulses, the velocity response process $\dot{Y}(t)$ reveals jumps at every infinitesimal time interval dt, hence it is discontinuous and consequently the displacement response process $Y(t)$ is non-differentiable. If the impulses have zero-mean values, i.e. $E[P] = 0$, and at the same time their magnitudes tend to zero in such a way that $\nu E[P^2]$ is kept constant, the Poisson impulse process

$$F(t) = \sum_{i=1}^{N(t)} P_i \delta(t - t_i) \tag{27}$$

tends asymptotically to the **Gaussian white noise** and the equations of motion become Itô's stochastic differential equations. In the case of additive excitation

$$d\mathbf{Y}(t) = \mathbf{c}(\mathbf{Y}(t), t)dt + \mathbf{b}dW(t), \quad \mathbf{Y}(0) = \mathbf{y}_0, \tag{28}$$

where $W(t)$ is the **Wiener process**.

In the asymptotic case the generalized Itô's differential rule (13) reduces to the classical Itô's differential rule

$$\begin{aligned} dV(t, \mathbf{Y}(t)) &= \frac{\partial V(t, \mathbf{Y}(t))}{\partial t} dt \\ &+ \sum_{i=1}^{n} \frac{\partial V(t, \mathbf{Y}(t))}{\partial Y_i} \left(c_i\left(\mathbf{Y}(t), t\right) dt + b_i dW(t)\right) + \frac{1}{2} \nu E[P^2] \sum_{i,j=1}^{n} b_i b_j \frac{\partial^2 V(t, \mathbf{Y}(t))}{\partial Y_i Y_j} dt \end{aligned} \tag{29}$$

In the asymptotic case the Kolmogorov-Feller integro-differential equation (26) reduces to the classical Fokker - Planck - Kolmogorov partial differential equation

$$\frac{\partial}{\partial t} q_{\{\mathbf{Y}\}}(\mathbf{y}, t) = -\sum_i \frac{\partial}{\partial y_i} [c_i(\mathbf{y}, t) q_{\{\mathbf{Y}\}}(\mathbf{y}, t)] + \frac{1}{2} \nu E[P^2] \sum_{i,j=1}^{n} b_i b_j \frac{\partial^2 q_{\{\mathbf{Y}\}}(\mathbf{y}, t)}{\partial y_i y_j} \tag{30}$$

4. Non-Markov problems

4.1. Exact conversion of a non-Markov problem into a Markov one

Consider a dynamic system under a renewal process impulse process where the occurrence times of the impulses are driven by a renewal process $R(t)$. The increments $dR(t)$ **are not independent**. The state vector $\mathbf{Y}(t)$ governed by

$$d\mathbf{Y}(t) = \mathbf{c}(\mathbf{Y}(t), t)dt + \mathbf{b}(\mathbf{Y}(t), t))P(t)dR(t), \quad \mathbf{Y}(0) = \mathbf{y}_0, \tag{31}$$

is not a Markov process. Exact conversion of the original non-Markov problem into a Markov one is performed by recasting of the original impulse process (replacement valid with probability 1) as [9,11]

$$\sum_{i,R=1}^{R(t)} P_{i,R} \delta(t - t_{i,R}) = \sum_{i=1}^{N(t)} \rho(N(t_i)) P_i \delta(t - t_i), \tag{32}$$

where $\rho(N(t_i)$ is an **auxiliary, pure jump, zero-one stochastic process**, which selects only some impulses from the train driven by a Poisson process $N(t)$.

As an auxiliary, pure-jump stochastic process consists of negative-exponential distributed "phases", the states of sojourn of the process in these "phases" are **Markov states** $S(t) = 1, 2, ..., m$. The jump process must be defined in such a way: the actual impulse (i.e. the jump in the velocity response $\dot{Y}(t)$) only occurs if there is a transition between some particular Markov states. Other transitions are not associated with a jump in the velocity response.

Original state variables of the dynamic system and the states of the auxiliary jump process are **jointly Markovian**.

Accordingly, the mixed-type joint probability density-discrete distribution function of the continuous-jump processes $Y(t)$, $\dot{Y}(t)$ and of $\mathbf{m}$ Markov states $S(t)$ of the pure jump process is defined as

$$q_j(\mathbf{y},t)d\mathbf{y} = \Pr\{\mathbf{Y}(t) \in (\mathbf{y}+d\mathbf{y}) \wedge S(t)=j\}, \quad j=1,2,...,m \tag{33}$$

This function is governed by the set of integro-differential forward Chapman-Kolmogorov equation

$$\begin{aligned}
&\frac{\partial}{\partial t} q_j(\mathbf{y},t) = \mathcal{K}_j[\mathbf{q}(\mathbf{y},t)] \\
&\mathcal{K}_j[\mathbf{q}(\mathbf{y},t)] = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\Big[c_i(\mathbf{y},t)q_j(\mathbf{y},t)\Big] + \\
&\sum_{l=1,l\neq j}^{m} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Bigg[J_{\{\mathbf{Y}\}}(\mathbf{y},j|\mathbf{x},l,t)q_l(\mathbf{x},t) - J_{\{\mathbf{Y}\}}(\mathbf{x},l|\mathbf{y},j,t)q_j(\mathbf{y},t)\Bigg]d\mathbf{x},
\end{aligned} \tag{34}$$

where $\mathcal{K}_j[...]$ is the forward integro-differential Chapman-Kolmogorov operator, $j=1,2,...,m$ and $c_i(\mathbf{y},t)$ are the drift coefficients of the equation of motion.

The explicit integro-differential equations are obtained after the insertion of the suitably determined jump probability intensity function $J_{\{\mathbf{Y}\}}(\mathbf{y},j|\mathbf{x},l,t)$ and subsequent integration with respect to $\mathbf{x}$. The jump probability intensity function defined as

$$J_{\{\mathbf{Y}\}}(\mathbf{y},j|\mathbf{x},l,t) = \lim_{\Delta t\to 0} \frac{\Pr\{\mathbf{Y}(t+\Delta t)=\mathbf{y}, S(t+\Delta t)=j, |\mathbf{X}(t)=\mathbf{x}, S(t)=l\}}{\Delta t} \tag{35}$$

must be determined from the pertinent chain of Markov states.

The non-zero jump probability intensity functions are only defined for $j \neq l$, such that there is a transition in a Markov chain (jump in the auxiliary process) from $S(t)=l$ to $S(t+\Delta t)=j$. These functions assume two different forms: for the transitions in a Markov chain not associated by the jump in the velocity process and those associated by the jump in the velocity process. Let us determine the jump probability intensity function for a single-degree-of-freedom system (two state variables). If there is a transition from $S(t)=l$ to $S(t+\Delta t)=j$, but no actual impulse occurs (no jump in the velocity process)

$$J_{\{\mathbf{Y}\}}(\mathbf{y},j|\mathbf{x},l,t) = \pi(j|l)\delta(y_1-x_1)\delta(y_2-x_2), \tag{36}$$

where

$$\pi(j|l) = \frac{P_{j|l}(\Delta t)}{\Delta t}, \qquad j \neq l, \tag{37}$$

and $P_{j|l}(\Delta t) = \Pr\{S(t+\Delta t)=j\Big|S(t)=l\}$ is the transition probability in a pertinent Markov chain. If the transition from $S(t)=l$ to $S(t+\Delta t)=j$, is associated with the actual impulse (the jump in the velocity process $\dot{Y}(t)=Y_2(t)$)

$$J_{\{\mathbf{Z}\}}(\mathbf{y},j|\mathbf{x},l,t) = \pi(j|l)\delta(y_1-x_1)\int_{\mathcal{P}} \delta\Big(y_2-\Big(x_2+b(x_1,x_2)p\Big)\Big)f_P(p)dp \tag{38}$$

In an Erlang renewal process [20-22] with an integer parameter k the events are every kth Poisson events, hence it must be assumed that the actual impulse, and hence the jump in the velocity variable occurs when the jump is from the state k to 1. All other jumps in the auxiliary process occur from $j-1$ to j ($j=2,...k$), but there are no corresponding impulses. The auxiliary pure jump process and the corresponding Markov chain are shown in the figures 5 and 6.

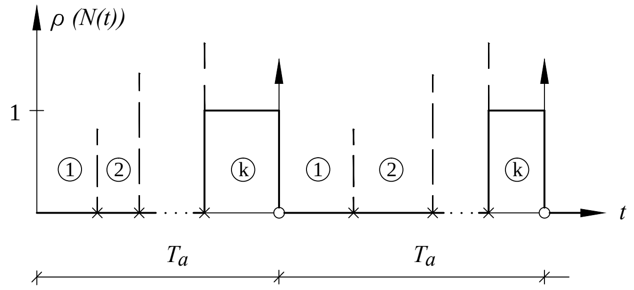

Figure 5. Auxiliary pure jump process

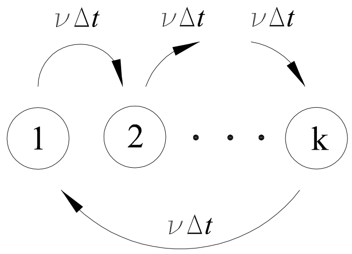

Figure 6. Corresponding Markov chain

The jump probability intensity function is determined as

$$J_{\mathbf{Y}}(y_1,y_2,j|x_1,x_2,l,t) = \begin{cases} \nu\delta(y_1-x_1)\int\limits_{\mathcal{P}}\delta(y_2-(x_2+b(x_1,x_2)p))f_P(p)dp, \\ \hfill j=1, l=k. \\ \nu\delta(y_1-x_1)\delta(y_2-x_2), \quad j=2,3,...,k, \ l=j-1 \end{cases} \tag{39}$$

If the excitation is purely external (additive), i.e. $\mathbf{b}(\mathbf{Y}(t),t) = const. = \mathbf{b}$, the integration with respect to $\mathbf{x}$ yields the following equations for the Erlang process with an arbitrary parameter k

$$\begin{aligned} \frac{\partial}{\partial t}q_1(\mathbf{y},t) &= -\sum_{i=1}^{n}\frac{\partial}{\partial y_i}\Big[c_i(\mathbf{y},t)q_1(\mathbf{y},t)\Big] + \nu\int\limits_{\mathcal{P}} q_k(\mathbf{y}-\mathbf{b}p,t)f_P(p)dp - \nu q_1(\mathbf{y},t) \\ &\vdots \\ \frac{\partial}{\partial t}q_j(\mathbf{y},t) &= -\sum_{i=1}^{n}\frac{\partial}{\partial y_i}\Big[c_i(\mathbf{y},t)q_j(\mathbf{y},t)\Big] + \nu q_{j-1}(\mathbf{y},t) - \nu q_j(\mathbf{y},t), \quad j=2,3,...k \end{aligned} \tag{40}$$

The generating equation for moments is derived for the expectations with respect to the joint probability density - discrete distribution function $q_j(\mathbf{y},t)$

$$E_j\Big[V(\mathbf{Y}(t),t)\Big] = \int\limits_{-\infty}^{\infty} V(\mathbf{y},t)q_j(\mathbf{y},t)d\mathbf{y}, \quad j=1,2,...,m. \tag{41}$$

Its time evolution is determined as

$$\begin{aligned} &\frac{d}{dt}E_j\Big[V(\mathbf{Y}(t),t)\Big] = \frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty} V(\mathbf{y}(t),t)q_j(\mathbf{y},t)d\mathbf{y} = \\ &E_j\left[\frac{\partial}{\partial t}V(\mathbf{Y}(t),t)\right] + \int\limits_{-\infty}^{\infty} V(\mathbf{y}(t),t)\mathcal{K}_j\Big[\mathbf{q}(\mathbf{y},t)\Big]d\mathbf{y}. \end{aligned} \tag{42}$$

After the integration by parts, suitable interchange of dummy variables and some rearrangements the generating equation for moments is arrived at in the form [14]

$$\begin{aligned}&\frac{d}{dt}E_j\Big[V(\mathbf{Y}(t),t)\Big] = E_j\left[\frac{\partial}{\partial t}V(\mathbf{Y}(t),t)\right] + \sum_{r=1}^{n} E_j\left[\frac{\partial V(\mathbf{Y}(t),t)}{\partial Y_r}c_r(\mathbf{Y}(t),t)\right]\\&+\sum_{l=1}^{m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[V(\mathbf{x}(t),t)J_{\{\mathbf{Y}\}}(\mathbf{x},j|\mathbf{y},l,t)q_i(\mathbf{y},t)\\&\qquad -V(\mathbf{y}(t),t)J_{\{\mathbf{Y}\}}(\mathbf{x},l|\mathbf{y},j,t)q_j(\mathbf{y},t)\Big]d\mathbf{x}d\mathbf{y},\quad j=1,2,...,m.\end{aligned} \tag{43}$$

As the usual-sense (marginal) joint probability density function is obtained by summation

$$q(\mathbf{y},t) = \sum_{j=1}^{m} q_j(\mathbf{y},t), \tag{44}$$

so is the usual-sense expectation

$$E\Big[V(\mathbf{Y}(t),t)\Big] = \sum_{j=1}^{m} E_j\Big[V(\mathbf{Y}(t),t)\Big] \tag{45}$$

4.2. Moment equations for an Erlang renewal impulse process

For an Erlang renewal impulse process the jump probability intensity function $J_{\{\mathbf{Y}\}}(x_1,x_2,j|y_1,y_2,l,t)$ determined with the aid of the Markov chain (Figure 6) equals

$$J_{\{\mathbf{Y}\}}(x_1,x_2,j|y_1,y_2,l,t) = \begin{cases}\nu\delta(x_1-y_1)\displaystyle\int_{\mathcal{P}}\delta\Big(x_2-(y_2+b(y_1,y_2)p)\Big)f_P(p)dp, & \\ \qquad\qquad j=1, l=k. & \\ \nu\delta(x_1-y_1)\delta(x_2-y_2),\quad j=2,3,...,k,\ l=j-1\end{cases} \tag{46}$$

After the insertion of the jump probability intensity function into the general generating equation (43) and integration with respect to $\mathbf{x}$ the problem-specific generating equations for moments are obtained as

$$\begin{aligned}&\frac{d}{dt}E_1\Big[V(\mathbf{Y}(t),t)\Big] = E_1\left[\frac{\partial}{\partial t}V(\mathbf{Y}(t),t)\right] + \sum_{r=1}^{n} E_1\left[\frac{\partial V(\mathbf{Y}(t),t)}{\partial Y_r}c_r(\mathbf{Y}(t),t)\right]\\&+\nu E_k\left[\int_{\mathbf{P}} V\Big(\mathbf{Y}(t)+\mathbf{b}(\mathbf{Y})p,t\Big)f_P(p)dp\right] - \nu E_1\Big[V(\mathbf{Y}(t),t)\Big],\\&\frac{d}{dt}E_j\Big[V(\mathbf{Y}(t),t)\Big] = E_j\left[\frac{\partial}{\partial t}V(\mathbf{Y}(t),t)\right] + \sum_{r=1}^{n} E_j\left[\frac{\partial V(\mathbf{Y}(t),t)}{\partial Y_r}c_r(\mathbf{Y}(t),t)\right]\\&+\nu E_{j-1}\Big[V(\mathbf{Y}(t),t)\Big] - \nu E_j\Big[V(\mathbf{Y}(t),t)\Big],\qquad j=2,3,...,k\end{aligned} \tag{47}$$

The explicit equations for first-, second-, third- and fourth-order moments are obtained as

$$\begin{aligned}\frac{d}{dt}E_1\Big[Z_i(t)\Big] &= E_1\Big[c_i(\mathbf{Z}(t))\Big]+\nu E_k\Big[Z_i(t)\Big]+\nu E[P]E_k\Big[b_i(\mathbf{Z}(t))\Big]-\nu E_1\Big[Z_i(t)\Big]\\ \frac{d}{dt}E_j\Big[Z_i(t)\Big] &= E_j\Big[c_i(\mathbf{Z}(t))\Big]+\nu E_{j-1}\Big[Z_i(t)\Big]-\nu E_j\Big[Z_i(t)\Big],\qquad j=2,3,...,k\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{d}{dt}E_1\Big[Z_i(t)Z_l(t)\Big] &= 2\Big\{E_1\Big[Z_ic_l(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_k\Big[Z_i(t)Z_l(t)\Big]+2\nu E[P]\Big\{E_k\Big[Z_ib_l(\mathbf{Z}(t))\Big]\Big\}_s\\ &\quad+\nu E[P^2]E_k\Big[b_i(\mathbf{Z}(t))b_l(\mathbf{Z}(t))\Big]-\nu E_1\Big[Z_i(t)Z_l(t)\Big]\\ \frac{d}{dt}E_2\Big[Z_i(t)Z_l(t)\Big] &= 2\Big\{E_j\Big[Z_ic_l(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_{j-1}\Big[Z_i(t)Z_l(t)\Big]-\nu E_j\Big[Z_i(t)Z_l(t)\Big],\quad j=2,3,...,k\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{d}{dt}E_1\Big[Z_i(t)Z_l(t)Z_m(t)\Big] &= 3\Big\{E_1\Big[Z_iZ_lc_m(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_k\Big[Z_i(t)Z_l(t)Z_m(t)\Big]\\ &\quad+3\nu E[P]\Big\{E_k\Big[Z_iZ_lb_m(\mathbf{Z}(t))\Big]\Big\}_s+3\nu E[P^2]\Big\{E_k\Big[Z_ib_l(\mathbf{Z}(t)b_m(\mathbf{Z}(t))\Big]\Big\}_s\\ &\quad+\nu E[P^3]E_k\Big[b_i(\mathbf{Z}(t))b_l(\mathbf{Z}(t))b_m(\mathbf{Z}(t)\Big]-\nu E_1\Big[Z_i(t)Z_l(t)Z_m(t)\Big]\\ \frac{d}{dt}E_j\Big[Z_i(t)Z_l(t)Z_m(t)\Big] &= 3\Big\{E_j\Big[Z_iZ_lc_m(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_{j-1}\Big[Z_i(t)Z_l(t)Z_m(t)\Big]\\ &\quad-\nu E_j\Big[Z_i(t)Z_l(t)Z_m(t)\Big]\qquad j=2,3,...,k\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{d}{dt}E_1\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big] &= 4\Big\{E_1\Big[Z_iZ_lZ_mc_n(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_k\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big]\\ &\quad+4\nu E[P]\Big\{E_k\Big[Z_iZ_lZ_mb_n(\mathbf{Z}(t))\Big]\Big\}_s\\ &\quad+6\nu E[P^2]\Big\{E_k\Big[Z_iZ_lb_m(\mathbf{Z}(t)b_n(\mathbf{Z}(t))\Big]\Big\}_s\\ &\quad+4\nu E[P^3]E_k\Big[Z_ib_l(\mathbf{Z}(t))b_m(\mathbf{Z}(t))b_n(\mathbf{Z}(t)\Big]\\ &\quad+\nu E[P^4]E_k\Big[b_i(\mathbf{Z}(t))b_l(\mathbf{Z}(t))b_m(\mathbf{Z}(t))b_n(\mathbf{Z}(t)\Big]\\ &\quad-\nu E_1\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big]\\ \frac{d}{dt}E_j\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big] &= 4\Big\{E_j\Big[Z_iZ_lZ_mc_n(\mathbf{Z}(t))\Big]\Big\}_s+\nu E_{j-1}\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big]\\ &\quad-\nu E_j\Big[Z_i(t)Z_l(t)Z_m(t)Z_n(t)\Big],\qquad j=2,3,...,k\end{aligned} \tag{51}$$

4.3. Example: response of a linear oscillator to an Erlang renewal impulse process with $k=2$

Consider a linear oscillator under external excitation, governed by the equation

$$\ddot{Y}+2\zeta\omega\dot{Y}+\omega^2Y=b\sum_{i=1}^{R(t)}P_i\delta(t-t_i), \tag{52}$$

or

$$
\begin{aligned}
dZ_1(t) &= Z_2(t)dt \\
dZ_2(t) &= \Big(-2\zeta\omega Z_2(t) - \omega^2 Z_1(t)\Big)\, dt + bP(t)dR(t),
\end{aligned}
\tag{53}
$$

hence $c_1(Z_1, Z_2, t) = Z_2$, $c_2(Z_1, Z_2, t) = -\omega^2 Z_1 - 2\zeta\omega Z_2$, $b_1 = 0$, $b_2(Z_1, Z_2, t) = \text{const.} = b$. The equations for the mean values $\mathbf{m}(t) = \Big[E_1[Z_1(t)], E_2[Z_1(t)], E_1[Z_2(t)], E_2[Z_2(t)]\Big]^T$ are obtained from (47) as

$$
\frac{d}{dt}\mathbf{m}(t) = \mathbf{A}_1\mathbf{m}(t) + \mathbf{f}_1(t),
\tag{54}
$$

where

$$
\mathbf{A}_1 = \begin{bmatrix} -\nu & \nu & 1 & 0 \\ \nu & -\nu & 0 & 1 \\ -\omega^2 & 0 & -(2\zeta\omega+\nu) & \nu \\ 0 & -\omega^2 & \nu & -(2\zeta\omega+\nu) \end{bmatrix}, \quad \mathbf{f}_1(t) = \begin{bmatrix} 0 \\ 0 \\ \nu b E[P]\mathcal{P}_2(t) \\ 0 \end{bmatrix}.
\tag{55}
$$

and $\mathcal{P}_2(t) = \Pr\{S(t) = 2\}$.

According to (45), the usual-sense mean values $E[Z_1]$ and $E[Z_2]$ are

$$
E[Z_1] = E_1[Z_1] + E_2[Z_1], \quad E[Z_2] = E_1[Z_2] + E_2[Z_2],
\tag{56}
$$

hence the equations for these mean values are obtained by pair-wise summation of the equations (54, 55), which yields

$$
\begin{aligned}
\frac{d}{dt}E[Z_1] &= E[Z_2] \\
\frac{d}{dt}E[Z_2] &= -\omega^2 E[Z_1] - 2\zeta\omega E[Z_2] + \nu b E[P]\mathcal{P}_2(t)
\end{aligned}
\tag{57}
$$

The asymptotic, stationary mean values $\mathbf{m} = \lim_{t\to\infty} \mathbf{m}(t) = \Big[E_1[Z_1], E_2[Z_1], E_1[Z_2], E_2[Z_2\Big]^T$ are governed by

$$
\mathbf{A}_1\mathbf{m} = -\mathbf{f}_1,
\tag{58}
$$

with the stationary value of $\mathcal{P}_2(t)$

$$
\mathcal{P}_2 = \frac{1}{2}
\tag{59}
$$

In particular, the stationary solution for the mean value of the displacement response equals (cf. [9])

$$
E[Z_1] = \frac{\nu b E[P]}{2\omega^2}
\tag{60}
$$

The equations for the ordinary second-order moments $\boldsymbol{\mu}(t) = \Big[E_1[Z_1^2], E_2[Z_1^2], E_1[Z_1Z_2], E_2[Z_1Z_2], E_1[Z_2^2], E_2[Z_2^2]\Big]^T$ are obtained from (47) as

$$
\frac{d}{dt}\boldsymbol{\mu}(t) = \mathbf{A}_2\boldsymbol{\mu}(t) + \mathbf{f}_2(t),
\tag{61}
$$

where

$$
\mathbf{A}_2 = \begin{bmatrix} -\nu & \nu & 2 & 0 & 0 & 0 \\ \nu & -\nu & 0 & 2 & 0 & 0 \\ -\omega^2 & 0 & -(2\zeta\omega+\nu) & \nu & 1 & 0 \\ 0 & -\omega^2 & \nu & -(2\zeta\omega+\nu) & 0 & 1 \\ 0 & 0 & -2\omega^2 & 0 & -(4\zeta\omega+\nu) & \nu \\ 0 & 0 & 0 & -2\omega^2 & \nu & -(4\zeta\omega+\nu) \end{bmatrix},
\tag{62}
$$

$$\mathbf{f}_2(t) = \begin{bmatrix} 0 \\ 0 \\ \nu b E[P] E_2[Z_1] \\ 0 \\ 2\nu b E[P] E_2[Z_2] + \nu b^2 E[P^2] \mathcal{P}_2(t) \\ 0 \end{bmatrix}. \tag{63}$$

The pair-wise summation of the equations (61 - 63) yields the equations for the usual-sense ordinary second-order moments $E\Big[Z_i(t)Z_l(t)\Big]$

$$\begin{aligned} \frac{d}{dt}E[Z_1^2] &= 2E[Z_1Z_2], \\ \frac{d}{dt}E[Z_1Z_2] &= -\omega^2 E[Z_1^2] - 2\zeta\omega E[Z_1Z_2] + E[Z_2^2] + \nu b E[P] E_2[Z_1], \\ \frac{d}{dt}E[Z_2^2] &= -2\omega^2 E[Z_1Z_2] - 4\zeta\omega E[Z_2^2] + 2\nu b E[P] E_2[Z_2] + \nu b^2 E[P^2] \mathcal{P}_2(t) \end{aligned} \tag{64}$$

In order to obtain the asymptotic, stationary ordinary second-order moments $E\Big[Z_rZ_s\Big]$ the stationary moments $E_2[Z_1]$ and $E_2[Z_2]$ must be determined from the equations (58). The results are

$$E_2[Z_1] = \frac{\nu^2 b E[P](\zeta\omega + \nu)}{\omega^2\left(\omega^2 + 4\nu(\zeta\omega + \nu)\right)}, \qquad E_2[Z_2] = -\frac{\nu^2 b E[P]}{2\left(\omega^2 + 4\nu(\zeta\omega + \nu)\right)}. \tag{65}$$

The stationary solution for the mean-square of the displacement response $E[Z_1^2]$ is

$$E[Z_1^2] = \frac{\nu b^2 E[P^2]}{8\zeta\omega^3} + \frac{\nu^3 b^2 E^2[P](\zeta\omega + \nu)}{\omega^2\left(\omega^2 + 4\nu(\zeta\omega + \nu)\right)} \left(\frac{\zeta\omega + \nu}{\omega^2} - \frac{1}{4\zeta\omega}\right) \tag{66}$$

Assume the data: $\omega = 1$ $[s^{-1}]$, $\zeta = 0.05$, $b = 1$ $[kg^{-1}]$, $\nu = 20$ $[s^{-1}]$, and assume that the impulses magnitudes P are Rayleigh distributed with parameter $\sigma = 0.1$ $[\frac{kg\cdot m}{s}]$. Hence $E[P] = \sigma\sqrt{\dfrac{\pi}{2}} = 0.12533$ $[\frac{kg\cdot m}{s}]$, $E[P^2] = 2\sigma^2 = 0.02$ $[\frac{kg^2\cdot m^2}{s^2}]$.

Variance of the stationary displacement response

$$var(Z_1) = E[Z_1^2] - E^2[Z_1] = 2.178 - 1.2533^2 = 0.607[\mathrm{m}^2] \tag{67}$$

Mean value and variance of the response to Erlang renewal impulse process excitation were determined in [9] with the aid of the method of state vector augmentation. The above value of $var(Z_1)$ is in perfect agreement with the numerical results given in [9].

5. Conclusions

The review of the methods for determination of the response of mechanical dynamic systems to Poisson and non-Poisson impulse process stochastic excitations is presented. Stochastic differential and integro-differential counterparts of the usual differential equations of motion are introduced. For systems driven by Poisson impulse process the following tools of the theory of non-diffusive Markov processes are presented: the generalized Itô's differential rule and the forward integro-differential Chapman-Kolmogorov equation. It is shown how these tools allow to derive the differential equations for response moments and the integro-differential equation (Kolmogorov-Feller equation) governing the probability density of the response. The relation of Poisson impulse process problems to the theory of diffusive Markov processes is also discussed. For systems driven by a class of non-Poisson (Erlang renewal) impulse processes an exact conversion of the original non-Markov problem into a Markov one is based on the appended Markov chain corresponding to the introduced auxiliary pure jump stochastic process. The derivation of the set of integro-differential equations for response probability density and also a moment equations technique are based on the forward integro-differential Chapman-Kolmogorov equation. An illustrating numerical example is also included.

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